Extensions of $L$-fuzzy closure spaces

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Abstract In this paper, we study the extension theory to $L$-fuzzy closure spaces, where $L$ is a strictly two-sided, commutative quantale lattice. We give new notions such as $L$-fuzzy stack, $L$-fuzzy $c$-grill and trace of a point. Also, we construct order relation and equivalence relation between two extensions. Also, We introduce the concept of a principal extension of $L$-fuzzy closure space and study some of its applications.

1. Introduction

In crisp topology, extension theory has been intensively studied for completely regular spaces and is fairly well developed for $T_0$ topological spaces (see for example [1]). Some basic concepts on extensions of closure spaces are introduced and results on embedding of closure spaces in cubes are studied by Gagrat and Thron [2], Čech [3] and Thron and Warren [4]. A general theory of extensions of $G_0$ closure spaces is introduced and investigated by Chattopadhyay and Thron [5]. In fuzzy setting, particular type of extension such as compactifications, completions of fuzzy topological spaces and fuzzy uniform spaces have been studied [6–8]. In [9], Chattopadhyay, Hazra and Samanta introduced for the first time, a general concept of extensions of fuzzy topological spaces and provided a method of construction of $T_0$ principal extension of a $T_0$ fuzzy topological space.

In this paper, we need to study the extension of $L$-fuzzy closure space which defined by Kim [10], as follows:

In Section 2, we introduce the notions of $L$-fuzzy grills, $L$-fuzzy $c$-grills and a homeomorphism between two $L$-fuzzy closure spaces and study some results on it. In addition, we define a base for the $r$-$L$-fuzzy closed sets in $L$-fuzzy closure space.

In Section 3, we study the extension theory to $L$-fuzzy closure spaces, where $L$ is a strictly two-sided, commutative quantale lattice. We give new notions such as $L$-fuzzy stack, $L$-fuzzy $c$-grill and trace of a point. We present the trace $T_{(x,y)}$ of the point $y$ with respect to the extension $E$ and the trace system $X^E$ of the extension $E$. Also, we study some of its properties. Furthermore, we define a principal extension of $L$-fuzzy closure space and provide some results on it.

2. Preliminaries

Through this paper, let $X$ be a non-empty set, $L = (L, \leq, \emptyset, \oslash, 0, 1)$ a complete lattice where $0$ and $1$ denote the least and the greatest elements in $L$, $L_0 = L - \{0\}$. 

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Definition 2.1 (8,11). A complete lattice $(L, \leq, \lor)$ is called a strictly two-sided, commutative quantal (sqc-lattice, for short) if it satisfies the following properties:

1. $(L, \lor)$ is a commutative semigroup.
2. $x = x \lor 1$, for each $x \in L$ and 1 is the universal upper bound.
3. $\lor$ is distributive over arbitrary joins, i.e.,
   \[
   \bigvee_{i \in I} (x_i \lor y) = \bigvee_{i \in I} (x_i \lor y).
   \]

Definition 2.2 (8,11). Let $(L, \leq, \lor)$ be an sqc-lattice. A mapping $f : L \to L$ is called a strong negation if it satisfies the following conditions:

(a) $(a')' = a$ for each $a \in L$.
(b) If $a \leq b$, then $a' \geq b'$, for each $a, b \in L$.

In this paper, we assume that $(L, \leq, \lor, ', \top)$ is an sqc-lattice, where $\top$ is defined as follows:

$x \lor y = (x' \lor y')'$.

In particular, the unit interval $([0, 1], \leq, \lor, ')$, where $\lor = \wedge, \top = v$, is sqc-lattice with a strong negation $a' = 1 - a$ for each $a \in [0, 1]$.

Lemma 2.3 12. Let $(L, \leq, \lor, ', \top)$ be an sqc-lattice with a strong negation. Then for each $x, y, z \in L$, $\{y_i; i \in I\} \subset L$, we have the following properties:

1. If $y \leq z$, then $(x \lor y) \leq (x \lor z)$.
2. If $y \leq z$, then $(x \lor y) \leq (x \lor z)$.
3. $x \lor y \leq x \lor y$.
4. $\forall \emptyset = 1, I' = 0$ and $\forall x \lor y \leq x \lor y$.
5. $\forall x, y \in L$, $\forall x \lor y = (x \lor y)'$.
6. $\forall x, y \in L$, $\forall x \lor y = (x \lor y)'$.
7. $\forall x, y \in L$, $\forall x \lor (x \lor y) = (x \lor y) = (x \lor y)$.

All algebraic operations on $L$ can be extended pointwise to the set $L^X$ as follows: for all $x \in X$ and $\lambda, \mu \in L^X$:

1. $(\lambda \lor \mu)(x) = \lambda(x) \lor \mu(x)$.
2. $(\lambda \lor \mu)(x) = \lambda(x) \lor \mu(x)$.

Definition 2.4 10. An $L$-fuzzy closure space is an ordered pair $(X, c)$, where $c : L^X \times L_0 \to L^X$ is a mapping satisfying the following axioms:

(CO1) $c(\emptyset, r) = \emptyset$ for all $r \in L_0$.
(CO2) $\lambda \leq c(\lambda, r)$ for all $\lambda \in L^X$, $r \in L_0$.
(CO3) If $\lambda \leq \mu$ and $r \leq s$, then $c(\lambda, r) \leq c(\mu, s)$ for all $\lambda, \mu \in L^X$.
(CO4) $c(\lambda \lor \mu, r) = c(\lambda, r) \lor c(\mu, r)$ for all $\lambda, \mu \in L^X$.
(CO5) $c(\lambda, r) = c(\lambda, r) \lor c(\lambda, r)$ for all $\lambda \in L^X$.

An $L$-fuzzy set $\lambda$ is called an $r$-$L$-fuzzy closed set if $c(\lambda, r) = \lambda$.

Let $(X, c_1)$ and $(Y, c_2)$ be two $L$-fuzzy closure spaces. A mapping $f(x, c_1) \to (Y, c_2)$ is called $LC$-fuzzy continuous if for each $\lambda \in L^X$, $r \in L_0$,$f(c_1(\lambda, r)) \leq c_2(f(\lambda), r)$.

Definition 2.5. Let $(X, c)$ be an $L$-fuzzy closure space, $r \in L_0$ and $\beta$ a family of $r$-$L$-fuzzy closed sets in $(X, c)$. Then $\beta$ is said to be a base for the $r$-$L$-fuzzy closed sets in $(X, c)$ if each $r$-$L$-fuzzy closed set in $(X, c)$ can be expressed as an infimum of a subset of $\beta$.

Definition 2.6. Let $(X, c)$ be an $L$-fuzzy closure space and $A \subset X$. Define $c_4 : L^X \times L_0 \to L^X$ by $c_4(\lambda, r)(x) = c(\lambda, r)(x)$ for all $x \in A$ and $\lambda \in L^X$, $r \in L_0$. Then $(A, c_4)$ is an $L$-fuzzy closure space and is called a subspace of $(X, c)$.

Definition 2.7. An $L$-fuzzy closure space $(X, c)$ is called $T_0$ if for each $x, y \in X$, $x \neq y$ and $r \in L_0$, there exists an $r$-$L$-fuzzy closed set $\lambda$ such that $\lambda(x) \neq \lambda(y)$.

Definition 2.8. Let $(X, c)$ and $(Y, c')$ be two $L$-fuzzy closure spaces. A mapping $f(x, c) \to (Y, c')$ is called a homeomorphism if $f$ is bijective and $f^{-1}$ are $LC$-fuzzy continuous mappings.

Theorem 2.9. Let $(X, c)$ and $(Y, c')$ be two $L$-fuzzy closure spaces. A bijective mapping $h : (X, c) \to (Y, c')$ is a homeomorphism if

$h(c(\lambda, r)) = c'(h(\lambda), r)$ for all $\lambda \in L^X$, $r \in L_0$.

In the following, we define the notions of $L$-fuzzy stack, $L$-fuzzy grill and $L$-fuzzy $c$-grill in $L$-fuzzy closure space.

Definition 2.10. Let $X$ be a nonempty set. A mapping $S : L^X \to L$ satisfying

$S(\lambda_1) \supseteq S(\lambda_2)$ if $\lambda_1 \geq \lambda_2$ for all $\lambda_1, \lambda_2 \in L^X$ is called an $L$-fuzzy stack on $X$.

Definition 2.11. An $L$-fuzzy grill $G : L^X \to L$ is an $L$-fuzzy stack on $X$ such that it satisfies the following conditions:

(i) $G(\emptyset) = 0$.
(ii) $G(\lambda_1 \cup \lambda_2) = G(\lambda_1) \cup G(\lambda_2)$ for all $\lambda_1, \lambda_2 \in L^X$.
(iii) $G(\emptyset) > 0$. An $L$-fuzzy grill $G$ on $X$ is said to be proper if $G(\emptyset) = 1$.

Definition 2.12. Let $G$ be an $L$-fuzzy grill in an $L$-fuzzy closure space $(X, c)$. Then $G$ is called an $L$-fuzzy $c$-grill in $(X, c)$ if $G(c(\lambda, r)) = G(\lambda)$ for all $\lambda \in L^X$, $r \in L_0$.

Definition 2.13. Let $(X, c)$ be an $L$-fuzzy closure space. For all $x \in X$, define $G(x) : L^X \to L$ by

$G_x(\lambda) = c(x, \lambda)(x)$ for all $\lambda \in L^X$, $r \in L_0$. 

3. Extension of $L$-fuzzy closure space

In this section, we define an extension of $L$-fuzzy closure space, a trace of the point with respect to the extension, a trace system of the extension and a principal extension of $L$-fuzzy closure space. Also, we provide some results on it.

**Definition 3.1.** Let $(X, c)$ and $(Y, c')$ be two $L$-fuzzy closure spaces and $\pi(X, c) \to (Y, c')$ be a mapping. Then $(\pi, (Y, c'))$ is said to be an embedding of $(X, c)$ if $\pi: (X, c) \to (\pi(X), c'_\pi(X))$ is a homeomorphism.

**Definition 3.2.** Let $(X, c)$ and $(Y, c')$ be two $L$-fuzzy closure spaces and $\pi(X, c) \to (Y, c')$ be a mapping. Then $(\pi, (Y, c'))$ is said to be an extension of $(X, c)$ if

(i) $(\pi, (Y, c'))$ is an embedding of $(X, c)$.

(ii) $c'(\pi(\lambda), r) = \frac{1}{r}$.

(iii) $\pi(\lambda \oplus \mu) = \pi(\lambda) \oplus \pi(\mu)$ for all $\lambda, \mu \in L^X$.

An extension $E = (\pi, (Y, c'))$ is said to be a principal extension of $(X, c)$ if $\{c'(\pi(\mu), r): \mu \in L^X, r \in L_0\}$ is a base for the $r$-$L$-fuzzy closed sets in $(Y, c')$.

Let $E_1 = (\pi_1, (Y_1, c'_1))$ and $E_2 = (\pi_2, (Y_2, c'_2))$ be two extensions of $(X, c)$. Then $E_1$ is said to be greater than or equal to $E_2$ (written as $E_1 \geq E_2$) if there exists an $LC$-$fuzzy$ continuous mapping $f$ from $(Y_1, c'_1)$ onto $(Y_2, c'_2)$ such that $f \circ \pi_1 = \pi_2$.

The extension $E_1 = (\pi_1, (Y_1, c'_1))$ is said to be equivalent to $E_2 = (\pi_2, (Y_2, c'_2))$ (written as $E_1 \approx E_2$) if there exists a homeomorphism $h$ from $(Y_1, c'_1)$ onto $(Y_2, c'_2)$ such that $h \circ \pi_1 = \pi_2$.

**Theorem 3.3.** Let $(X, c)$ and $(Y, c')$ be two $L$-fuzzy closure spaces and $\pi(X, c) \to (Y, c')$ an injective mapping. Then $(\pi, (Y, c'))$ is an embedding of $(X, c)$ iff $\pi(c(\lambda, r)) = c'(\pi(\lambda), r)$ for all $\lambda \in L^X, r \in L_0$.

**Proof.** Let $(\pi, (Y, c'))$ be an embedding of $(X, c)$. Then $\pi: (X, c) \to (\pi(X), c'_\pi(X))$ is a homeomorphism. From Theorem 2.9, $\pi(c(\lambda, r)) = c'(\pi(\lambda), r) \land \pi(1)$ for all $\lambda \in L^X, r \in L_0$.

Conversely, let $\pi(X, c) \to (Y, c')$ be an injective mapping and $\pi(c(\lambda, r)) = c'(\pi(\lambda), r) \land \pi(1)$ for all $\lambda \in L^X, r \in L_0$. Then, $\pi: (X, c) \to (\pi(X), c'_\pi(X))$ is a homeomorphism. Therefore, $(\pi, (Y, c'))$ is an embedding of $(X, c)$.

In view of the above theorem, the following result holds.

If $(X, c)$ and $(Y, c')$ are two $L$-fuzzy closure spaces and $\pi(X, c) \to (Y, c')$ is an injective mapping, then $(\pi, (Y, c'))$ is an extension of $(X, c)$ iff

(i) $\pi(c(\lambda, r)) = c'(\pi(\lambda), r) \land \pi(1)$ for all $\lambda \in L^X, r \in L_0$.

(ii) $c'(\pi(\lambda), r) = \frac{1}{r}$.

(iii) $\pi(\lambda \oplus \mu) = \pi(\lambda) \oplus \pi(\mu)$ for all $\lambda, \mu \in L^X$. □

**Definition 3.4.** Let $E = (\pi, (Y, c'))$ be an extension of $(X, c)$ and $y \in Y$. Define the trace $T_{(y, E)}$ of the point $y$ with respect to the extension $E$ by

$T_{(y, E)}(\lambda) = c'(\pi(\lambda), r)(y)$ for all $\lambda \in L^X, r \in L_0$.

where there is no chance of confusion, we shall simply write $T_y$ for $T_{(y, E)}$. The trace system $X^K$ of the extension $E$ is defined by $X^K = \{T_y : y \in Y\}$.

**Theorem 3.5.** Let $E = (\pi, (Y, c'))$ be an extension of $(X, c)$, $x$ is order-preserving and let $(Y, c')$ be a topological $L$-fuzzy closure space. Then,

(i) $T_x$ is an $L$-fuzzy c-grill in $(X, c)$ for all $y \in Y$.

(ii) $T_{c(\lambda)} = G_\lambda$ for all $x \in X$.

(iii) If $E_1$ and $E_2$ are two equivalent extensions of $X$, then $X^K_1 = X^K_2$.

**Proof.**

(i) Let $y \in Y$. Then $T_y(\emptyset) = c'(\pi(\emptyset), r)(y) = 0$ and $T_y(1) = c'(\pi(1), r)(y) = 1$. Also, for any $\lambda, \mu \in L^X, r \in L_0$,

$T_y(\lambda \oplus \mu) = c'(\pi(\lambda \oplus \mu), r)(y) = c'(\pi(\lambda) \oplus \pi(\mu), r)(y) = (c'(\pi(\lambda), r) \oplus c'(\pi(\mu), r))(y)$

$= c'(\pi(\lambda), r) \land c'(\pi(\mu), r) = T_y(\lambda) \land T_y(\mu)$.

Thus, $T_y$ is an $L$-fuzzy grill in $(X, c)$. Now, let $\lambda \in L^X, r \in L_0$. Then
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$T_y(\lambda) = c'(x(\lambda), r)(y) = c'(c'(x(\lambda), r), r)(y)$
\[\geq c'(c'(x(\lambda), r), r)(y) = c'(c'(x(\lambda), r), r)(y)\]
\[= T_y(c'(x(\lambda), r), r)\].

Obviously, $T_y(x(\lambda), r) \leq T_y(c'(x(\lambda), r), r)$. Thus, $T_y(x(\lambda), r) \leq T_y(c'(x(\lambda), r), r)$. Hence, $T_y(x(\lambda), r)$ is an $L$-fuzzy c-grill in $(X, c)$ for all $y \in Y$.

(ii) Let $x \in X$ and $\lambda \in L^X_r \cap L_0$. Then,

$\mathcal{T}(x(\lambda), r) = c'(x(\lambda), r)(x(\lambda)) = c'(x(\lambda), r)(x(\lambda)) = \alpha(x(\lambda), r)(x(\lambda)) = \mathcal{G}_x(\lambda)$.  

Thus, $\mathcal{T}(x(\lambda), r) = \mathcal{G}_x$ for all $x \in X$.

(iii) Let $E_1 = \{x_1, (Y_1, c_1)\}$ and $E_2 = \{x_2, (Y_2, c_2)\}$ be two equivalent extensions of $(X, c)$. Then there exists a homeomorphism $h$ from $(Y_1, c_1)$ onto $(Y_2, c_2)$ such that $h \circ x_1 = x_2$. Let $y \in Y_1$ and $\lambda \in L^X_r \cap L_0$. Then,

$\mathcal{T}(E_1, E_2)(\lambda) = \alpha((h \circ x_1)(\lambda), r)(h(y)) = \alpha((h \circ x_1)(\lambda), r)(h(y)) = \alpha(\lambda, r)(h(y)) = \alpha(\lambda, r)(h(y)) = \mathcal{T}(h(E_1), E_2)(\lambda)$.  

Thus, $\mathcal{T}(E_1, E_2) = \mathcal{T}(h(E_1), E_2)$. Therefore

$X^{E_1} = \{\mathcal{T}(E_1, E_2) : y \in Y_1\} = \{\mathcal{T}(h(E_1), E_2) : y \in Y_1\} = \{\mathcal{T}(E_1, E_2) : y \in Y_2\} = X^{E_2}$, since $Y_2 = \{h(y) : y \in Y_1\}$. In what follows we give an example to show that the converse of the result (iii) of Theorem 3.5 need not hold.  

Example 3.6. Let $X, Y, Z$ be three infinite sets such that $X \subset Y \subset Z$ and $|X| < |Y| < |Z|$, where $|X|$ denotes the cardinal number of the set $X$. Let $(L, \leq, \land, \lor, \neg)$ = $(I, \leq, \land, \lor, \neg)$ and let $c : I^X \rightarrow I^X$ be defined by

$c(\lambda, r) = \begin{cases} 0 & \text{if } 0 = \lambda \in I^X_r, \ r \in I_0 \\ 1 & \text{otherwise}. \end{cases}$

Clearly, $(Z, c)$ is an $L$-fuzzy closure space. Let $(X, c_X)$ and $(Y, c_Y)$ be subspaces of $(Z, c)$. Let $i_1 : (X, c_X) \rightarrow (Z, c)$ and $i_2 : (Y, c_Y) \rightarrow (Y, c_Y)$ be the inclusion mappings. Clearly, $E_1 = \{i_1 \circ (Z, c)\}$ is an extension of $(X, c_X)$ and $E_2 = \{i_2 \circ (Y, c_Y)\}$ is also an extension of $(X, c_X)$. Note that for each $x \in X$, $\mathcal{T}(E_1, E_2)(x) = \mathcal{G}_x = \mathcal{T}(E_2, E_1)(x)$. Define $G : I^X \rightarrow I$ by

$G(\lambda) = \begin{cases} 1 & \text{if } \lambda(a) = 1 \text{ for all } a \in X \\ 0 & \text{otherwise}. \end{cases}$

Then it is easy to check that $\mathcal{T}(E_1, E_2) = G$, for all $\zeta \in Z - X$ and $\mathcal{T}(E_2, E_1) = G$ for all $\zeta \in Y - X$. Hence $X^{E_1} = X^{E_2}$. But $E_1 \approx E_2$, as $|Y| < |Z|$.  

Theorem 3.7. Let $E = (x, (Y, c^{*}(x, c^{*}))$ be an extension of $(X, c)$. Then $G_{x_1} \leq G_{x_2}$ implies $T_{x_1} \leq T_{x_2}$ for all $y_1, y_2 \in Y$.

Proof. Suppose that $G_{x_1} \leq G_{x_2}$ for all $y_1, y_2 \in Y$. Let $\mu \in L^X_r, r \in L_0$. Then we have

$T_{x_1}(\mu) = c'(x(\mu), r)(y_1) = G_{x_1}(x(\mu)) = c'(x(\mu), r)(y_2)$ (since $x(\mu) \in L^X$) = $T_{x_2}(\mu)$.

Thus, $T_{x_1} \leq T_{x_2}$.  

Theorem 3.8. Let $(Y, c^{*})$ be a topological $L$-fuzzy closure space such that $(z, (Y, c^{*}))$ is a principal extension of $(X, c)$. Then $T_{y_1} \leq T_{y_2}$ iff $G_{y_1} \leq G_{y_2}$ for all $y_1, y_2 \in Y$.

Proof. From Theorem 3.7, we have $G_{y_1} \leq G_{y_2}$ implies $T_{y_1} \leq T_{y_2}$. Conversely, suppose that $T_{y_1} \leq T_{y_2}$. Thus, $c'(x(\mu), r)(y_1) = c'(x(\mu), r)(y_2)$ for all $\mu \in L^X$. Let $\lambda \in L^X$. Since $(Y, c^{*})$ is topological and $c'(x(\mu), r) \mu \in L^X, r \in L_0$ is a base for the $r$-$L$-fuzzy closed sets in $(Y, c^{*})$, then, we have

$G_{y_1}(\lambda) = c'(\lambda, r)(y_1) = \bigwedge (c'(\lambda, r) : \mu \in L^X, r \in L_0) = c'(\lambda, r)(y_2) = \bigwedge (c'(\lambda, r) : \mu \in L^X, r \in L_0) = c'(\lambda, r)(y_2) = G_{y_2}(\lambda)$.  

Thus, $G_{y_1} \leq G_{y_2}$.  

The following corollary is an easy consequence of the above theorem.

Corollary 3.9. Let $(Y, c^{*})$ be a topological $L$-fuzzy closure space and $(z, (Y, c^{*}))$ be a principal extension of $(X, c)$. Then $T_{y_1} \leq T_{y_2}$ iff $G_{y_1} = G_{y_2}$ for all $y_1, y_2 \in Y$.

Theorem 3.10. Let $(Y, c^{*})$ be a topological $L$-fuzzy closure space such that $(z, (Y, c^{*}))$ is a principal extension of $(X, c)$. Then $(Y, c^{*})$ is $T_0$ iff $T_{x_1} = T_{y_1}$ implies $x = y$ for all $x, y \in Y$.

Proof. Let $(Y, c^{*})$ be a $T_0$ and $x, y \in Y$ be such that $T_{x_1} = T_{y_1}$. Thus, $G_x = G_y$ and hence $x = y$.

Conversely, suppose that $T_{x_1} = T_{y_1}$ implies $x = y$ for all $x, y \in Y$. Let $x, y \in Y$ be such that $x \neq y$. Then $T_{x_1} \neq T_{y_1}$ and hence $G_x \neq G_y$. Thus, there exists $\lambda \in L^X$ such that $c'(\lambda, r)(x) \neq c'(\lambda, r)(y)$. Since $(Y, c^{*})$ is topological, then $\mu \in L^X$ is a $r$-$L$-fuzzy closed set such that $\mu(x) \neq \mu(y)$. Therefore, $(Y, c^{*})$ is $T_0$.  

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