MATHEMATICS

INDECOMPOSABLE ABELIAN GROUPS

BY

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(Communicated by Prof. H. FREUDENTHAL at the meeting of October 27, 1956)

1. *Introduction*

In this note all groups are *abelian*, written additively. We refer to KUROSH [6] for definitions and theorems used without reference.

A group is p.d. *indecomposable*, if it is not the direct sum of two proper subgroups. Indecomposable groups that are periodic (all elements of finite order) are cyclic or quasi-cyclic primary groups. Since every mixed group (i.e. a group having both elements $\neq 0$ of finite and infinite order) is decomposable, the more complicated indecomposable groups can only be found among the torsion-free groups (all elements $\neq 0$ have infinite order). Indeed, LEVI [3], PONTRJAGIN [7] and KUROSH [4, 5] constructed torsion-free groups of finite rank (the rank denotes the number of linearly independent elements), and BAER [1] proved that the additive group of p-adic integers P is indecomposable and that every serving subgroup of P is indecomposable as well. So we are furnished with examples of indecomposable groups up to continuous rank. Recently ERDÖS [9] gave a simple example of an indecomposable group of rank 2, which is essentially the same as our first example in section 2.

We present here in section 2 and 3 another simple method for *constructing indecomposable groups up to continuous potency*. Our groups are directly *embedded in the additive group of real numbers* and this "representation" is used to obtain simple proofs. Since, however, every torsion-free group of at most continuous potency is isomorphic to an additive group of real numbers, it may be observed that the possibility of this embedding is not significant in itself.

It is easy to see that our construction gives $2^{\aleph}$ (that is the cardinal of the family of all sets of real numbers) mutually non-isomorphic indecomposable groups. However, taking some care in the construction of our groups we shall even prove in section 4 the following theorem.

Theorem I. *There exists a family of $2^{\aleph}$ additive groups of real numbers such that no element of the family can be mapped homomorphically (and non-degenerately)* [1]) *in (or on) any other element of the family.*

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[1]) A homomorphic map on the null-element is called degenerate.

We note that it is easy to determine families of potency $\aleph$ satisfying the propositions of our theorem. Indeed, take a family of continuously many subgroups of rational numbers with incomparable types.

It is an unsolved problem whether there exist indecomposable abelian groups of a cardinal greater than that of the continuum (KUROSH [6] I, p. 218). As far as the author knows, no answer has as yet been found to the following question either: Does there exist for every infinite cardinal $\mathfrak{m}$ a family of $2^{\mathfrak{m}}$ mutually non-isomorphic torsion-free abelian groups? For arbitrary groups both questions (omitting the word "abelian") have been answered in the affirmative.

One could say that the groups of the family in our theorem are "completely different" from each other. Two groups are more alike if they are *equivalent* (cf. [2]), that is each is isomorphic to a subgroup of the other. We shall see in section **5** that it is easy to prove the following corollary.

Corollary. *There exists a family of* $2^{\aleph}$ *additive groups of real numbers such that any two different elements of the family are equivalent but not isomorphic.*

The number of automorphisms of a periodic group tends to infinity, if the order of the group tends to infinity (this is also true in non-abelian groups; for detailed information see LEDERMANN and NEUMANN [10]).

How is the situation if we consider abelian groups of infinite order (in the non-abelian case one has to take into account the inner automorphisms)? If the periodic part of such a group is infinite we can always split off an indecomposable periodic direct summand D, whether the group is mixed or not. If D is quasi-cyclic, D and therefore the group itself has already continuously many automorphisms. If, however, D is finite we can split off another direct summand, and so on. Thus in any case the group has an infinite number of automorphisms, if the periodic part is infinite [2]), and the number of automorphisms tends to infinity if the order of the periodic part tends to infinity (in a given sequence of groups).

Hence, there remains only the torsion-free case. The infinite cyclic group $\{a\}$ has only two *trivial automorphisms* $a \to a$ and $a \to -a$. But this is a group of rank one. However, it appears that this is not essential, since we shall prove in section **6**:

Theorem II. *There exist* $2^{\aleph}$ *(mutually non-isomorphic) additive groups of real numbers of continuous rank such that the automorphism group of each of them is trivial.*

A similar theorem holds for the endomorphisms, where the trivial endomorphisms are the maps $g \to kg$ with a fixed arbitrary integer k.

The example of Baer, previously mentioned, has the property that each

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[2]) BOYER [11] proved the theorem that a countable periodic group has continuously many automorphisms.

of its serving subgroups is indecomposable. Let us call such groups *absolutely indecomposable.* This shows already that our examples are essentially new. Indeed, they are, with a single trivial exception – not absolutely indecomposable, as the reader himself may easily show. Therefore, they cannot be embedded isomorphically in the example of Baer since the property of being a serving subgroup is transitive. This proves

Theorem III. *There exist absolutely indecomposable abelian groups up to continuous rank (Baer) and there exist indecomposable but not absolutely indecomposable groups for all ranks r satisfying* $3 \leqq r \leqq \aleph$.

In section 7 we construct *new types of absolutely indecomposable groups* up to at most countable rank.

It would be interesting to develop constructions for indecomposable groups by means of the theory of limit-groups (cf. FREUDENTHAL [8]). There are, however, a number of difficulties to overcome. E.g. the author thinks it is possible to construct a decomposable limit-group, the members of which are all indecomposable.

In section 8, at last, we construct absolutely indecomposable groups similar to those of section 7. However, the groups constructed here, have the following additional property.

Theorem IV. *There exist (absolutely indecomposable) abelian groups of countable rank in which the automorphisms of every serving subgroup are trivial.*

It may be noted that a serving subgroup of rank 1 of a group defined in section 7 has apparently non-trivial automorphisms.

In this paper we have only given more or less general examples and sets of examples of indecomposable groups. Let us realize first that there exist indecomposable groups of a completely different structure (cf. PONTRJAGIN [7] and KUROSH [4]), and secondly that we are far from any coherent theory concerning the indecomposable groups in general. Indeed, remarks like "a torsion-free group with only two automorphisms is necessarily indecomposable" and "a torsion-free group in which every two linearly independent elements have incomparable types is absolutely indecomposable" are of some use, but trivial in themselves.

2. *Indecomposable groups of finite rank*

Our construction is based upon the following indecomposable group of rank 2, previously introduced by the author [2] [3]).

If τ is a transcendental number, put

$$a_1 = \tau, \; b_1 = \tau^2$$

[3]) We repeat the proof since the method of proof is also used implicitly in other sections of this paper.

and define G as the additive group of real numbers, generated by the elements

$$\frac{a_1}{p^n}, \frac{b_1}{q^n}, \frac{a_1+b_1}{r^n}$$

where p, q and r are distinct prime numbers and the n are variable integers.

G is clearly a countable, torsion-free group of rank 2. To prove the indecomposability we can proceed e.g. in the following way. Using the transcendency of τ we see that G only contains elements of four different types, i.c. the type 0 (of the infinite cyclic group) and the types α, β and γ of the elements a_1, b_1 and a_1+b_1 respectively. Moreover, we see in the same way that only the elements of the serving subgroup $A=\{a_1/p^n\}$ generated by the elements a_1/p^n have type α, while the corresponding statements are true for the serving subgroups

$$B=\left\{\frac{b_1}{q^n}\right\} \text{ and } C=\left\{\frac{a_1+b_1}{r^n}\right\}.$$

Suppose now that G is decomposable into a direct sum

$$G=P+Q.$$

Then one of the direct summands must contain C, since otherwise, as the reader may easily verify, both P and Q should contain elements of type γ in contradiction with the fact that only the serving subgroup C of rank 1 contains such elements. Therefore, say

$$C\subset P, \text{ so } a_1+b_1\in P.$$

From this and the fact that also A and B must belong to either one of the direct summands, follows that A and B can neither belong both to Q, nor belong to different direct summands. This means that A, B and C and therefore G are contained in P, which shows $Q=0$, q.e.d.

2.1. Following the same principle we can construct indecomposable groups G of arbitrary finite rank m by taking $m+1$ distinct prime numbers $p_1, p_2, \ldots, p_m, p_{m+1}$ and defining G as the group generated by

$$\frac{\tau}{p_1^n}, \frac{\tau^2}{p_2^n}, \ldots, \frac{\tau^m}{p_m^n}, \frac{\tau+\tau^2+\ldots+\tau^m}{p_{m+1}^n},$$

where n is again a variable integer.

3. *Indecomposable groups of infinite rank*

To every subgroup of the group of rational numbers, that is a group of rank 1, is attached a certain type, as is well-known (cf. KUROSH [6], p. 207), while two such groups have the same type if and only if they are isomorphic. We shall only use types which have corresponding characteristics with argument values 0 or ∞ (though this is not at all necessary). E.g.

(1) $$t=(0, 0, \infty, 0, \infty, \infty, 0, \ldots)$$

means briefly that we have a group of rank 1 with a "fixed" element given, where this element is divisible by all powers of the third, fifth, sixth, ... prime number (in the sequence of all prime numbers), but not divisible by any other prime number. Since this "fixed" element is not uniquely determined, the reader may observe that one can always find another one for which in the corresponding characteristic a finite number of argument values zero is replaced by arbitrary finite numbers. As usual, however, we shall mean, by the type of an element, the type of the serving subgroup (of rank 1) generated by that element.

We shall i.a. use the following types (1). First, λ will be a fixed type

$$\lambda = (\infty, 0, 0, \dots).$$

Types denoted by τ and τ^* will be of the form

$$\tau = (0, t_2, 0, t_4, \dots) \qquad (t_{2i} = 0 \text{ or } \infty)$$

and

$$\tau^* = (0, 0, t_3, 0, t_5, \dots) \qquad (t_{2i+1} = 0 \text{ or } \infty)$$

respectively.

Now starting with the construction of our group, let $\{a_\mu\}$, $\{b_\mu\}$ for continuously many $\mu = \mu_\nu$ be two sets of symbols. To each a_μ anf b_μ we attach (except for the null-element) disjoint groups A_μ and B_μ of rank 1 and types τ_μ and τ_μ^* respectively, where

$$a_\mu \in A_\mu, \; b_\mu \in B_\mu.$$

Consider the group generated by all A_μ and B_μ while the following continuously many defining relations are added.

$$(2) \qquad a_{\mu_1} + b_{\mu_1} = a_{\mu_2} + b_{\mu_2} = \dots = a_{\mu_\nu} + b_{\mu_\nu} = \dots.$$

We require that the set of the types $T = \{\tau_\mu\}$ satisfies the following additional condition. For any pair of types of this set the first will contain a certain argument value 0, whereas the second has an ∞ in the corresponding place in (1), and conversely. This can obviously be done, since there are continuously many distinct types τ. We require analogous properties for $T^* = \{\tau_\mu^*\}$.

Observe that all these types τ_μ, τ_μ^* and λ are incomparable elements in the lattice of types.

Now we define a (minimal) group G which contains the group just mentioned and in which moreover the element

$$c = a_{\mu_\nu} + b_{\mu_\nu}$$

has type λ (the existence of such a group G follows e.g. from the embedding below).

G is an indecomposable group of continuous rank, and we can construct in the same way indecomposable groups of any rank with a cardinal smaller than that of the continuum.

To prove this, we define G *effectively* as an additive group of real numbers as follows. Choose a system of algebraically independent irrational real numbers $\{a_\mu\}$ of potency of the continuum (that this can be done effectively has been proved by J. VON NEUMANN). Put

$$b_\mu = 1 - a_\mu.$$

The set of elements $\{a_\mu, b_\mu\}$ satisfies therefore (2), but is otherwise linearly independent. Now we divide each of the real numbers a_μ, b_μ and $1 = a_\mu + b_\mu$ by powers of suitable prime numbers such that the a_μ, b_μ and 1 become of types τ_μ, τ_μ^* and λ respectively. Consider the group G generated by the totality of these real numbers. To prove the indecomposability of G observe that it follows from the linear independency and the choice of the types τ_μ, τ_μ^* and λ, that 1 is also *in* G of type λ and the a_μ and b_μ are also *in* G of type τ_μ and τ_μ^* respectively. Further observe that there occur elements of numerous other types (e.g. an element $a_{\mu_1} - a_{\mu_2}$), but such a type is (the type of the null-element excepted) always *smaller than or incomparable with* any of the types τ_μ, τ_μ^*, λ (in the lattice of types). From this follows that we can use exactly the same method of proof as in section 2, which shows the indecomposability of G.

4. *Proof of Theorem I*

There are $\aleph$ incomparable types τ_μ. As is well-known from set theory, one can determine in this set $T = \{\tau_\mu\}$ a system $\{S\}$ of $2^\aleph$ subsets S of T (each of potency $\aleph$) such that for every pair of distinct S and S' each contains an element not belonging to the other set. Form the corresponding T^* and $\{S^*\}$ (if the $2i$-th coordinate in (1) is 0 or ∞ for τ_μ, the same will hold for the $(2i+1)^{\text{th}}$ coordinate of a certain τ_μ^*; so T is mapped one to one on T^*). To each corresponding pair S, S^* one can determine an indecomposable group of continuous rank as described in the preceding sections, S and S^* replacing T and T^*. Thus we have a family F of $2^\aleph$ indecomposable additive groups of real numbers, each element $f \in F$ corresponding to a pair S, S^*. Let there be given a homomorphic map φ of f in $f' \in F$, the latter corresponding to S', S'^*. A homomorphic map of a group of rank 1 in a torsion-free group is clearly either isomorphic or degenerate. There is a τ_μ with

$$\tau_\mu \in S, \ \ \tau_\mu \notin S'.$$

There is an element a_μ in the group f of type τ_μ. We shall prove

$$\varphi a_\mu = 0.$$

Indeed, otherwise, φa_μ would have type $\geqq \tau_\mu$, which is impossible, since S' does not contain such types and therefore f' does not contain subgroups of rank 1 of such a type. Since the corresponding statements hold for S^* and S'^* we have

$$\varphi b_\mu = 0.$$

But then $\varphi c = 0$ and therefore

$$\varphi a_{\mu_\nu} = -\varphi b_{\mu_\nu} \quad \text{for all } \mu_\nu.$$

Hence the type of φa_{μ_ν} is at least $\tau_{\mu_\nu} + \tau^*_{\mu_\nu}$ in f'. Such elements do not occur in f', the null-element excepted. So

$$\varphi a_{\mu_\nu} = 0 \quad \text{for all } \mu_\nu,$$

this means $\varphi f = 0$, so φ is degenerate, which proves the theorem.

5. *Proof of the corollary*

Starting from the family $\{f\} = F$ in our theorem we construct a new family F^*, in which each element is the direct sum of an f with R, the additive group of real numbers

$$\{f + R\} = F^*.$$

Two groups $f + R$ and $f' + R'$ are equivalent since each torsion-free group with continuously many elements can be embedded isomorphically in R. However, $f + R$ and $f' + R'$ are not isomorphic, since an isomorphic map ψ of $f + R$ on $f' + R'$ would carry the uniquely determined and (with respect to automorphisms) invariantly defined maximal complete subgroup R onto R'. Hence

$$\psi(f + R) = \psi f + R' = f' + R',$$

so

$$\psi f \simeq f' + R'/R' \simeq f',$$

which is impossible, since f is not isomorphic to f'.

We remark that also a simple proof of this corollary can be given without making use of our theorem. Indeed, form the (restricted) direct sum D of continuously many subgroups of the additive group of rational numbers with incomparable types $\{t\} = T$. There are $2^{\aleph}$ subsets $S, S', \ldots,$ of T such that for every pair of different sets S, S' each contains a type not contained in the other. To each S a subgroup $G \subset D$ corresponds uniquely. The system of $2^{\aleph}$ groups $\{G + R\}$ satisfies the conditions required.

6. *Proof of Theorem II*

The family F of section 4 satisfying theorem I, also satisfies the requirements of theorem II. Indeed, since the element c of a group $f \in F$ and the elements linearly dependent of c are the only ones in f of type λ, a non-trivial automorphism of f must necessarily map

$$c \to \pm\, 2^n c,$$

where n is some integer $\neq 0$.

From the linear independence of the a_μ and b_μ, relation (2) excepted, and the fact that a group A_μ or B_μ necessarily maps on itself it follows

$$a_\mu \to \pm\, 2^n\, a_\mu, \quad b_\mu \to \pm\, 2^n\, b_\mu.$$

But such a map is not automorphic on A_μ or B_μ. Thus the automorphisms of f are trivial.

7. *Construction of countable, absolutely indecomposable groups*

We take a countable number of linearly independent real numbers $a_1, a_2, \ldots$, and enumerate all those (finite) linear forms in the a_i with integer coefficients

$$n_1 a_{i_1} + n_2 a_{i_2} + \ldots + n_k a_{i_k} \quad (i_1 < i_2 < \ldots < i_k)$$

for which

$$(|n_1|, |n_2|, \ldots, |n_k|) = 1$$

and

$$n_1 > 0.$$

Let $b_1, b_2, \ldots$ be an enumeration of these linear forms. We define a minimal group of real numbers in which $b_i (i = 1, 2, \ldots)$ becomes of type λ_i where λ_i has argument values 0 in (1), the i^{th} argument value excepted, which equals ∞. Observe that this construction is possible indeed, and that each non-null-element of the group has a certain type λ_i, and any two elements which are linearly independent, have different, even incomparable, types. This group is absolutely indecomposable. Indeed, a serving subgroup has the same properties concerning its types as mentioned. Hence a proper decomposition is impossible as the reader may easily verify.

This construction method yields $\aleph$ different absolutely indecomposable groups of finite or countable rank.

8. *Proof of Theorem IV*

We decompose the set of prime numbers $\{p_k\}$ into a countable number of infinite disjoint subsets $P_1, P_2, \ldots$.

Type $\mu_i (i = 1, 2, \ldots)$ is defined by means of a characteristic (1), in which the k^{th} argument t_k will have the following value

$$\begin{cases} t_k = 1 \text{ if } p_k \in P_i, \\ t_k = 0 \text{ otherwise.} \end{cases}$$

Now we consider the construction of section 7, in which the types λ_i are replaced by the μ_i.

This group is also absolutely indecomposable and one proves easily that every serving subgroup has only trivial automorphisms.

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