Separation of variables and exact solutions to nonlinear diffusion equations with $x$-dependent convection and absorption

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Abstract

This paper considers nonlinear diffusion equations with $x$-dependent convection and source terms: $u_t = (D(u)u_x)_x + Q(x,u)u_x + P(x,u)$. The functional separation of variables of the equations is studied by using the generalized conditional symmetry approach. We formulate conditions for such equations which admit the functionally separable solutions. As a consequence, some exact solutions to the resulting equations are constructed. Finally, we consider a special case for the equations which admit the functionally separable solutions when the convection and source terms are independent of $x$.

1. Introduction

In this paper, we are concerned with the separation of variables and exact solutions to the following $(1+1)$-dimensional nonlinear diffusion equations with $x$-dependent convection and source terms

$$u_t = (D(u)u_x)_x + Q(x,u)u_x + P(x,u)$$

where $D(u)$ is the given diffusion coefficient; $Q(x,u)$ and $P(x,u)$ are respectively the convection and source terms. They are smooth functions of the indicated variables. These types of equation arise in several important physical applications including engineering [1], physics [2], the theory of chemical reactions [3] and biology [4], etc.

In the case that $Q(x,u)$ and $P(x,u)$ are independent of $x$, a number of methods relating to symmetry group have been developed successfully to find exact solutions and symmetry reductions of various special forms of (1). These methods are the Lie-point symmetry method [7,17], the nonclassical method [5,8,10,13,36,37], the direct method [9,12,21], the generalized conditional symmetry (GCS) method [11,15,16,22–35,39], the sign-invariant and invariant-subspace methods [18–20] and Painlevé analysis [6,14]. Many exciting results have been derived. There are also

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many well-developed mathematical theories that consider the qualitative analysis of the equations, such as existence, uniqueness, blow-up and asymptotic behavior, etc. Among these methods, the symmetry group methods have been proved to be effective to classify equations or specify the functions appeared in the equations with respect to given symmetries.

In the case that \( Q(x, u) \) and \( P(x, u) \) depend on \( x \), such equations also have important physical applications (see [27,38] and references therein), but there are fewer works on Eq. (1).

In [26], Changzheng Qu et al. had firstly used the generalized conditional symmetry approach to study the separation of variables of quasilinear diffusion equations with the nonlinear source. They had obtained a complete list of canonical forms for such equations which admit the functionally separable solutions. In this paper, we apply the generalized conditional symmetry approach to study the separation of variables and exact solutions of (1).

Suppose a nonlinear second-order evolution equation of the form

\[
  u_t = E(x, u, u_x, u_{xx}),
\]

(2)

where \( E \) is smooth function of the indicated variables.

For a nonlinear partial differential equation (PDE) with one dependent variable \( u \) and two independent variables \( x \) and \( t \), the solutions

\[
  u = \phi(t) \psi(x) \quad \text{or} \quad u = \phi(t) + \psi(x)
\]

are known as the product or additive separable solutions, respectively. They are also called the ordinary separable solutions. A generalization to the ordinary separable solutions is

\[
  f(u) = \phi(t) + \psi(x).
\]

A solution of (2) is said to be functionally separable if there exist some functions \( f \), \( \phi \) and \( \psi \) of the indicated single variable such that (3). It has been shown that many nonlinear PDEs have this type of separable solutions. (3) is the additive separable solutions when \( f(u) = u \), and (3) is the product separable solutions when \( f(u) = \ln u \).

The outline of the paper is as follows. In Section 2, we present some necessary definitions and fundamental theorem on the GCS. In Section 3, we formulate conditions for Eq. (1) which possesses the functionally separable solution (3), and give some special cases to each condition. Some new equations are obtained, and their corresponding exact solutions are derived. Some examples are given in Section 4. In Section 5, we formulate conditions for Eq. (1) which admits the functionally separable solution (3) when the convection and source terms are independent of \( x \). Finally, we conclude the paper.

2. Basic definitions and theorem

Let

\[
  V = \eta(t, x, u, u_1, u_2, \ldots, u_j) \frac{\partial}{\partial u}
\]

(4)

be an evolutionary vector field and \( \eta \) be its characteristic, where \( u_i = \frac{\partial^j u}{\partial x_j} \), \( j = 1, 2, \ldots \).

**Definition 1.** The evolutionary vector field (4) is said to be a generalized symmetry of Eq. (2) if and only if

\[
  V^{(2)}(u_t - E)|_L = 0,
\]

where \( L \) is the solution set of Eq. (2), and \( V^{(2)} \) is a second order prolongation of \( V \).

**Definition 2.** The evolutionary vector field (4) is said to be a generalized conditional symmetry (GCS) of Eq. (2) if and only if

\[
  V^{(2)}(u_t - E)|_{L \cap W} = 0,
\]

(5)

where \( W \) is the set of equations \( D^i \eta = 0 \), \( i = 0, 1, 2, \ldots \).

It follows from (5) that Eq. (2) admits the GCS (4) if and only if

\[
  D_t \eta = 0,
\]
where $D_t \eta$ is the total derivative in $t$. Moreover, if $\eta$ does not depend on time $t$ explicitly, then
\[ \eta'_E |_{L \cap W} = 0, \]
where
\[ \eta'(u) E = \lim_{\varepsilon \to 0} \frac{d}{d\varepsilon} \eta(u + \varepsilon E) \]
denotes the Gateaux derivative of $\eta$ along the direction $E$.

It follows from (3) that $f_{xt} = 0$, that is $f_{xt} = (f' u_x)_t = f'' u_x u_t + f' u_{xt} = 0$.

Considering that $f' \neq 0$, let $V = (u_{xt} + g(u) u_x u_t) \frac{\partial}{\partial u}$ be an evolutionary vector field, and $\eta \equiv u_{xt} + g(u) u_x u_t$.

The following theorem (that has been proved in [26]) is useful.

**Theorem 1.** Equation (2) possesses the functionally separable solution (3) if and only if it admits the GCS
\[ V = (u_{xt} + g(u) u_x u_t) \frac{\partial}{\partial u}. \]

It can be noted that $g(u) = f''(u) / f'(u) = 0$ when $f(u) = u$, and $g(u) = -u^{-1}$ when $f(u) = \ln u$.

**Corollary 1.** Equation (2) possesses the additive separable solution
\[ u = \phi(t) + \psi(x) \]
if and only if it admits the GCS
\[ V = u_{xt} \frac{\partial}{\partial u}. \]

**Corollary 2.** Equation (2) possesses the product separable solution
\[ u = \phi(t) \psi(x) \]
if and only if it admits the GCS
\[ V = (u_{xt} - u^{-1} u_x u_t) \frac{\partial}{\partial u}. \]

3. Conditions for Eq. (1) which possesses the functionally separable solution (3)

**Theorem 2.** Equation (1) possesses the functionally separable solution (3) if and only if the coefficient functions $D(u)$, $Q(x,u)$, $P(x,u)$ and $g(u)$ satisfy the following system of PDEs:

\[
\begin{align*}
    h_1 &\equiv 3D''D - 3(D')^2 - 3D'Dg - 2D^2(g' - g^2) = 0, \\
    h_2 &\equiv DQ_u - D'Q = 0, \\
    h_3 &\equiv D''D - D''D' - D''Dg - (D')^2g - D'D(2g' - g^2) - D^2(g'' - 2g'g) = 0, \\
    h_4 &\equiv DQ_{uu} - D'Q_u - D'Qg = 0, \\
    h_5 &\equiv DP_{uu} - D'P_u + DP_{ag} - D'Pg + DP'g + DQ_{xu} - D'Q_x = 0, \\
    h_6 &\equiv DP_{xu} + DP_x g - D'P_x = 0,
\end{align*}
\]

or

\[
\begin{align*}
    p' + g' - pg - g^2 &= 0, \\
    q' + qg &= 0, \\
    u_{xx} &= pu_x^2 + au_x + q, \\
    u_t &= Du_{xx} + D'u_x + Qu_x + P,
\end{align*}
\]

where $a$ is an arbitrary constant, $p(u)$ and $q(u)$ are to be determined.
Proof. By Theorem 1 and Definition 2, Eq. (1) possesses the functionally separable solution (3) iff $D_x \eta = 0$, where

$$\eta = u_{xt} + g(u)u_{tx}.$$  

Considering that $u_t = D u_{xx} + D' u_t^2 + Q u_x + P$, we get

$$u_{tx} = D u_{xxx} + 3 D' u_x u_{xx} + Q u_{xxx} + D'' u^3_x + Q u u_x^2 + P u_x + Q u_x u_x + P_x,$$

$$u_{tt} = D u_{xxt} + 2 D' u_x u_{xt} + Q u_{xxt} + D' u_{xx} u_t + D'' u^2_x + Q u u_x u_t + P u_{xt},$$

$$u_{ttt} = D u_{xxtt} + 3 D' u_x u_{xxt} + Q u_{xxtt} + 3 D'' u_x^2 u_{xt} + 3 D' u_{xx} u_{xt} + Q u u_{xt} u_x + Q u_u u_x u_t + P u_{xxt}$$

$$\quad + D' u_{xxx} u_t + 3 D'' u_x u_{xx} u_t + Q u_{xx} u_{xx} u_t + D'' u^3_x u_t + Q u u_{xxx} u_x u_t + P u_{xx} u_{xt} + Q u_x u_x u_t + P_{xx} u_{xt}.$$

In view of $\eta = 0$, $D_x \eta = 0$, $D_x^2 \eta = 0$, we have

$$\eta = u_{xt} + g u_x u_t = 0,$$

$$D_x \eta = u_{xxt} + g' u_x^2 u_t + g u_{xxx} u_t + g u_x u_{xt} = 0,$$

$$D_x^2 \eta = u_{xxx} + g u_{xx} u_{xt} + 2 g u_{xxx} u_t + 2 g' u_x^2 u_{xt} + g u_{xxx} u_t + 3 g' u_x u_x u_t + g'' u^3_x u_t = 0,$$

and

$$D u_{xxx} = -(3 D' u_x u_{xx} + g D u_{xxx} u_x + Q u_{xxx} + D'' u^3_x + g D' u_x^3 + Q u u_x^2 + g Q u_x^2 + P u_x + g P u_x$$

$$\quad + Q u_x u_x + P_x).$$

Reducing $D_{x} \eta = u_{xxt} + g'(u) u_x^2 u_t + g(u) u_{xxx} u_t + g(u) u_x u_{xt} = 0$ gives

$$D_x \eta = h_1(x, u) u_x u_t + h_2(x, u) u_{xxx} u_t + h_3(x, u) u_{xxx} u_t + h_4(x, u) u_x^2 u_t + h_5(x, u) u_x u_t + h_6(x, u) u_t = 0.$$

That is

$$h_i(x, u) = 0, \quad i = 1, 2, 3, 4, 5, 6,$$

where $h_i = h_i(x, u), i = 1, \ldots, 6$, shown just as (8).

If $u_{xx}$ is linearly independent of $u_x^2$ and $u_x$ and any functions of $u$, then the above expression of $D_x \eta = 0$ yields the system (8). Otherwise, there are two functions $p(u), q(u)$ and the constant $a$ satisfy $u_{xx} = p u_x^2 + a u_x + q$.

In view of

$$\eta = u_{xt} + g u_x u_t = 0,$$

$$D_x \eta = u_{xxt} + g' u_x^2 u_t + g u_{xxx} u_t + g u_x u_{xt} = 0$$

and

$$u_{xxt} = p' u_x^2 u_t + 2 p u_x u_{xt} + a u_x + q' u_t,$$

we arrive at

$$D_x \eta = (p' + g' - p g - g^2) u_x^2 u_t + (q' + q g) u_t = 0,$$

that is

$$p' + g' - p g - g^2 = 0, \quad q' + q g = 0.$$

Considering that $u_{xx} = p u_x^2 + a u_x + q$ and $u_t = D u_{xx} + D' u_x^2 + Q u_x + P$, finally, system (9) is obtained.  \hfill \square

Remark. In system (9), there are two zeros among $a$, $p$ and $q$, otherwise, only can a trivial separable solution be obtained. $p(u)$ and $q(u)$ can be determined when $g(u)$ is given. The additive separable solution is considered when $g = 0$, and the product separable solution is considered when $g(u) = -u^{-1}$.

Theorem 3. Equation (1) possesses the additive separable solution (6) if and only if the coefficient functions $D(u), Q(x, u)$ and $P(x, u)$ satisfy the following system of PDEs:
or
\[ D'' + 3bD' + 2b^2D - (bx + c)(Q_u + bQ) + (bx + c)^2(Q_x + P_u) - (bx + c)^3P_x = 0, \]
where \( c, b \neq 0 \) are two arbitrary constants;
or
\[ D''a^3e^{3ax} + (3aD' + Q_u)a^2c^2e^{2ax} + (Q_x + aQ + P_u + a^2D)ace^{ax} + P_x = 0, \]
where \( ac \neq 0 \) are two arbitrary constants;
or
\[ 8a^3(x + b)^3D'' + 4a^2(x + b)^2Q_u + 2a(x + b)(6aD' + Q_x + P_u) + 2aQ + P_x = 0, \]
where \( b, a \neq 0 \) are two arbitrary constants.

The additive separable solutions (6) to Eq. (1) with (11), (12) and (13) respectively are given by:

(i) \( \psi(x) = -\frac{1}{b} \ln |bx + c|, \) \( \phi'(t) = \frac{bD + D'}{c^2} - \frac{1}{c}Q(0, u) + P(0, u) \) \( (c \neq 0); \)
(ii) \( \psi(x) = ce^{ax}, \) \( \phi'(t) = a^2c^2D' + a^2cD + acQ(0, u) + P(0, u) \) \( (ac \neq 0); \)
(iii) \( \psi(x) = a(x + b)^2, \) \( \phi'(t) = 4a^2b^2D' + 2abQ(0, u) + 2aD + P(0, u) \) \( (a \neq 0). \)

Some special cases to Eq. (1) with (11), (12) and (13), respectively, are as follows:

(i) \( D(u) = -\frac{c_1}{b}e^{-2bu} + c_2e^{-bu}, \) \( Q(x, u) = (bx + c)F_1(\xi), \) \( P(x, u) = F_2(\xi), \)
where \( \xi = u + \frac{\ln |bx + c|}{b}, \) \( F_1 \) and \( F_2 \) are two arbitrary smooth functions, \( b \neq 0, c, c_1, c_2 \) are arbitrary constants. \( \phi(t) \) satisfies \( \phi'(t) = c_1e^{-2b\phi(t)} + F_2(\phi) - F_1(\phi). \)
(ii) \( D(u) = \frac{c_1}{a^2}u + c_2, \) \( P(x, u) = 2c^2c_1e^{2ax} - a^2cc_2e^{ax} - cc_1ue^{ax} - ace^{ax}F_1(\xi) + F_2(\xi), \)
\( Q(x, u) = F_1(\xi) - \frac{3cc_1}{a}e^{ax}, \)
where \( \xi = u - ce^{ax}, \) \( F_1 \) and \( F_2 \) are two arbitrary smooth functions, \( ac \neq 0, c_1, c_2 \) are arbitrary constants. \( \phi(t) \) satisfies \( \phi'(t) = F_2(\phi). \)
(iii) \( D(u) = \frac{c_1}{4a}u + c_2, \) \( Q = 0, \) \( P(x, u) = -\frac{3}{2}ac_1x^2 - 3abc_1x + F(u - ax^2 - 2abx), \)
where \( F \) is an arbitrary smooth function, \( a \neq 0, b, c_1, c_2 \) are arbitrary constants. \( \phi(t) \) satisfies \( \phi'(t) = \frac{3}{2}ab^2c_1 + 2ac_2 + \frac{c_1}{2}F(ab^2 + \phi). \)

Solving the system (10), we obtain
\[ D(u) = c_1e^{c_2u}, \]
\[ Q(x, u) = Q_1(x)e^{c_2u}, \]
\[ P(x, u) = P_1(x)e^{c_2u} + c_3 \quad (c_2 \neq 0), \]
\[ P(x, u) = P_1(x) + c_3u \quad (c_2 = 0), \]

where \( P_1(x) \) and \( Q_1(x) \) are two arbitrary smooth functions, \( c_1, c_2 \) and \( c_3 \) are three arbitrary constants.

Moreover, Eq. (1) with (14) has the additive separable solution (6) which satisfies:

1. When \( c_2 \neq 0 \), then
   \[ e^{-c_2\phi(t)}[\phi'(t) - c_3] = \lambda, \]
   \[ c_1e^{c_2\psi(x)}\psi''(x) + c_1c_2e^{c_2\psi(x)}[\psi'(x)]^2 + Q_1(x)e^{c_2\psi(x)}\psi'(x) + P_1(x)e^{c_2\psi(x)} = \lambda. \]

2. When \( c_2 = 0 \), then
   \[ \phi'(t) - c_3\phi(t) = \lambda, \]
   \[ c_1\psi''(x) + Q_1(x)\psi'(x) + c_3\psi(x) + P_1(x) = \lambda, \]

where and henceforth \( \lambda \) denotes separable constant.

**Theorem 4.** Equation (1) possesses the product separable solution (7) if and only if the coefficient functions \( D(u), Q(x, u) \) and \( P(x, u) \) satisfy the following system of PDEs:

\[
\begin{align*}
D''D - (D')^2 + D'Du^{-1} = 0, \\
DQ_u - D'Q = 0, \\
DP_{xu} - DP_{x}u^{-1} - D'P_x = 0, \\
DQ_{uu} - D'Q_u + D'Qu^{-1} = 0, \\
DP_{uu} - D'P_u - DP_{u}u^{-1} + D'Pu^{-1} + DP_{u}^{-2} + DQ_{xu} - D'Q_x = 0.
\end{align*}
\]

or

\[
\begin{align*}
(u^3D'' - u^2D')c^3e^{3cx} + (3cu^2D' + u^2Q_u - cuD - uQ)c^2e^{2cx}(c_0 + e^{cx}) \\
+ (uQ_x + uP_u + c^2uD + cuQ - P)ce^{cx}(c_0 + e^{cx})^2 + P_1(c_0 + e^{cx})^3 = 0.
\end{align*}
\]

where \( c_0, c \neq 0 \) are two arbitrary constants;

or

\[
\begin{align*}
u^3D'' + (3c_1 - 1)u^2D' + (2c_1^2 - 2c_1)uD - (u^2Q_u + c_1uQ - uQ)\xi + (uQ_x + uP_u - P)\xi^2 - P_3\xi^3 \ &= 0,
\end{align*}
\]

where \( \xi = (c_1 - 1)x + c, c, c_1 \neq 1 \) are two arbitrary constants;

or

\[
\begin{align*}
c^3(u^2D'' - uD')\frac{\xi^3}{\xi^3} + c^2(uQ_u - Q)\frac{\xi^2}{\xi^2} + \left(3c^3u'D' + cQ_x + cP_u - \frac{cP}{u}\right)\frac{\xi_1}{\xi} + c^2Q + \frac{P_3}{u} = 0,
\end{align*}
\]

where \( \xi = c_1e^{cx} + c_2e^{-cx}, \xi_1 = c_1e^{cx} - c_2e^{-cx}, c \neq 0, c_1c_2 \neq 0 \) are arbitrary constants;

or

\[
\begin{align*}
c^3(u^2D'' - uD')\frac{\xi^3}{\xi^3} + c^2(uQ_u - Q)\frac{\xi^2}{\xi^2} + \left(-3c^3u'D' + cQ_x + cP_u - \frac{cP}{u}\right)\frac{\xi_1}{\xi} - c^2Q + \frac{P_3}{u} = 0,
\end{align*}
\]

where \( \xi = c_1\cos cx + c_2\sin cx, \xi_1 = -c_1\sin cx + c_2\cos cx, c \neq 0, c_1, c_2 \) are arbitrary constants.
The product separable solutions (7) to Eq. (1) with (16)–(19), respectively, are given by:

(i) \( \psi(x) = c_0 + e^{cx}, \quad \frac{\phi'(t)}{\phi(t)} = \frac{c^2 u D' + Q(0, u) + P(0, u)}{c_0 + 1} \) \( (c \neq 0, c_0 + 1 \neq 0). \)

(ii) \( \psi(x) = [(c_1 - 1)x + c]^{\frac{1}{1-c}}, \quad \frac{\phi'(t)}{\phi(t)} = c^{-2}(u D' + c_1 D) - c^{-1}Q(0, u) + \frac{P(0, u)}{u}, \)

where \( c_1 \neq 1, c \neq 0. \)

(iii) \( \psi(x) = c_1 e^{cx} + c_2 e^{-cx}, \quad \frac{\phi'(t)}{\phi(t)} = c^2 (c_1 - c_2)^2 u D' + c_1 - c_2 \frac{Q(0, u)}{c_1 + c_2} + \frac{P(0, u)}{u}, \)

where \( c \neq 0, c_1 + c_2 \neq 0. \)

(iv) \( \psi(x) = c_1 \cos cx + c_2 \sin cx, \quad \frac{\phi'(t)}{\phi(t)} = \frac{c^2 c_2}{c_1} u D' - c^2 D + \frac{cc_2}{c_1} Q(0, u) + \frac{P(0, u)}{u}, \)

where \( c \neq 0, c_1 \neq 0. \)

Some special cases to Eq. (1) with (16)–(19), respectively, are as follows:

(1) \( D(u) = c_1 + c_2 \ln u, \quad Q(x, u) = -c_2 c^2 x + F_1(ue^{-cx}), \quad P(x, u) = e^{cx} F_2(ue^{-cx}), \)

where \( F_1 \) and \( F_2 \) are arbitrary smooth functions, \( c \neq 0, c_1, c_2 \) are arbitrary constants. \( \phi(t) \) satisfies \( \phi'(t) = (c_1 c^2 + c_2^2) \phi + c_2 e^2 \phi \ln |\phi| + c \phi F_1(\phi) + F_2(\phi). \)

(2) \( D(u) = c_1 c_2 u^{2-2c_1}, \quad Q(x, u) = \xi F_1(ue^{\frac{1}{1-\xi}}) + \frac{c_1 c_2}{\xi}, \quad P(x, u) = \xi^{\frac{1}{1-c}} F_2(ue^{\frac{1}{1-\xi}}), \)

where \( F_1 \) and \( F_2 \) are arbitrary smooth functions, \( \xi = (c_1 - 1)x + c, c_1 \neq 1, c_2, c_3 \) are arbitrary constants. \( \phi(t) \) satisfies \( \phi'(t) = (2c_3 - c_1 c_3) \phi^{3-2c_1} - c F_1(\phi) + F_2(\phi). \)

(3) \( D(u) = c_2 u^{-c_1} + c_3 u^{2-2c_1}, \quad Q(x, u) = \xi F_1(ue^{\frac{1}{1-\xi}}), \quad P(x, u) = \xi^{\frac{1}{1-c}} F_2(ue^{\frac{1}{1-\xi}}), \)

where \( F_1 \) and \( F_2 \) are arbitrary smooth functions, \( \xi = (c_1 - 1)x + c, c_1 \neq 1, 2, c_2, c_3 \) are arbitrary constants. \( \phi(t) \) satisfies \( \phi'(t) = (2c_3 - c_1 c_3) \phi^{3-2c_1} - c F_1(\phi) + F_2(\phi). \)

(4) \( D(u) = c_1 + c_2 u^{3-2c_1}, \quad Q(x, u) = \xi F_1(ue^{\frac{1}{1-\xi}}) + c_1 c_2 u^{c_1} + c_1 c_3 u^{3-c_1}, \)

\[ P(x, u) = \xi^{\frac{1}{1-c}} F_2(ue^{\frac{1}{1-\xi}}), \]

where \( F_1 \) and \( F_2 \) are arbitrary smooth functions, \( \xi = (c_1 - 1)x + c, c_1 \neq 1, c_2, c_3 \) are arbitrary constants. \( \phi(t) \) satisfies \( \phi'(t) = (3c_3 - 2c_1 c_3) \phi^{4-2c_1} - c F_1(\phi) + F_2(\phi). \)

(5) \( D = b, \quad Q(x, u) = \xi F_1(ue^{\frac{1}{1-\xi}}) \),

\[ \phi(t) = \xi F_2(ue^{\frac{1}{1-\xi}}), \]

where \( F_1 \) and \( F_2 \) are arbitrary smooth functions, \( \xi = c_1 e^{cx} + c_2 e^{-cx}, b, c_1 \neq 0, c_1 c_2 \neq 0 \) are arbitrary constants. \( \phi(t) \) satisfies \( \phi'(t) = b c^2 \phi + c F_1(\phi) + F_2(\phi). \)

(6) \( D = b, \quad Q(x, u) = \xi F_1(ue^{\frac{1}{1-\xi}}) \),

\[ \phi(t) = \xi F_2(ue^{\frac{1}{1-\xi}}), \]

where \( F_1 \) and \( F_2 \) are arbitrary smooth functions, \( \xi = c_1 \cos cx + c_2 \sin cx, b, c_1 \neq 0, c_1, c_2 \) are arbitrary constants. \( \phi(t) \) satisfies \( \phi'(t) = -b c^2 \phi - c F_1(\phi) + F_2(\phi). \)

Remark. The special cases (2), (3) and (4) above are ones of (17).
Solving the system (15), we obtain

\[ D(u) = c_1 u^{c_2}, \]
\[ Q(x, u) = Q_2(x) u^{c_2}, \]
\[ P(x, u) = P_2(x) u^{c_2 + 1} + c_3 u \quad (c_2 \neq 0), \]
\[ P(x, u) = P_2(x) u + c_4 u \ln u \quad (c_2 = 0), \]  

where \( P_2(x) \) and \( Q_2(x) \) are two arbitrary smooth functions, \( c_1, c_2, c_3 \) and \( c_4 \) are arbitrary constants.

Moreover, Eq. (1) with (20) has the product separable solution (7) which satisfies:

1. When \( c_2 \neq 0 \), then
   \[ \phi(t)^{c_2 - 1} \left[ \phi'(t) - c_3 \phi(t) \right] = \lambda, \]
   \[ \psi(x)^{c_2 - 1} \left[ c_1 \psi''(x) + Q_2(x) \psi'(x) + c_1 c_2 \psi(x)^{-1} \psi'(x)^2 + P_2(x) \psi(x) \right] = \lambda. \]

2. When \( c_2 = 0 \), then
   \[ \phi^{-1}(t) \left[ \phi'(t) - c_4 \phi(t) \ln |\phi(t)| \right] = \lambda, \]
   \[ c_1 \psi^{-1}(x) \psi''(x) + Q_2(x) \psi^{-1}(x) \psi'(x) + c_4 \ln |\psi(x)| + P_2(x) = \lambda. \]

4. Examples

Equation (1) when \( D(u) = u^m \) is considered, and some examples are given.

**Example 1.** Equation \( u_t = u_{xx} + Q_1(x) u_x + a u + P_1(x) \) has the additive separable solution (6) which satisfies:

\[ \phi'(t) - a \phi(t) = \lambda, \]
\[ \psi''(x) + Q_1(x) \psi'(x) + a \psi(x) + P_1(x) = \lambda, \]

where \( P_1 \) and \( Q_1 \) are arbitrary smooth functions, \( a \) is an arbitrary constant.

**Remark.** The potential form of the well-known Burgers equation, \( w_t = w_{xx} + w^2 \) can be obtained from \( u_t = u_{xx} \) by the transformation \( u = e^{aw} \).

**Example 2.** Equation \( u_t = (uu_x)_x + Q(x, u) u_x + P(x, u) \) has the additive separable solution (6) which satisfies:

\[ \psi(x) = ce^{ax}, \quad \phi'(t) = F_2(\phi), \]

where

\[ Q(x, u) = F_1(u - ce^{ax}) - 3ace^{ax}, \]
\[ P(x, u) = 2a^2 c^2 e^{2ax} - a^2 cue^{ax} - ace^{ax} F_1(u - ce^{ax}) + F_2(u - ce^{ax}), \]

where \( F_1 \) and \( F_2 \) are arbitrary smooth functions, \( a, c \neq 0 \) are arbitrary constants.

**Example 3.** Equation \( u_t = (uu_x)_x - 6a(ax^2 + 2abx) + F(u - ax^2 - 2abx) \) has the additive separable solution (6) which satisfies:

\[ \psi(x) = a(x + b)^2, \quad \phi'(t) = 2a \phi(t) + F(ab^2 + \phi) + 6a^2 b^2, \]

where \( F \) is an arbitrary smooth function, \( b, a \neq 0 \) are arbitrary constants.

**Example 4.** Equation \( u_t = (u^m u_x)_x + Q_2(x) u^m u_x + P_2(x) u^{m+1} + bu \) \( (m \neq 0) \) has the product separable solution (7) which satisfies:
\[ \phi(t)^{-m-1}[\phi'(t) - b\phi(t)] = \lambda, \]
\[ \psi(x)^{-m-1}[\psi''(x) + m\psi(x)^{-1}[\psi'(x)]^2 + Q_2(x)\psi'(x) + P_2(x)\psi(x)] = \lambda, \]
where \( P_2 \) and \( Q_2 \) are arbitrary smooth functions, \( b \) is an arbitrary constant.

**Example 5.** Equation \( u_t = u_{xx} + Q_2(x)u_x + P_2(x)u + bu \ln u \) has the product separable solution (7) which satisfies:
\[ \phi^{-1}(t)\phi'(t) - b \ln|\phi(t)| = \lambda, \]
\[ \psi^{-1}(x)\psi''(x) + Q_2(x)\psi^{-1}(x)\psi'(x) + b \ln|\psi(x)| + P_2(x) = \lambda, \]
where \( P_2 \) and \( Q_2 \) are arbitrary smooth functions, \( b \) is an arbitrary constant.

**Example 6.** Equation
\[ u_t = \left(u^{-2k}u_x\right)_x + (kx + c)F_1(u(kx + c)^\frac{1}{2})u_x + (kx + c)^\frac{1}{2}F_2(u(kx + c)^\frac{1}{2}) \quad (k \neq 0) \]
has the product separable solution (7) which satisfies:
\[ \psi(x) = (kx + c)^{-\frac{1}{2}}, \]
\[ \phi'(t) = (1 - k)\phi^{1-2k} - \phi F_1(\phi) + F_2(\phi), \]
where \( F_1 \) and \( F_2 \) are arbitrary smooth functions, \( c, k \neq 0 \) are arbitrary constants.

**Example 7.** Equation
\[ u_t = u_{xx} + \frac{c_1e^{cx} + c_2e^{-cx}}{c_1e^{cx} - c_2e^{-cx}}F_1\left(\frac{u}{c_1e^{cx} + c_2e^{-cx}}\right)u_x + \left(c_1e^{cx} + c_2e^{-cx}\right)F_2\left(\frac{u}{c_1e^{cx} + c_2e^{-cx}}\right) \]
has the product separable solution (7) which satisfies:
\[ \psi(x) = c_1e^{cx} + c_2e^{-cx}, \]
\[ \phi'(t) = c^2\phi + c\phi F_1(\phi) + F_2(\phi), \]
where \( F_1 \) and \( F_2 \) are arbitrary smooth functions, \( c \neq 0, c_1, c_2 \neq 0 \) are arbitrary constants.

**Example 8.** Equation
\[ u_t = u_{xx} + \frac{c_1 + c_2 \tan cx}{c_1 \tan cx - c_2}F_1\left(\frac{u}{c_1 \cos cx + c_2 \sin cx}\right)u_x + \left(c_1 \cos cx + c_2 \sin cx\right)F_2\left(\frac{u}{c_1 \cos cx + c_2 \sin cx}\right) \]
has the product separable solution (7) which satisfies:
\[ \psi(x) = c_1 \cos cx + c_2 \sin cx, \]
\[ \phi'(t) = -c^2\phi(t) - c\phi F_1(\phi) + F_2(\phi), \]
where \( F_1 \) and \( F_2 \) are arbitrary smooth functions, \( c \neq 0, c_1, c_2 \) are arbitrary constants.

Next, we consider a special example. That is, the nonlinear diffusion equation with the source term
\[ u_t = (D(u)u_x)_x + B(x)P(u), \quad B_x \neq 0, \quad (21) \]
which has a wide range of physical applications such as in microwave heating, in the theory of chemical reactions, in mathematical biology and in soil science.

**Theorem 5.** Equation (21) possesses the additive separable solution (6) if and only if the coefficient functions \( D(u) \), \( P(u) \) and \( B(x) \) satisfy the following system
\[ B(x) \text{ is an arbitrary function,} \quad D = c_1e^{ku}, \quad P = c_2e^{ku}, \quad (22) \]
where \( c_1, c_2, k \) are arbitrary constants;
or
\[(ax + b)^{-3}(D'' + 3aD' + 2a^2 D) + (ax + b)^{-1} BP' - B' P = 0,\]
where \(b, a \neq 0\) are two arbitrary constants;

or
\[(2ax + b)^3 D'' + (2ax + b)(6a D' + BP') + B' P = 0,\]
where \(a, b\) are two arbitrary constants;

or
\[4(bx + c)^{3/2} D'' + b^2 (bx + c)^{-3/2} D + 4(bx + c)^{1/2} B P' + 4B' P = 0,\]
where \(c, b \neq 0\) are two arbitrary constants.

Theorem 6. Equation (21) with (22) has the additive separable solution (6) which satisfies:
\[\begin{align*}
eq -\frac{1}{a} \ln|ax + b|, & \quad \phi'(t) = b^{-2}(D' + aD) + B(0) P, \text{ where } b \neq 0, a \neq 0. \\
\psi(x) = ax^2 + bx, & \quad \phi'(t) = b^2 D' + 2aD + B(0) P. \\
\psi(x) = 2/(3b)(bx + c)^{3/2}, & \quad \phi'(t) = cD' + \frac{bD}{2\sqrt{c}} + B(0) P, \text{ where } b \neq 0, c > 0. 
\end{align*}\]

Equation (21) with (22) has the additive separable solution (6) which satisfies:
\[\begin{align*}
eq e^{-k\phi(x)} \phi'(t) = \lambda, \\
c_1 e^{k\phi(x)} \psi''(x) + c_1 k e^{k\phi(x)} [\psi'(x)]^2 + c_2 e^{k\phi(x)} B(x) = \lambda.
\end{align*}\]

Theorem 6. Equation (21) possesses the product separable solution (7) if and only if the coefficient functions \(D(u), P(u)\) and \(B(x)\) satisfy the following system:
\[B(x) \quad \text{is an arbitrary function}, \quad D(u) = c_1 u^k, \quad P(u) = c_1 c_2 u^{k+1},\]
\[\text{or}
\begin{align*}
eq (u^2 D'' - uD' + 3c_1 uD' + 2c_1^2 D - 2c_1 D) \left[(1 - c_1)x + c_0 \right]^{-3} \\\n+ (P' B - u^{-1} P B) \left[(1 - c_1)x + c_0 \right]^{-1} + u^{-1} P B' = 0, 
\end{align*}\]
where \(c_0, c_1 \neq 1\) are two arbitrary constants;

or
\begin{align*}
eq k^3 \left(u^2 D'' - uD' \right) \xi_3^3 + \left(3k^3 uD' + kP'B - ku^{-1} P B \right) \xi_1 \xi_3 + u^{-1} P B' = 0, 
\end{align*}\]
where \(\xi = c_1 e^{kx} + c_2 e^{-kx}, \xi_1 = c_1 e^{kx} - c_2 e^{-kx}, k \neq 0, c_1, c_2\) are arbitrary constants;

or
\begin{align*}
eq k^3 \left(u^2 D'' - uD' \right) \xi_3^3 - \left(3k^3 uD' - kP'B + ku^{-1} P B \right) \xi_1 \xi_3 - u^{-1} P B' = 0, 
\end{align*}\]
where \(\xi = c_1 \cos(kx) + c_2 \sin(kx), \xi_1 = c_1 \sin(kx) - c_2 \cos(kx), k \neq 0, c_1, c_2\) are arbitrary constants.

The product separable solutions (7) to Eq. (21) with (27)–(29), respectively, are given by:

(i)  \[\psi(x) = c_2 \left[(1 - c_1)x + c_0 \right]^{-1}, \quad \frac{\phi'(t)}{\phi(t)} = c_0^{-2} (uD' + c_1 D) + u^{-1} P B(0),\]
where \(c_0 \neq 0, c_1 \neq 1.\]
(ii) \( \psi(x) = c_1 e^{kx} + c_2 e^{-kx}, \quad \frac{\phi'(t)}{\phi(t)} = k^2 \frac{(c_1 - c_2)^2}{(c_1 + c_2)^2} u D' + k^2 D + u^{-1} P B(0), \)
where \( k \neq 0, c_1 + c_2 \neq 0. \)

(iii) \( \psi(x) = c_1 \cos(kx) + c_2 \sin(kx), \quad \frac{\phi'(t)}{\phi(t)} = \frac{k^2 c_1^2}{c_1^2} u D' - k^2 D + u^{-1} P B(0), \)
where \( k \neq 0, c_1 \neq 0. \)

Equation (21) with (26) has the product separable solution (7) which satisfies:
\[
\phi'(t) = \lambda \phi^{k+1}(t),
\]
\[
c_1 \psi^{k-1}(x) \psi''(x) + c_1 k \psi^{k-2}(x) \left[ \psi'(x) \right]^2 + c_1 c_2 \psi^k(x) B(x) = \lambda.
\]

5. Conditions for Eq. (1) when the convection and source terms are independent of \( x \)

In this section, it is considered that Eq. (1) admits functionally separable solution (3) when \( P \) and \( Q \) are independent of variable \( x \). That is the following equation
\[
u_t = \left( D(u) u_x \right)_x + Q(u) u_x + P(u).
\] (30)

**Theorem 7.** Equation (30) possesses the additive separable solution (6) if and only if the coefficient functions \( D(u), Q(u) \) and \( P(u) \) satisfy the following system:
\[
D = c_1 e^{cu}, \quad Q = c_2 e^{cu}, \quad P = c_3 + c_4 e^{cu} \quad (c \neq 0), \quad P = c_3 + c_4 u \quad (c = 0),
\] (31)

where \( c, c_1 \neq 0, c_2, c_3, c_4 \) are arbitrary constants; or
\[
D = c_4 e^{-bu} - \frac{c_3}{b} e^{-2bu}, \quad Q = c_1 e^{-bu}, \quad P = c_2,
\] (32)

where \( b \neq 0, c_1, c_2, c_3 \neq 0, c_4 \) are arbitrary constants; or
\[
D = \frac{c_3}{a^2} u + c_4, \quad Q = -\frac{3c_3}{a} u + \frac{c_2}{a} - ac_4, \quad P = c_3 u^2 - c_2 u + c_1,
\] (33)

where \( a \neq 0, c_1, c_4, c_2^2 + c_3^2 \neq 0 \) are arbitrary constants; or
\[
D = \frac{c_2}{4a} u + c_3, \quad Q = 0, \quad P = c_1 - 2ac_3 - \frac{3}{2} c_2 u,
\] (34)

where \( a \neq 0, c_1, c_2^2 + c_3^2 \neq 0 \) are arbitrary constants.

Equation (30) with (31) has the additive separable solution (6) which satisfies:

1. When \( c \neq 0 \), then
\[
\phi'(t) - c_3 = \lambda e^{\phi(t)},
\]
\[
c_1 e^{\psi(x)} \psi''(x) + c_1 e^{\psi(x)} \left[ \psi'(x) \right]^2 + c_2 e^{\psi(x)} \psi'(x) + c_4 e^{\psi(x)} = \lambda.
\]

2. When \( c = 0 \), then
\[
\phi'(t) - c_4 \phi(t) - c_3 = \lambda, \quad c_1 \psi''(x) + c_2 \psi'(x) + c_4 \psi(x) = \lambda.
\]

Equation (30) with (32) has the additive separable solution (6) which satisfies
\[
\psi(x) = -\frac{1}{b} \ln|bx + c|, \quad e^{2b\phi(t)} \phi'(t) = c_2 e^{2b\phi(t)} - c_1 e^{b\phi(t)} + c_3,
\]
where \( c \) is an arbitrary constant.
Equation (30) with (33) has the additive separable solution (6) which satisfies
\[ \psi(x) = ce^{ax}, \quad \phi'(t) = c_3\phi^2(t) - c_2\phi(t) + c_1, \]
where \( c \neq 0 \) is an arbitrary constant.

Equation (30) with (34) has the additive separable solution (6) which satisfies
\[ \psi(x) = a(x + b)^2, \quad \phi'(t) = c_1 - c_2\phi(t), \]
where \( b \) is an arbitrary constant.

**Remark.** In [26], the separation of variables of Eq. (30) had been studied by applying the generalized conditional symmetry approach when \( Q(u) \equiv 0 \). It had been classified under what conditions it admits the functionally separable solution, and a large number of exact solutions for the resulting equations had been obtained by the approach of separation of variables. Moreover, another two types of equations related to it had been considered.

**Theorem 8.** Equation (30) possesses the product separable solution (7) if and only if the coefficient functions \( D(u) \), \( Q(u) \) and \( P(u) \) satisfy the following system:
\[ \begin{align*}
D &= c_1u, \quad Q = c_2u^c, \quad P = c_3u + c_4u^{c+1} \quad (c \neq 0), \\
&= c_3u + c_4u\ln u \quad (c = 0),
\end{align*} \]
(35)
where \( c, c_1, c_2, c_3, c_4 \) are arbitrary constants; or
\[ \begin{align*}
D(u) \quad \text{and} \quad Q(u) \quad \text{are arbitrary smooth functions},
\end{align*} \]
(36)
(37)
where \( c \neq 0, c_1 \) are arbitrary constants; or
\[ \begin{align*}
D &= \frac{c_2}{2}u^{2-2c_1} + c_5u^{-c_1} \quad (c_1 \neq 2), \\
Q &= c_3u^{1-c_1}, \quad P = c_4u,
\end{align*} \]
(38)
where \( c_1 \neq 1, c_2, c_3 \neq 0, c_4, c_5 \) are arbitrary constants; or
\[ \begin{align*}
D &= c_2u^2 + c_3, \quad Q = -5ac_2u^2 + c_4u - ac_3, \\
P &= 2a^2c_2u^3 - ac_4u^2 + c_5u,
\end{align*} \]
(39)
where \( a \neq 0, c_2, c_3, c_4 \neq 0, c_5 \) are arbitrary constants; or
\[ \begin{align*}
D &= c_2u^2 + c_3, \quad Q = 0, \\
P &= -3a^2c_2u^3 + c_4u,
\end{align*} \]
(40)
where \( a \neq 0, c_2c_3 \neq 0, c_4 \) are arbitrary constants; or
\[ \begin{align*}
D &= c_2, \quad Q = 0, \\
P &= c_3u,
\end{align*} \]
where \( c_2 \neq 0, c_3 \) are arbitrary constants.

**Remark.** It is noted that (35) when \( c \neq 0, (38)-(40) \) are the special cases to (36), of course, the product separable solutions of Eq. (30) with (36) is also the product separable solutions of Eq. (30) with them.

Equation (30) with (36) has one of the product separable solutions (7) which satisfies
\[ \begin{align*}
\psi(x) &= e^{cx}, \\
\phi'(t) &= c_1\phi(t),
\end{align*} \]
where \( c \neq 0 \) is an arbitrary constant.

Equation (30) with (37) has the product separable solution (7) which satisfies
\[ \begin{align*}
\psi(x) &= [(c_1 - 1)x + c]^{-1/c_1}, \\
\phi'(t) &= c_2\phi^{3-2c_1}(t) - c_3\phi^{2-c_1}(t) + c_4\phi(t),
\end{align*} \]
where \( c \) is an arbitrary constant.
Equation (30) with (38) has the product separable solution (7) which satisfies
\[ \psi(x) = c_1 + e^{ax}, \quad \phi'(t) = 2a^2c_1^2c_2\phi^3(t) - ac_1c_4\phi^2(t) + c_5\phi(t), \]
where \( c_1 \neq 0 \) is an arbitrary constant.

Equation (30) with (39) has the product separable solution (7) which satisfies
\[ \psi(x) = e^{ax} + c_1e^{-ax}, \quad \phi'(t) = (a^2c_3 + c_4)\phi(t) - 8a^2c_1c_2\phi^3(t), \]
where \( c_1 \neq 0 \) is an arbitrary constant.

Equation (30) with (40) has the product separable solution (7) which satisfies
\[ \psi(x) = \cos ax + c_1 \sin ax, \quad \phi'(t) = (c_3 - a^2c_2)\phi(t), \]
or
\[ \psi(x) = e^{ax} + c_1e^{-ax}, \quad \phi'(t) = (c_3 + a^2c_2)\phi(t), \]
where \( c_1 \neq 0, a \neq 0 \) are arbitrary constants.

Equation (30) with (35) has the product separable solution (7) which satisfies
(1) When \( c \neq 0 \), then
\[ \phi'(t) - c_3\phi(t) = \lambda \phi^{c+1}(t), \]
\[ c_1\psi^{c-1}\psi' + c_1c\psi^{c-2}(\psi')^2 + c_2\psi^{c-1}\psi' + c_4\psi^c = \lambda. \]
(2) When \( c = 0 \), then
\[ \phi' - c_4\phi \ln|\phi| = \lambda \phi, \]
\[ c_1\psi'' + 2c_2\psi' + c_3\psi + c_4\psi \ln|\psi| = \lambda \psi. \]

6. Conclusions

In this paper, we have applied the generalized conditional symmetry approach to study the separation of variables and exact solutions to nonlinear diffusion equations with \( x \)-dependent convection and source. We formulate conditions for nonlinear diffusion equations which possess the functionally separable solution. Many new equations that depend on the \( x \)-dependent convection and source terms are obtained, and their corresponding exact solutions are derived. Meanwhile, we formulate conditions for nonlinear diffusion equations which admit the functionally separable solution when the convection and source terms are independent of \( x \).

References


