Available online at www.sciencedirect.com
ScienceDirect
ELSEVIER
Journal of Computational and Applied Mathematics 220 (2008) 364–372
JOURNAL OF COMPUTATIONAL AND APPLIED MATHEMATICS
www.elsevier.com/locate/cam

Remarks on the existence and uniqueness of the solutions to stochastic functional differential equations with infinite delay

Yong Ren[a,*], Shiping Lu[a], Ningmao Xia[b]

[a] Department of Mathematics, Anhui Normal University, Wuhu 241000, PR China
[b] Department of Mathematics, East China University of Science and Technology, Shanghai 200237, PR China

Received 21 July 2007; received in revised form 22 August 2007

Abstract

In this paper, we obtain some results on the existence and uniqueness of solutions to stochastic functional differential equations with infinite delay at phase space $BC((-\infty, 0]; R^d)$ which denotes the family of bounded continuous R^d-value functions φ defined on $(-\infty, 0]$ with norm $\|\varphi\| = \sup_{-\infty<\theta\leqslant 0} |\varphi(\theta)|$ under non-Lipschitz condition with Lipschitz condition being considered as a special case and a weakened linear growth condition. The solution is constructed by the successive approximation.

Keywords: Stochastic functional differential equations; Infinite delay; Existence; Uniqueness

1. Introduction

Stochastic differential equations (SDEs in short) are well known to model problems from many areas of science and engineering, wherein quite often the future state of such systems depends not only on the present state but also on its past history (delay) leading to stochastic functional differential equations with delay rather than SDEs. In the recent years, there is an increasing interest in stochastic evolution equations with finite delay under less restrictive conditions than Lipschitz condition; on this topic, one can see Liu [4], Govindan [3], Boukfaoui and Erraoui [2], Taniguchi [6] and references therein for details. Mao [5] discussed stochastic functional differential equations with finite delay under uniform Lipschitz condition and linear growth condition. Following this way, Wei and Wang [7] considered one such class of the so-called stochastic functional differential equations with infinite delay (ISFDEs in short) at phase space $BC((-\infty, 0]; R^d)$ to be described below. And they obtained the existence and uniqueness of solutions to ISFDEs under uniform Lipschitz condition and a weakened linear growth condition.

Motivated by the above works, in this paper we will generalize the existence and uniqueness of the solutions to ISFDEs under non-Lipschitz condition with Lipschitz condition being considered as a special case. The solution is constructed by the successive approximation.

The paper is organized as follows. In Section 2, we formulate the problem and introduce some notations. Section 3 is devoted to the proof of existence and uniqueness of solutions.

* Corresponding author. Tel.: +86 553 5768036; fax: +86 553 5910639.
E-mail address: brightry@hotmail.com (Y. Ren).

0377-0427/$ - see front matter
doi:10.1016/j.cam.2007.08.022

2. Preliminaries and statements of the main result

Let $|\cdot|$ denote the Euclidean norm in R^n. If A is a vector or a matrix, its transpose is denoted by A^{T}; if A is a matrix, its Frobenius norm is represented by $|A|=\sqrt{\mathrm{trace}(A^{\mathrm{T}}A)}$. Let t_0 be a positive constant and $(\Omega,\mathscr{F},P)$ be a complete probability space with a filtration $\{\mathscr{F}_t\}_{t\geqslant t_0}$ satisfying the usual conditions. Assume that $B(t)$ is an m-dimensional Brownian motion defined on $(\Omega,\mathscr{F},P)$, that is, $B(t)=(B_1(t),B_2(t),\ldots,B_m(t))^{\mathrm{T}}$. Let $\mathrm{BC}((-\infty,0];R^d)$ denote the family of bounded continuous R^d-value functions φ defined on $(-\infty,0]$ with norm $\|\varphi\|=\sup_{-\infty<\theta\leqslant 0}|\varphi(\theta)|$. We denote by $\mathscr{M}^2((-\infty,0];R^d)$ the family of all $\mathscr{F}_{t_0}$-measurable, $\mathbb{R}^d$-valued process $\psi(t)=\psi(t,\omega)$, $t\in(-\infty,0]$, such that $E\int_{-\infty}^0|\psi(t)|^2\,\mathrm{d}t<\infty$.

With all the above preparation, consider the following d-dimensional stochastic functional differential equations:

$$\mathrm{d}X(t)=f(t,X_t)\,\mathrm{d}t+g(t,X_t)\,\mathrm{d}B(t),\quad t_0\leqslant t\leqslant T, \tag{1}$$

where $X_t=\{X(t+\theta):-\infty<\theta\leqslant 0\}$ can be regarded as a $\mathrm{BC}((-\infty,0];R^d)$-value stochastic process, where $f:[t_0,T]\times\mathrm{BC}((-\infty,0];R^d)\to R^d$ and $g:[t_0,T]\times\mathrm{BC}((-\infty,0];R^d)\to R^{d\times m}$ are Borel measurable.

Next, we give the initial data of (1) as follows:

$$X_{t_0}=\xi=\{\xi(\theta):-\infty<\theta\leqslant 0\}\text{ is }\mathscr{F}_{t_0}\text{-measurable, }\mathrm{BC}((-\infty,0];R^d)\text{-value random variable such that }\xi\in\mathscr{M}^2((-\infty,0];R^d). \tag{2}$$

Definition 1. R^d-value stochastic process $X(t)$ defined on $-\infty<t\leqslant T$ is called the solution of (1) with initial data (2), if

(i) $X(t)$ is continuous and for all $t_0\leqslant t\leqslant T$, $X(t)$ is $\mathscr{F}_t$-adapted;
(ii) $\{f(t,X_t)\}\in\mathscr{L}^1([t_0,T];R^d)$ and $\{g(t,X_t)\}\in\mathscr{L}^2([t_0,T];R^{d\times m})$;
(iii) $X_{t_0}=\xi$, for each $t_0\leqslant t\leqslant T$, $X(t)=\xi(0)+\int_{t_0}^t f(s,X_s)\,\mathrm{d}s+\int_{t_0}^t g(s,X_s)\,\mathrm{d}B(s)$ a.s.

$X(t)$ is called as a unique solution, if any other solution $\bar{X}(t)$ is not distinguishable with $X(t)$, that is,

$$P(X(t)=\bar{X}(t),\ \text{for all }-\infty<t\leqslant T)=1.$$

In order to attain the solution of (1) with initial data (2), we propose the following condition:

(H1) For all $\varphi,\psi\in\mathrm{BC}((-\infty,0];R^d)$ and $t\in[t_0,T]$, it follows that

$$|f(t,\varphi)-f(t,\psi)|^2\vee|g(t,\varphi)-g(t,\psi)|^2\leqslant\kappa(\|\varphi-\psi\|^2), \tag{3}$$

where $\kappa(\cdot)$ is a concave nondecreasing function from $\mathbb{R}_+$ to $\mathbb{R}_+$ such that $\kappa(0)=0$, $\kappa(u)>0$ for $u>0$ and $\int_{0+}\mathrm{d}u/\kappa(u)=\infty$.

(H2) $f(t,0),g(t,0)\in L^2([t_0,T])$ and for all $t\in[t_0,T]$, it follows that

$$|f(t,0)|^2\vee|g(t,0)|^2\leqslant K, \tag{4}$$

where $K>0$ is a constant.

Remark 2. To demonstrate the generality of our results, let us illustrate it using a concrete function $\kappa(\cdot)$. Let $K>0$ and let $\delta\in(0,1)$ be sufficiently small. Define

$$\kappa_1(u)=Ku,\quad u\geqslant 0.$$

$$\kappa_2(u)=\begin{cases}u\log(u^{-1}), & 0\leqslant u\leqslant\delta,\\ \delta\log(\delta^{-1})+\kappa_2'(\delta-)(u-\delta), & u>\delta,\end{cases}$$

$$\kappa_3(u)=\begin{cases}u\log(u^{-1})\log\log(u^{-1}), & 0\leqslant u\leqslant\delta,\\ \delta\log(\delta^{-1})\log\log(\delta^{-1})+\kappa_3'(\delta-)(u-\delta), & u>\delta.\end{cases}$$

They are all concave nondecreasing functions satisfying $\int_{0+} \mathrm{d}u/\kappa_i(u) = +\infty$ $(i = 1, 2, 3)$. In particular, we see that the Lipschitz condition is a special case of our proposed condition. In other words, in this paper we obtain a more general result than that of Wei and Wang [7].

Now we give the existence and uniqueness theorem to (1) with initial data (2) under the above non-Lipschitz condition and the weakened linear growth condition.

Theorem 3. *Assume that* (H1) *and* (H2) *hold. Then, there exists a unique solution to* (1) *with initial data* (2).

In order to obtain the uniqueness of solutions, we give Bihari inequality which appeared in [1].

Lemma 4 (*Bihari inequality*). *Let* $T > 0$ *and* $u_0 \geqslant 0$, $u(t)$, $v(t)$ *be continuous functions on* $[0, T]$. *Let* $\kappa : \mathbb{R}_+ \to \mathbb{R}_+$ *be a concave continuous and nondecreasing function such that* $\kappa(r) > 0$ *for* $r > 0$. *If*

$$u(t) \leqslant u_0 + \int_0^t v(s)\kappa(u(s))\,\mathrm{d}s$$

for all $0 \leqslant t \leqslant T$, *then*

$$u(t) \leqslant G^{-1}\left(G(u_0) + \int_0^t v(s)\,\mathrm{d}s\right)$$

for all such $t \in [0, T]$ *that*

$$G(u_0) + \int_0^t v(s)\,\mathrm{d}s \in \mathrm{Dom}(G^{-1}),$$

where $G(r) = \int_0^r \mathrm{d}s/\kappa(s)$, $r > 0$, *and* G^{-1} *is the inverse function of* G. *In particular, if, moreover,* $u_0 = 0$ *and* $\int_{0+} \mathrm{d}s/\kappa(s) = \infty$, *then* $u(t) = 0$, *for all* $t \in [0, T]$.

3. Existence and uniqueness of solutions

In order to obtain the existence of solutions to (1) with initial data (2), we define $X_{t_0}^0 = \xi$ and $X^0(t) = \xi(0)$, for $t_0 \leqslant t \leqslant T$. Let $X_{t_0}^n = \xi$, $n = 1, 2, \ldots$ and define the Picard sequence:

$$X^n(t) = \xi(0) + \int_{t_0}^t f(s, X_s^{n-1})\,\mathrm{d}s + \int_{t_0}^t g(s, X_s^{n-1})\,\mathrm{d}B(s), \quad t_0 \leqslant t \leqslant T. \tag{5}$$

Lemma 5. *Under condition* (H1) *and* (H2), *for all* $t \in (-\infty, T]$, $n \geqslant 1$,

$$E|X^n(t)|^2 \leqslant C_1, \tag{6}$$

where C_1 *is a positive constant.*

Proof. Obviously, $X^0(t) \in \mathscr{M}^2((-\infty, T]; R^d)$. By induction, $X^n(t) \in \mathscr{M}^2((-\infty, T]; R^d)$, in fact,

$$|X^n(t)|^2 \leqslant 3|\xi(0)|^2 + 3\left|\int_{t_0}^t f(s, X_s^{n-1})\,\mathrm{d}s\right|^2 + 3\left|\int_{t_0}^t g(s, X_s^{n-1})\,\mathrm{d}B(s)\right|^2.$$

From Hölder inequality, we have

$$\begin{aligned} E|X^n(t)|^2 &\leqslant 3E|\xi(0)|^2 + 3E\left|\int_{t_0}^t f(s, X_s^{n-1})\,\mathrm{d}s\right|^2 + 3E\left|\int_{t_0}^t g(s, X_s^{n-1})\,\mathrm{d}B(s)\right|^2 \\ &\leqslant 3E\|\xi\|^2 + 3(t - t_0)E\int_{t_0}^t |f(s, X_s^{n-1}) - f(s, 0) + f(s, 0)|^2)\,\mathrm{d}s \\ &\quad + 3E\int_{t_0}^t |g(s, X_s^{n-1}) - g(s, 0) + g(s, 0)|^2)\,\mathrm{d}s. \end{aligned} \tag{7}$$

Using the elementary inequality $(u+v)^2 \leqslant u^2+v^2$, (H1) and (H2), we have

$$\begin{aligned} E|X^n(t)|^2 \leqslant\, & 3E\|\xi\|^2 + 3(t-t_0)E\int_{t_0}^{t}[2|f(s,X_s^{n-1})-f(s,0)|^2+2|f(s,0)|^2])\,\mathrm{d}s \\ & + 3E\int_{t_0}^{t}[2|g(s,X_s^{n-1})-g(s,0)|^2+2|g(s,0)|^2])\,\mathrm{d}s \\ \leqslant\, & 3E\|\xi\|^2 + 3(t-t_0+1)E\int_{t_0}^{t}[2\kappa(\|X_s^{n-1}\|^2)+2K])\,\mathrm{d}s \\ \leqslant\, & 3E\|\xi\|^2 + 6(T-t_0+1)(T-t_0)K \\ & + 6(T-t_0+1)E\int_{t_0}^{t}\kappa(\|X_s^{n-1}\|^2)\,\mathrm{d}s. \end{aligned} \tag{8}$$

Given that $\kappa(\cdot)$ is concave and $\kappa(0)=0$, we can find a pair of positive constants a and b such that

$$\kappa(u) \leqslant a + bu \quad \text{for all } u \geqslant 0.$$

So, we have

$$E|X^n(t)|^2 \leqslant c_1 + 6b(T-t_0+1)E\int_{t_0}^{t}\|X_s^{n-1}\|^2)\,\mathrm{d}s,$$

where $c_1 = 3E\|\xi\|^2 + 6(T-t_0+1)(T-t_0)(K+a)$. Furthermore,

$$\begin{aligned} E|X^n(t)|^2 & \leqslant c_1 + 6b(T-t_0+1)\int_{t_0}^{t}E(\sup_{t_0\leqslant r\leqslant s}|X^{n-1}(r)|^2)\,\mathrm{d}s \\ & \leqslant c_1 + 6b(T-t_0+1)\int_{t_0}^{t}E|X^{n-1}(s)|^2)\,\mathrm{d}s. \end{aligned} \tag{9}$$

Hence, for any $k \geqslant 1$, we can derive that

$$\max_{1\leqslant n\leqslant k} E|X^n(t)|^2 \leqslant c_1 + 6b(T-t_0+1)\int_{t_0}^{t}\max_{1\leqslant n\leqslant k}E|X^{n-1}(s)|^2)\,\mathrm{d}s.$$

Note that

$$\begin{aligned} & \max_{1\leqslant n\leqslant k} E|X^{n-1}(s)|^2)\,\mathrm{d}s \\ & \quad = \max\{E|\xi(0)|^2, E|X^1(s)|^2, \ldots, E|X^{k-1}(s)|^2\} \\ & \quad \leqslant \max\{E|\xi(0)|^2, E|X^1(s)|^2, \ldots, E|X^{k-1}(s)|^2, E|X^k(s)|^2\} \\ & \quad = \left\{E\|\xi\|^2, \max_{1\leqslant n\leqslant k} E|X^n(s)|^2\right\} \\ & \quad \leqslant E\|\xi\|^2 + \max_{1\leqslant n\leqslant k} E|X^n(s)|^2. \end{aligned} \tag{10}$$

So, we have

$$\begin{aligned} \max_{1\leqslant n\leqslant k} E|X^n(t)|^2 & \leqslant c_1 + 6b(T-t_0+1)\int_{t_0}^{t}\left(\max_{1\leqslant n\leqslant k}E|X^n(s)|^2\right)\mathrm{d}s \\ & \leqslant c_2 + 6b(T-t_0+1)\int_{t_0}^{t}\left(\max_{1\leqslant n\leqslant k}E|X^n(s)|^2\right)\mathrm{d}s, \end{aligned} \tag{11}$$

where $c_2 = c_1 + 6b(T-t_0+1)(T-t_0)E\|\xi\|^2$.

From Gronwall inequality, we derive that

$$\max_{1\leqslant n\leqslant k} E|X^n(t)|^2 \leqslant c_2 \mathrm{e}^{6b(T-t_0+1)(T-t_0)}.$$

Since k is arbitrary, we have that

$$E|X^n(t)|^2 \leqslant c_2 \mathrm{e}^{6b(T-t_0+1)(T-t_0)}, \quad t_0 \leqslant t \leqslant T, \quad n \geqslant 1.$$

So, the desired result holds with $C_1 = c_2 \mathrm{e}^{6b(T-t_0+1)(T-t_0)}$. □

Lemma 6. *Under condition* (H1) *and* (H2), *there exists a positive constant* C_2 *such that*

$$E\left[\sup_{t_0 \leqslant s \leqslant t} |X^{n+m}(s) - X^n(s)|^2\right] \leqslant C_2 \int_{t_0}^t \kappa\left(E \sup_{t_0 \leqslant r \leqslant s} |X^{n+m-1}(r) - X^{n-1}(r)|^2\right) \mathrm{d}s \tag{12}$$

for all $t_0 \leqslant t \leqslant T, n, m \geqslant 1$.

Proof. From (5), we can derive that

$$\begin{aligned} X^{n+m}(t) - X^n(t) &= \int_{t_0}^t [f(s, X_s^{n+m-1}) - f(s, X_s^{n-1})]) \,\mathrm{d}s \\ &\quad + \int_{t_0}^t [g(s, X_s^{n+m-1}) - g(s, X_s^{n-1})] \,\mathrm{d}B(s). \end{aligned} \tag{13}$$

So,

$$\begin{aligned} E|X^{n+m}(t) - X^n(t)|^2 &\leqslant 2E\left|\int_{t_0}^t [f(s, X_s^{n+m-1}) - f(s, X_s^{n-1})]) \,\mathrm{d}s\right|^2 \\ &\quad + 2E\left|\int_{t_0}^t [g(s, X_s^{n+m-1}) - g(s, X_s^{n-1})] \,\mathrm{d}B(s)\right|^2 \\ &\leqslant 2(t-t_0)E\int_{t_0}^t |f(s, X_s^{n+m-1}) - f(s, X_s^{n-1})|^2) \,\mathrm{d}s \\ &\quad + 2E\int_{t_0}^t |g(s, X_s^{n+m-1}) - g(s, X_s^{n-1})|^2) \,\mathrm{d}s. \end{aligned} \tag{14}$$

Thus, we derive that

$$E\left[\sup_{t_0 \leqslant s \leqslant t} |X^{n+m}(s) - X^n(s)|^2\right] \leqslant 2(T-t_0+1)E\int_{t_0}^t \kappa(\|X_s^{n+m-1} - X_s^{n-1}\|^2) \,\mathrm{d}s.$$

From Jensen inequality, we have that

$$\begin{aligned} E\left[\sup_{t_0 \leqslant s \leqslant t} |X^{n+m}(s) - X^n(s)|^2\right] &\leqslant 2(T-t_0+1)\int_{t_0}^t \kappa(E\|X_s^{n+m-1} - X_s^{n-1}\|^2) \,\mathrm{d}s \\ &\leqslant 2(T-t_0+1)\int_{t_0}^t \kappa\left(E \sup_{t_0 \leqslant r \leqslant s} |X^{n+m-1}(r) - X^{n-1}(r)|^2\right) \mathrm{d}s. \end{aligned} \tag{15}$$

If we choose $C_2 = 2(T-t_0+1)$, we can derive that the lemma holds. □

Lemma 7. *Under condition* (H1) *and* (H2), *there exists a positive constant* C_3 *such that*

$$E\left[\sup_{t_0\leqslant s\leqslant t}|X^{n+m}(s)-X^n(s)|^2\right]\leqslant C_3(t-t_0) \tag{16}$$

for all $t_0\leqslant t\leqslant T, n, m\geqslant 1$.

Proof. From Lemmas 5 and 6, we have that

$$\begin{aligned} &E\left[\sup_{t_0\leqslant s\leqslant t}|X^{n+m}(s)-X^n(s)|^2\right]\\ &\leqslant C_2\int_{t_0}^t \kappa(E\sup_{t_0\leqslant r\leqslant s}|X^{n+m-1}(r)-X^{n-1}(r)|^2)\,\mathrm{d}s\\ &\leqslant C_2\int_{t_0}^t \kappa(2C_1)\,\mathrm{d}s\\ &\leqslant C_2\kappa(2C_2)(T-t_0)=C_3(t-t_0). \end{aligned} \tag{17}$$

The proof is complete. □

Define

$$\varphi_1(t)=C_3(t-t_0),$$

$$\varphi_{n+1}(t)=C_2\int_{t_0}^t \kappa(\varphi_n(s))\,\mathrm{d}s,\quad n\geqslant 1,$$

$$\varphi_{n,m}(t)=E[\sup_{t_0\leqslant r\leqslant t}|X^{n+m}(r)-X^n(r)|^2], n, m\geqslant 1,$$

Choose $T_1\in[t_0, T)$ such that

$$C_2\kappa(C_3(t-t_0))\leqslant C_3\quad\text{for all } t_0\leqslant t\leqslant T_1.$$

Lemma 8. *There exists a positive* $t_0\leqslant T_1<T$ *such that for all* $n, m\geqslant 1$,

$$0\leqslant\varphi_{n,m}(t)\leqslant\varphi_n(t)\leqslant\varphi_{n-1}(t)\leqslant\cdots\leqslant\varphi_1(t) \tag{18}$$

for all $t_0\leqslant t\leqslant T_1$.

Proof. We prove this lemma by induction on n. By Lemma 7, we have that

$$\varphi_{1,m}(t)=E\left[\sup_{t_0\leqslant r\leqslant t}|X^{1+m}(r)-X^1(r)|^2\right]\leqslant C_3(t-t_0)=\varphi_1(t).$$

By Lemma 6,

$$\begin{aligned} \varphi_{2,m}(t)&=E\left[\sup_{t_0\leqslant r\leqslant t}|X^{2+m}(r)-X^2(r)|^2\right]\\ &\leqslant C_2\int_{t_0}^t\kappa\left(E\sup_{t_0\leqslant r\leqslant s}|X^{1+m}(r)-X^1(r)|^2\right)\mathrm{d}s\\ &\leqslant C_2\int_{t_0}^t\kappa(\varphi_{1,m}(s))\,\mathrm{d}s\\ &\leqslant C_2\int_{t_0}^t\kappa(\varphi_1(s))\,\mathrm{d}s=\varphi_1(t). \end{aligned} \tag{19}$$

So, we also have that

$$\begin{aligned}\varphi_2(t) &= C_2\int_{t_0}^{t}\kappa(\varphi_1(s))\,\mathrm{d}s\\ &\leqslant C_2\int_{t_0}^{t}\kappa(C_3 s)\,\mathrm{d}s\\ &\leqslant C_2\int_{t_0}^{t}C_3\,\mathrm{d}s=\varphi_1(t).\end{aligned}\tag{20}$$

We have already shown that

$$\varphi_{2,m}(t)\leqslant\varphi_2(t)\leqslant\varphi_1(t)\quad\text{for all } t_0\leqslant t\leqslant T_1.$$

Now, we assume that (18) holds for some $n\geqslant 1$. Then, using the same inequalities as above yields

$$\begin{aligned}\varphi_{n+1,m}(t) &= C_2\int_{t_0}^{t}\kappa\left(E\sup_{t_0\leqslant r\leqslant s}|X^{n+m}(r)-X^n(r)|^2\right)\mathrm{d}s\\ &\leqslant C_2\int_{t_0}^{t}\kappa(\varphi_{n,m}(s))\,\mathrm{d}s\\ &\leqslant C_2\int_{t_0}^{t}\kappa(\varphi_n(s))\,\mathrm{d}s=\varphi_{n+1}(t)\end{aligned}\tag{21}$$

for all $t_0\leqslant t\leqslant T_1$. On the other hand, we have that

$$\varphi_{n+1}(t)=C_2\int_{t_0}^{t}\kappa(\varphi_n(s))\,\mathrm{d}s\leqslant C_2\int_{t_0}^{t}\kappa(\varphi_{n-1}(s))\,\mathrm{d}s=\varphi_n(t)$$

for all $t_0\leqslant t\leqslant T_1$. This completes the proof. □

Proof of Theorem 3. *Uniqueness*: Let $X(t)$ and $\bar{X}(t)$ be two solutions of (1). Note that

$$\begin{aligned}&X(t)-\bar{X}(t)\\ &\quad=\int_{t_0}^{t}[f(s,X_s)-f(s,\bar{X}_s)])\,\mathrm{d}s\\ &\qquad+\int_{t_0}^{t}[g(s,X_s)-g(s,\bar{X}_s)]\,\mathrm{d}B(s),\end{aligned}\tag{22}$$

So,

$$\begin{aligned}&E|X(t)-\bar{X}(t)|^2\\ &\quad\leqslant 2E\left|\int_{t_0}^{t}[f(s,X_s)-f(s,\bar{X}_s)])\,\mathrm{d}s\right|^2\\ &\qquad+2E\left|\int_{t_0}^{t}[g(s,X_s)-g(s,\bar{X}_s)]\,\mathrm{d}B(s)\right|^2\\ &\quad\leqslant 2(t-t_0)E\int_{t_0}^{t}|f(s,X_s)-f(s,\bar{X}_s)|^2)\,\mathrm{d}s\\ &\qquad+2E\int_{t_0}^{t}|g(s,X_s)-g(s,\bar{X}_s)|^2)\,\mathrm{d}s.\end{aligned}\tag{23}$$

Thus, we derive that

$$E\left[\sup_{t_0\leqslant s\leqslant t}|X(s)-\bar{X}(s)|^2\right]\leqslant 2(T-t_0+1)E\int_{t_0}^{t}\kappa(\|X_s-\bar{X}_s\|^2)\,\mathrm{d}s.$$

From Jensen inequality, we have that

$$E\left[\sup_{t_0\leqslant s\leqslant t}|X(s)-\bar{X}(s)|^2\right]$$
$$\leqslant 2(T-t_0+1)\int_{t_0}^{t}\kappa(E\|X_s-\bar{X}_s\|^2)\,\mathrm{d}s$$
$$\leqslant 2(T-t_0+1)\int_{t_0}^{t}\kappa\left(E\sup_{t_0\leqslant r\leqslant s}|X(r)-\bar{X}(r)|^2\right)\mathrm{d}s. \tag{24}$$

Bihari inequality yields

$$E[\sup_{t_0\leqslant s\leqslant t}|X(s)-\bar{X}(s)|^2]=0, \quad t_0\leqslant t\leqslant T. \tag{25}$$

The above expression means that $X(t)=\bar{X}(t)$ for all $t_0\leqslant t\leqslant T$. Therefore, for all $-\infty<t\leqslant T$, $X(t)=\bar{X}(t)$ a.s. This establishes the uniqueness.

Existence: We claim that

$$E\sup_{t_0\leqslant s\leqslant t}|X^{n+m}-X^n(s)|^2\to 0 \tag{26}$$

for all $t_0\leqslant t\leqslant T_1$, as $n,m\to\infty$. Note that φ_n is continuous on $[t_0,T_1]$. Note also that for each $n\geqslant 1$, $\varphi_n(\cdot)$ is decreasing on $[t_0,T_1]$, and, for each t, $\varphi_n(t)$ is a decreasing sequence. Therefore, we can define the function $\varphi(t)$ as

$$\varphi(t)=\lim_{n\to\infty}\varphi_n(t)=\lim_{n\to\infty}C_2\int_{t_0}^{t}\kappa(\varphi_{n-1}(s))\,\mathrm{d}s=C_2\int_{t_0}^{t}\kappa(\varphi(s))\,\mathrm{d}s \tag{27}$$

for all $t_0\leqslant t\leqslant T_1$. Bihari inequality implies that $\varphi(t)=0$ for all $t_0\leqslant t\leqslant T_1$. Now, from Lemma 8, we have that

$$\varphi_{n,n}(t)\leqslant\sup_{t_0\leqslant t\leqslant T_1}\varphi_n(t)\leqslant\varphi_n(T_1)\to 0 \tag{28}$$

as $n\to\infty$. That is, $X^n(t)$ is a Cauchy sequence in L^2 on $(-\infty,T_1]$. From Lemma 5, we can easily derive that

$$E|X(t)|^2\leqslant C,$$

where C is a positive constant.

Using condition (H1) and the property of the function $\kappa(\cdot)$, we can obtain that, for all $t_0\leqslant t\leqslant T_1$,

$$E\left|\int_{t_0}^{t}[f(s,X_s^n)-f(s,X_s)])\,\mathrm{d}s\right|^2\to 0 \quad \text{as } n\to\infty,$$

$$E\left|\int_{t_0}^{t}[g(s,X_s^n)-g(s,X_s)]\,\mathrm{d}B(s)\right|^2\to 0 \quad \text{as } n\to\infty.$$

For all $t_0\leqslant t\leqslant T_1$, taking limits on both the sides of (5), we obtain that

$$\lim_{n\to\infty}X^n(t)=\xi(0)+\lim_{n\to\infty}\int_{t_0}^{t}f(s,X_s^{n-1})\,\mathrm{d}s+\lim_{n\to\infty}\int_{t_0}^{t}g(s,X_s^{n-1})\,\mathrm{d}B(s). \tag{29}$$

That is,

$$X(t)=\xi(0)+\int_{t_0}^{t}f(s,X_s)\,\mathrm{d}s+\int_{t_0}^{t}g(s,X_s)\,\mathrm{d}B(s). \tag{30}$$

The above expression demonstrates that $X(t)$ is one solution of (1) with initial data (2) on $[t_0,T_1]$. By iteration, the existence of solutions to (1) on $[t_0,T]$ can be obtained. □

Acknowledgements

The authors wish to thank the anonymous referee for his or her valuable comments, correcting errors and improving written language. And the work was supported by the Key Science and Technology Project of Ministry of Educational (China 207047).

References

[1] I. Bihari, A generalization of a lemma of Bellman and its application to uniqueness problem of differential equations, Acta Math. Acad. Sci. Hungar. 7 (1956) 71–94.
[2] Y.El. Boukfaoui, M. Erraoui, Remarks on the existence and approximation for semilinear stochastic differential in Hilbert spaces, Stochastic Anal. Appl. 20 (2002) 495–518.
[3] T.E. Govindan, Stability of mild solution of stochastic evolution equations with variable delay, Stochastic Anal. Appl. 21 (2003) 1059–1077.
[4] K. Liu, Lyapunov functionals and asymptotic of stochastic delay evolution equations, Stochastics and Stochastic Rep. 63 (1998) 1–26.
[5] X.R. Mao, Stochastic Differential Equations and Applications, Horwood Publication, Chichester, 1997.
[6] T. Taniguchi, Successive approximations to solutions of stochastic differential equations, J. Differential Equations 96 (1992) 152–169.
[7] F.Y. Wei, K. Wang, The existence and uniqueness of the solution for stochastic functional differential equations with infinite delay, J. Math. Anal. Appl. 331 (2007) 516–531.