Four Methods of Approximate Reasoning with Interval-Valued Fuzzy Sets

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ABSTRACT

An interval-valued fuzzy set approach is proposed for approximate reasoning. There are a number of possible interpretations of the interval-valued fuzzy modus ponens. Four of these are discussed. Sufficiency conditions are identified in order to reach a conclusion similar to the classical result. Numerical examples are given where appropriate. A prototype study is reviewed that shows an application of the interval-valued fuzzy set approach to aggregate production planning. The results of this approach are shown to fairly well approximate the results of currently accepted methods of aggregate production planning. Furthermore, the approach provides a much more "user friendly" interface.

KEYWORDS: ***approximate reasoning, crisp connectives, interval-valued fuzzy sets, linguistic connectives, modus ponens, sufficiency conditions, production planning***

INTRODUCTION

A framework for approximate reasoning can be formulated around the notion of "interval-valued" as opposed to "point-valued" fuzzy sets. "Interval-valued" fuzzy sets (IVFS) are causally connected to the real world in at least two ways: First,

- Imprecise knowledge obtained from domain experts usually gives rise to an "interval" specification of membership values.

That is, experts prefer to specify a range of values rather than a "point value" (Turksen [1], Adlassnig [2]). Second,

- Experts are at times vague about their meaning of linguistic connectives such as AND, OR, and IF ... THEN.

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That is, "AND" does not always precisely correspond to a "crisp" "intersection" operator; "OR" is not usually equivalent to a "crisp" "union" operator; "IF ... THEN" cannot always be modeled with a "crisp" "implication" operator (Turksen [3, 4]).

We will be concerned with both of these causes to the degree that they affect four methods of reasoning to be examined in this paper.

Several alternative approaches can be proposed for interval-valued fuzzy reasoning. Some of these are:

1. Truth-valued restriction (Zadeh [5], Baldwin and Guild [6]).
2. Compositional rule of inference with crisp operators (Zadeh [5]).
3. Compositional rule of inference with linguistic connectives (Turksen [7]).
4. Linguistic pattern-matching inference (Turksen and Zhong [8]).

With these alternatives, interval-valued fuzzy reasoning can be formulated for different modes of reasoning such as modus ponens, modus tollens, denial, and confirmation (Bandler and Kohout [9]). We will discuss four methods of inference based on an interpretation of alternatives 2 and 3 for generalized modus ponens.

Generalized modus ponens is expressed as

Premise:	IF P THEN Q	(Expert rule)
Premise:	P	(Observation)
Conclusion:	$Q' = P \circ (P \to Q)$	(Advice)

where $\circ$ is the compositional rule of inference.

With the interval-valued fuzzy set representation of linguistic compositions to be presented in the next section, we can formulate at least four methods for interval-valued fuzzy modus ponens. These four possible formulations are the result of two options we have for each of the following two factors:

1. The observation P could be interpreted as either (a) a "point-valued" or (b) an "interval-valued" fuzzy set
2. The compositional rule of inference "$\circ$" that combines the expert rule $P \to Q$ with the observation P could either be "crisp" or "linguistic."

In this paper, it is assumed that a representation of P and Q is given as "point-valued" fuzzy sets to start out our analysis. Further, it is assumed that an experts' rule, $P \to Q$, is represented as an "interval-valued" fuzzy set due to the linguistic interpretation of the implication. This will be discussed in the next section. Thus we are confronted with the four possible interpretations as a result of two options on P and two options on the compositional rule of inference $\circ$.

As a background, we first briefly review linguistic interpretations of the combined propositions that form the basis of interval-valued fuzzy sets considered in this paper. Afterwards, we turn our attention to the interval-valued fuzzy inference and discuss the four possible methods of generalized modus ponens. Finally, we review an application of the interval-valued fuzzy set approach to aggregate production planning.

INTERVAL-VALUED FUZZY SETS

I have suggested previously that the linguistic connectives AND, OR, IF ... THEN should be interpreted in a way that leads to an interval-valued representation of the linguistic propositions (Turksen [3]). More recently it was shown that there is a theoretical foundation for a class of interval-valued fuzzy sets (IVFSs) (Turksen [4]). This class of IVFSs is based on the disjunctive and conjunctive normal forms, DNF and CNF, respectively, of a fuzzy proposition as an extension of the canonical forms of Boolean logic. For example, DNF and CNF for the three basic linguistic propositions are

(I) Linguistic AND proposition:

$$\mathrm{DNF}[A \text{ AND } B] \triangleq A \cap B$$

$$\mathrm{CNF}[A \text{ AND } B] \triangleq (A \cup B) \cap (A \cup B^c) \cap (A^c \cup B)$$

(II) Linguistic OR proposition:

$$\mathrm{DNF}[A \text{ OR } B] \triangleq (A \cap B) \cup (A \cap B^c) \cup (A^c \cap B)$$

$$\mathrm{CNF}[A \text{ OR } B] \triangleq A \cup B$$

(III) Linguistic IF ... THEN ($\rightarrow$) proposition:

$$\mathrm{DNF}[A \rightarrow B] \triangleq (A \cap B) \cup (A^c \cap B) \cup (A^c \cap B^c)$$

$$\mathrm{CNF}[A \rightarrow B] \triangleq A^c \cup B$$

In this framework, linguistic affirmation of an observed system fact A in a specific domain (or relative to B) can also be interpreted with its DNF and CNF as follows:

(IV) Linguistic affirmation A:

$$\mathrm{DNF}(A) = (A \cap B) \cup (A \cap B^c)$$

$$\mathrm{CNF}(A) = (A \cup B) \cap (A \cup B^c)$$

It should be noted that in this paper there are three distinct but related levels of representational schemes: (1) the propositional level, (2) the fuzzy set (symbolic) form level, and (3) the fuzzy set (numeric) membership level.

At the top level, the propositions are represented as A AND B, A OR B, IF A THEN B ($A \rightarrow B$, for short), etc., where the logical connectives are linguistic, i.e., AND, OR, IF ... THEN ($\rightarrow$), etc.

At the midlevel, the interval-valued fuzzy set representation of these propositions is expressed with dual forms of DNF(●) and CNF(●) as shown in (I), (II), (III), and (IV).

The normal forms intern are defined, on the right-hand side, with the use of fuzzy set generalized connectives for intersection, union, and complementation as $\cap$, $\cup$, and ^{c}, respectively. We need to emphasize the distinction between these generalized connectives that we use in the definition of symbolic forms, DNF and CNF, and their corresponding particular representations with a choice of t-norms, t-conorms, and pseudo or strong complementation operators for the computations in the membership domain.

However, before we proceed to the bottom-level representation in the membership domain, let us clarify the definition of the IVFSs at the midlevel. The essential result of my previous work (Turksen [3, 4, 7, 10]) is that for certain classes of the conjugate pairs of t-norms and t-conorms each DNF is contained in its corresponding CNF for every linguistic proposition in general, and in particular, for the three linguistic propositions and affirmations specified above. As a consequence, it is suggested that every linguistic proposition and affirmation be represented by an IVFS, $S(\bullet) \triangleq \{S\}$ such that

$$S(\bullet) \triangleq \{S|\text{DNF}[\bullet] \subseteq S \subseteq \text{CNF}[\bullet]\}.$$

For example, for the three linguistic propositions considered above, $S(\bullet)$ is to be defined as follows:

$$\begin{aligned} S(A \text{ AND } B) &\triangleq \{S|A \cap B \subseteq S \subseteq (A \cup B) \\ &\qquad \cap (A \cup B^c) \cap (A^c \cup B)\} \end{aligned} \tag{1a}$$

$$\begin{aligned} S(A \text{ OR } B) &\triangleq \{S|(A \cap B) \cup (A \cap B^c) \\ &\qquad \cup (A^c \cap B) \subseteq S \subseteq A \cup B\} \end{aligned} \tag{1b}$$

$$\begin{aligned} S(A \rightarrow B) &\triangleq \{S|(A \cap B) \cup (A^c \cap B) \\ &\qquad \cup (A^c \cap B^c) \subseteq S \subseteq A^c \cup B\} \end{aligned} \tag{1c}$$

and the linguistic affirmation of A relative to B as

$$S(A) = \{S|(A \cap B) \cup (A \cap B^c) \subseteq S \subseteq (A \cup B) \cap (A \cup B^c)\} \tag{1d}$$

Observe that the "point-valued" definition of A AND B in current literature takes only the form $A \cap B$, which is the lower bound of $S(A \text{ AND } B)$. On the other hand, the "point-valued" definition of A OR B in current literature takes only the form $A \cup B$, which is the upper bound of $S(A \text{ OR } B)$. The case of $A \rightarrow B$ is defined in many ways in the current literature. It is sufficient to say that $S(A \rightarrow B)$ contains some of the "point-valued" definitions that are identified below.

However, before I can explain the meaning of my last statement, we need to discuss the membership level representation that is required, as stated above, for computations in the membership domain. At this level, the fuzzy set generalized combination operations intersection, union, and complementation—i.e., $\cap$, $\cup$, and ^{c}—are replaced by the corresponding t-norms, t-conorms, and pseudocomplementation, e.g., $(\wedge, \vee, -)$. Now, in order to make this distinction clear, the membership level representation scheme of the IF … THEN proposition is expressed as follows:

$$\underset{S(A\to B)}{\mu(\wedge, \vee, -)} \triangleq \{\mu_S \mid \underset{\mathrm{DNF}(A\to B)}{\mu(\wedge, \vee, -)} \le \mu_S \le \underset{\mathrm{CNF}(A\to B)}{\mu(\wedge, \vee, -)}\}$$

where the choice of t-norm, t-conorm, and pseudocomplementation is identified as well as the IVFS membership expression of the linguistic proposition $A \to B$.

With this membership level representation, we can compare the interval-valued fuzzy set representation of implication with the "point-valued" expressions of fuzzy implication that may be found, in part, in Bandler and Kohout [9] and elsewhere.

R_5:	Lukasiewicz	$a \to {}_5 b = \min(1, 1 - a + b)$
$R_{5.5}$:	Kleene–Dienes–Lukasiewicz	$a \to {}_{5.5} b = 1 - a + ab$
R_6:	Kleene–Dienes	$a \to {}_6 b = (1 - a) \vee b$
R_7:	Early Zadeh	$a \to {}_7 b = (a \wedge b) \vee (1 - a)$
R_8:	Willmott	$a \to {}_8 b = (a \to {}_7 b) \vee (b \wedge (1 - b))$

It is observed, for example, that

1. The Lukasiewicz implication $a \to {}_5 b$ is the upper bound of the interval-valued fuzzy implication $S(A \to B)$ defined by bold operations, $(\dot{\wedge}, \dot{\vee})$; i.e.,

$$(a \wedge b)\ \dot{\vee}\ [(1-a)\ \dot{\wedge}\ b]\ \dot{\vee}\ [(1-a)\ \dot{\wedge}\ (1-b)] \to \le \mu_S \le (1-a)\ \dot{\vee}\ b$$

where $(a\ \dot{\wedge}\ b) \triangleq \max(0, a + b - 1)$ and $(a\ \dot{\vee}\ b) \triangleq \min(1, a + b)$ are the bold intersection and bold union, respectively.

2. The Kleene–Dienes implication $a \to {}_6 b$ is the upper bound of the interval-valued fuzzy implication $S(A \to B)$ defined by min-max operators $(\wedge, \vee)$, i.e.,

$$(a\ \dot{\wedge}\ b) \vee [(1-a) \wedge b] \vee [(1-a) \vee (1-b)] \le \mu_S \le (1-a) \vee b$$

Upon further investigation, it can be shown that

$$\underset{\mathrm{DNF}(A\to B)}{\mu(\dot{\wedge}, \vee, -)} \le \{R_6, R_7, R_8\} \le \underset{\mathrm{CNF}(A\to B)}{\mu(\wedge, \vee, -)} \tag{2}$$

where, for example, R^6 stands for $a \to {}_6 b$ and so on.

$$\underset{\mathrm{DNF}(A\to B)}{\mu(\bullet, \hat{+}, -)} \le \{R_5\} \le \underset{\mathrm{CNF}(A\to B)}{\mu(\bullet, \hat{+}, -)} \tag{3}$$

where $a \hat{+} b = a + b - ab$ and $a \bullet b = ab$;

$$\underset{\text{DNF}(A \to B)}{\mu(\hat{\wedge}, \hat{\vee}, -)} \leq \{R_5, R_{5.5}, R_6\} \leq \underset{\text{CNF}(A \to B)}{\mu(\hat{\wedge}, \hat{\vee}, -)} \tag{4}$$

$$\underset{\text{DNF}(A \to B)}{\mu(T_w, S_w, -)} \leq \{R_5, R_{5.5}, R_6\} \leq \underset{\text{CNF}(A \to B)}{\mu(T_w, S_w, -)} \tag{5}$$

where

$$T_w(a, b) = \begin{cases} a \wedge b & \text{if } a \vee b = 1 \\ 0 & \text{otherwise} \end{cases}$$

and

$$S_w(a, b) = \begin{cases} a \vee b & \text{if } a \wedge b = 0 \\ 1 & \text{otherwise} \end{cases}$$

These results show that IVFS representation of implication is a cover for certain other "point-valued" fuzzy implication expressions. Thus, it may be more appropriate to use the IVFS representations of implication for certain domains of application where there exists a second-order imprecision.

INTERVAL-VALUED FUZZY MODUS PONENS

Let us recall the four possible interpretations of the generalized modus ponens

$$Q' = P \circ (P \to Q) \tag{6}$$

1. P is a "point-valued" fuzzy set, $P \to Q$ is an IVFS given by Eq. (1c), and
   (a) $\circ$ is a compositional rule of inference with "crisp" operators as given by Zadeh [5], or
   (b) $\circ$ is a linguistic compositional rule of inference where crisp operators are replaced by linguistic connectives; and
2. P is an "interval-valued" fuzzy set based on linguistic affirmation (1d), $P \to Q$ is an IVFS given by (1c), and
   (a) $\circ$ is a crisp compositional rule of inference, or
   (b) $\circ$ is a linguistic compositional rule of inference.

Let us now investigate each of these four possible methods for the distributive class of operators, e.g., min-max.

Method 1. P Point-Valued and $\circ$ Crisp

Let the interval-valued fuzzy set Q' be identified by its lower-bound fuzzy set Q'_L and its upper-bound set Q'_U. With our proposed interpretation of $(P \to Q)$ given in (1c), we rewrite it as

$$Q'_L = P \circ \text{DNF}[P \to Q] \qquad \text{and} \qquad Q'_U = P \circ \text{CNF}[P \to Q] \tag{7}$$

In the following we substitute the membership value expressions for DNF and CNF and interpret the composition $\circ$ in accordance with Zadeh [5].

$$q'_{jL} = \bigvee_i (p_i \wedge [(p_i \wedge q_j) \vee (\bar{p}_i \wedge q_j) \vee (\bar{p}_i \wedge \bar{q}_j)]) \qquad \text{and}$$

$$q'_{jU} = \bigvee_i (p_i \wedge (\bar{p}_i \vee q_j)) \tag{8}$$

for all $i \in I$ and $j \in J$ where the pi and q_j are the membership values in the finite support sets of I and J of the fuzzy sets P and Q, respectively. Furthermore, the q'_{jL} and q'_{jU} are the lower and upper bounds of the membership values for the resultant Q'.

It can be shown that

$$q'_{jL} = q'_{jU} = \bigvee_i [(p_i \wedge \bar{p}_i) \vee (p_i \wedge q_j)] \tag{9}$$

for the distributive class of conjugate pairs of operators $(\wedge, \vee)$ together with the pseudocomplement $(-)$.

It is clear from (9) that, in general $Q' \neq Q$. However, it can be shown that the following two sufficient conditions are required for $Q' = Q$:

(i) P must be normalized. (10)

(ii) We must take into consideration only those values of Q that are at or above an α cutoff, where $\alpha = \max(p_i \wedge \bar{p}_i)$. (11)

EXAMPLE 1. Consider two fuzzy sets,

$$P = 0.3/X_1 + 0.5/X_2 + 1/X_3 \qquad \text{and} \qquad Q = 0.2/Y_1 + 0.6/Y_2 + 0.8/Y_3$$

First, P satisfies condition (i), i.e., it is normalized. Second,

$$\alpha = \max(0.3, 0.5, 0) = 0.5$$

Therefore

$$Q_\alpha = 0.6/Y_2 + 0.8/Y_3$$

The interval-valued interpretation of IF P THEN Q is shown in Table 1 for

Table 1. IF P THEN Q ($P \to Q$)

	Q	
P	0.6	0.8
0.3	[0.6, 0.7]	[0.7, 0.8]
0.5	[0.5, 0.6]	[0.5, 0.8]
1	0.6	0.8

the conjugate pair of min-max operators and the pseudocomplement. Using Zadeh's composition, $P \circ (P \to Q)$, in accordance with the interpretation (8), we get back $Q_\alpha = 0.6/Y_2 + 0.8/Y_3$ as a result of the sufficiency conditions (10) and (11).

Method 2. P Point-Valued but $\circ$ Linguistic

With the linguistic composition interpreted in analogy to Zadeh's composition, we write the following four possible formulations:

DNF[P AND DNF[$P \to Q$]] DNF[P AND CNF[$P \to Q$]]

CNF[P AND DNF[$P_i \to Q_j$]] CNF[P AND CNF[$P \to Q$]]

Upon investigation, we find that

$$q'_{jL} = \bigvee_i \{(p_i \wedge q_j) \vee (p_i \wedge \bar{p}_i)\} \qquad \text{and}$$

$$q'_{jU} = \bigvee_i \{(p_i \wedge q_j) \vee (p_i \wedge \bar{p}_i) \vee (\bar{p}_i \wedge q_j \wedge \bar{q}_j)\}$$

for all $i \in I$ and $j \in J$

In order to obtain the classical result, i.e., $Q' = Q$, it can be observed that the lower boundary condition

$$(p_i \wedge q_j) \vee (p_i \wedge \bar{p}_i)$$

gives the same two conditions found in Method 1. Furthermore, it can be observed that

$$\bar{p}_i \leqslant q_j \wedge \bar{q}_j$$

collapses the upper boundary

$$(p_i \wedge q_j) \vee (p_i \wedge \bar{p}_i) \vee (\bar{p}_i \wedge q_j \wedge \bar{q}_j)$$

to the lower boundary together with conditions (10) and (11) of Method 1.

Hence we have the third condition:

(iii) We must now take into consideration only those values of P^c that are at or below a β cutoff, where

$$\beta = \min_i (q_j \wedge \bar{q}_j) \tag{12}$$

EXAMPLE 2. Consider the same two fuzzy sets P and Q given above. Now condition (12) produces a β cutoff for P^c; i.e., $\beta = \min(0.4, 0.2) = 0.2$. Thus $P_\beta = 1/X_3$, and hence

$$Q_{\alpha\beta} = Q'_{\alpha\beta} = 0.6/Y_2 + 0.8/Y_3$$

Method 3. *P* Interval-Valued and ∘ Crisp

In this case, we need to start out with the linguistic affirmation of P as

$$\mathrm{DNF}(P) \triangleq (P \cap Q) \cup (P \cap Q^c) \qquad \text{and}$$
$$CNF(P) \triangleq (P \cup Q) \cap (P \cup Q^c)$$

based on (1d). In accordance with Zadeh's crisp composition, we write the following four possible formulations:

$$\begin{array}{ll} \mathrm{DNF}(P) \circ \mathrm{DNF}[P \to Q) & \mathrm{DNF}(P) \circ \mathrm{CNF}[P \to Q) \\ \mathrm{CNF}(P) \circ \mathrm{DNF}[P \to Q) & \mathrm{CNF}(P) \circ \mathrm{CNF}[P \to Q) \end{array} \tag{13}$$

Upon investigation, we find that

$$q'_{qL} = \bigvee_i [(p_i \wedge q_j) \vee (p_i \wedge \bar{p}_i)] \qquad \text{and}$$

$$q'_{qU} = \bigvee_i [(p_i \wedge q_j) \vee (p_i \wedge \bar{p}_i) \vee (q_j \wedge \bar{q}_j)]$$

$$\text{for all } i \in I \text{ and } j \in J$$

It turns out that this will lead to the same conditions as in Method 1 for $Q' = Q$.

EXAMPLE 3. Consider the same two fuzzy sets P and Q. Now we obtain again $Q_\alpha = Q'_\alpha = 0.6/Y_2 + 0.8/Y_3$ as in Example 1.

Method 4. *P* Interval-Valued and ∘ Linguistic

In this case, we need to investigate eight expressions from

$$\mathrm{DNF}[\mathrm{DNF}(P) \text{ AND } \mathrm{DNF}[P \to Q]]$$

to

$$\text{CNF[CNF}(P)\text{ AND CNF}[P \rightarrow Q]]$$

For the sake of brevity, I leave the writing of these eight expressions as an exercise for the reader. Upon investigation, it turns out that

$$q'_{jL} = \bigvee_i [(p_i \wedge q_j) \vee (p_i \wedge \bar{p}_i)] \qquad \text{and}$$

$$q'_{jU} = \bigvee_i [(p_i \wedge q_j) \vee (p_i \wedge \bar{p}_i) \vee (q_j \wedge \bar{q}_j)]$$

$$\text{for all } i \in I \text{ and } j \in J$$

which are the same results as those obtained in Method 3 leading to the same sufficient conditions for $Q' = Q$.

It is clear that Method 1 is the most primitive and simplest, and Method 4 is the most comprehensive and most complex of all the four methods of approximate reasoning with interval-valued fuzzy sets considered in this paper.

We will next review the results of a prototype study where I applied Method 1 with the use of max-min operators. It should be recalled that in Method 1 the fuzzy sets P and Q are point-valued at the start. Then $P \rightarrow Q$ leads to an interval-valued representation and therefore the composition $P \circ (P \rightarrow Q)$ is also interval-valued even though the composition rule of inference $\circ$ is crisp in Zadeh's sense. Thus the result of the generalized modus ponens is an IVFS.

AGGREGATE PRODUCTION PLANNING—AN APPLICATION

Aggregate production planning is mainly concerned with decisions on production and work force levels and the management of inventories. More specifically, aggregate production planning attempts to balance costs: costs of regular payrolls associated with rates of production; costs of changes in the size of the work force (hiring/layoff costs); under- or overutilization costs of the work force (idle time or overtime costs); and inventory costs (carrying charges, backorder costs, lost sales costs); etc. For the factories of the future, some of this classic terminology will have to be modified to account for robots replacing some or all of the human operators and inventories adjusted to just-in-time schemes. However, with these modifications, the essential nature of the problem does not change but becomes much more complex, requiring a method of approximate reasoning methodology that was introduced in the previous sections.

Management Rules versus OR

In our quest for precision in operations research methodologies, a multitude of mathematical models of aggregate production planning were developed in the past 30 years of operational management literature (e.g., Bowman [11, 12]; Haussman and Hess [13]; Holt et al. [14–16]; Jones [17]). However, the implementation of any of these models in industry is practically nonexistent (Buffa and Taubert [18], Lee and Khumawala [19]). This may be attributed to the fact that these operations research models enforce an artificial precision inappropriate to many real world situations, in particular to such complex systems involving humans and robots. It appears that most managers use their own heuristics, which does not guarantee mathematical optimality. Available empirical evidence (Bowman [12]; Vergine [20]; Eilon [21]) suggests that these managers' (experts') judgmental models do remarkably well. It seems that experienced managers (experts) capture the knowledge of aggregate production planning in the form of rules that contain linguistic variables that are many-valued rather than two-valued. A collection of such expert rules is given in Table 2. For example, rule 4 should be read as

IF the current period sales forecast, S_t, is HIGH,

AND the previous period inventory level, I_{t-1}, is AVERAGE

AND the previous period work force level, W_{t-1}, is HIGH,

THEN the current period production rate, P_t, should be HIGH and the change in the work force level, ΔW_t, should be AVERAGE.

Variables of the Prototype

It is well known from the research of the past 30 years on operations management that the current period sales forecast, the previous period work force, and the previous inventory level are essential independent variables for the determination of the dependent variables that are production rate and work force level for the current period of concern in the planning. The following definitions are usually stated in the literature of operations management.

$S_t \quad \triangleq$ the sales forecast for period t

$W_{t-1} \quad \triangleq$ the work force level in period $t - 1$

$I_{t-1} \quad \triangleq$ the inventory level at the end of period $t - 1$

$\Delta W_t \quad \triangleq$ the change in the work force level at the beginning of period t

$P_t \quad \triangleq$ the production rate of period t

In quantitative models of operations management, the relationships between these variables are expressed in the form of linear and/or nonlinear equations. In approximate reasoning, these relationships are expressed in propositions containing linguistic (fuzzy) variables in forms of linguistic (fuzzy) statements as shown in the above example. Such statements form the core of a metalanguage of approximate reasoning providing a representational structure for human

Table 2. Rules for Production Level and Change in Work Force

Rule No.	Independent Variables			Decision Variables[a]	
	S_t	I_{t-1}	W_{t-1}	P_t	ΔW_t
1	H	H	H	SH	RL
2	H	H	A	A	RH
3	H	H	L	A	RH
4	H	A	H	H	A
5	H	A	A	RH	RH
6	H	A	L	SH	VH
7	H	L	H	H	A
8	H	L	A	RH	RH
9	H	L	L	SH	H
10	A	H	H	SH	RL
11	A	H	A	A	A
12	A	H	L	A	RH
13	A	A	H	SH	RL
14	A	A	A	A	A
15	A	A	L	A	RH
16	A	L	H	SH	RL
17	A	L	A	A	A
18	A	L	L	A	RH
19	L	H	H	SL	L
20	L	H	A	RL	RL
21	L	H	L	RL	A
22	L	A	H	SL	VL
23	L	A	A	RL	RL
24	L	A	L	L	A
25	L	L	H	SH	RL
26	L	L	A	A	A
27	L	L	L	A	RH

[a] See Table 3 for translation of these linguistic terms.

expert knowledge (see Table 2). In addition, propositions about observed system states and conclusions obtained with modes of inference are also represented in such linguistic expressions. The terms high (**H**), low (**L**), average (**A**), etc., in such linguistic expressions are known as linguistic (fuzzy) variables. It is clear that such linguistic expressions are a part of the natural language and should have a better appeal for managers.

The attractiveness and potential power of approximate reasoning rest on the observations that (1) the thought processes of managers (experts) incorporate such linguistic variables in expressing their knowledge of system interactions and (2) linguistic rule-based models of planning have the potential of producing a better cognitive simulation of managerial thought processes than ones that use precise mathematical models of operations research, which, it appears, have so far not appealed to managers.

Knowledge Base for the Prototype

The knowledge base for an aggregate production planning prototype should contain

1. A rule base that consists of management decision rules.
2. A data base that consists of all other information needed to support the inference system in order to infer an expert system advice.

Rule Base for the Prototype

Management decision rules for aggregate production planning should be expressed in the metalanguage representation as

IF X_1 IS $\mathbf{A}_1$ AND X_2 is $\mathbf{A}_2$ AND X_3 is $\mathbf{A}_3$

THEN Y is (should be) $\mathbf{B}$

where X_1 is the base variable of sales forecast S_t, X_2 is the inventory level I_{t-1}, and X_3 is the work force level W_{t-1}. Furthermore, the $\mathbf{A}_i$ are the linguistic variables describing these independent base variables with the allowable linguistic variables such as those given in Table 3.

In a previous study, Rinks [22] investigated a knowledge base with 40 such rules (see Table 4). I have hypothesized, however, that 27 such rules should be sufficient for aggregate production planning. My hypothesis is based on

Table 3. Membership Functions of the Allowable Linguistic Terms

Linguistic Term	Acronym	Membership Function Expression[a]
Very high	VH	HIGH * HIGH
High	H	$1 - \exp[(-0.5/\|1 - x\|)^{2.5}]$
Rather high	RH	$1 - \exp[(-0.25/\|0.7 - x\|)^{2.5}]$
Sort of high	SH	$1 - \exp[(-0.25/\|0.4 - x\|)^{2.5}]$
Average	A	$e^{-5\|x\|}$
Sort of low	SL	$1 - \exp[(-0.25/\|-0.4 - x\|)^{2.5}]$
Rather low	RL	$1 - \exp[(-0.25/\|-0.7 - x\|)^{2.5}]$
Low	L	$1 - \exp[(-0.5/\|-1 - x\|)^{2.5}]$
Very low	VL	LOW * LOW
At least average	ALA	$e^{-5\|x\|}, \quad -1 \le x \le 0$
		$1, \quad 0 < x \le 1$
At most average	AMA	$1, \quad -1 \le x \le 0$
		$e^{-5\|x\|}, \quad 0 < x \le 1$

Table 4. Rules for Production Level and Change in Work Force

Rule No.	Independent Variables			Decision Variables[a]	
	S_t	I_{t-1}	W_{t-1}	P_t	ΔW_t
1	H	AMA	H	H	A
2	H	AMA	A	RH	RH
3	H	AMA	L	SH	VH
4	SH	L	H	H	A
5	SH	L	A	RH	RH
6	SH	L	L	SH	VH
7	SH	SH	H	SH	RL
8	SH	SH	A	A	A
9	SH	SH	L	A	RH
10	A	A	H	SH	RL
11	A	A	A	A	A
12	A	A	L	A	RH
13	SL	SL	H	SH	RL
14	SL	SL	A	A	A
15	SL	SL	L	SL	RH
16	RL	L	H	SH	RL
17	RL	L	A	A	A
18	RL	L	L	A	RH
19	L	ALA	H	SL	VL
20	L	ALA	AL	RL	RL
21	L	ALA	L	L	A
22	SL	H	H	SL	VL
23	SL	H	A	RL	RL
24	SL	H	L	RL	A
25	H	AMA	SH	H	SH
26	H	AMA	SL	SH	H
27	SH	L	SH	H	SH
28	SH	L	SL	SH	H
29	SH	SH	SH	A	A
30	SH	SH	SL	A	SH
31	A	A	SH	A	SL
32	A	A	SL	A	SH
33	SL	SL	SH	A	SL
34	SL	SL	SL	A	A
35	RL	L	SH	A	SL
36	RL	L	SL	A	A
37	L	ALA	SH	RL	L
38	L	ALA	SL	L	SL
39	SL	H	SH	RL	L
40	SL	H	SL	RL	A

1. The knowledge that aggregate production is a robust system for both planning and execution. This has been suggested by earlier studies (Holt et al. [15], Eilon [21]).
2. The fact that approximate reasoning based on fuzzy logic is a robust inference mechanism in the sense that a system observation $A' \neq A$ can be combined with a rule $A \rightarrow B$ to provide a conclusion B'.

Thus my hypothesis suggests that it would be sufficient to use just the basic anchor variables known as "high," "low," and "medium" for all three base variables, sales forecast, inventory level, and work force level, to determine both the rate of production and the level of work force for the planned period of production.

On the basis of this hypothesis, I have decided that the linguistic management rules depicted in Table 2 should form a reasonable rule base in contrast to those of Table 4 provided by Rinks [22]. (A comparison of these two rule bases will be made toward the end of this paper.)

From the foregoing discussion it is to be noted that in Table 1 the independent system variables of aggregate production planning are the following:

1. The sales forecasts for period t. There will be a need to determine two sales forecasts, one for production decision rules, S_t^P, and one for work force decision rules, S_t^W. These will be further explained in the "Data Base" section.
2. The inventory level at the end of period $t - 1$, I_{t-1}.
3. The work force level in period $t - 1$, W_{t-1}.

The dependent decision variables are

4. The production rate for period t, P_t.
5. The change in work force level at the beginning of period t, ΔW_t.

Data Base for the Prototype

The data base consists of three basic types of components:

1. Weights for the sales forecast: one set for S_t^P and another for S_t^W.
2. Membership values for the linguistic variables allowed to be used in the system.
3. Descriptions of observed system characteristics.

In general, components 1 and 2 would be permanent components of the data base, which may require adjustments from time to time to accommodate environmental changes, whereas 3 would be temporary within the span of a particular system diagnosis or system planning for the period of concern.

Forecasting Weights

In order to determine the effects of future sales forecasts on the current period planning decisions, the decision maker needs to choose two weighted forecasting functions, one for the work force level and the other for the production level

(Rinks [22]). Let F_j denote the sales forecast for the jth time period obtained by a forecasting method, say regression analysis; then for use in determining the work force level by a rule, we need to compute

$$S_t^W = \sum a_i F_{t+i-1}$$

and for use in determining the production rate by a rule, we need to compute

$$S_t^P = \sum b_i F_{t+i-1}$$

For these weighting functions, the decision maker needs to choose a_i's and b_i's appropriately. It is suggested by Rinks [22] that these weights be given by the following formulas:

$$a_i = \frac{1}{4+i} \sum \frac{1}{4+i}, \qquad i = 1, \cdots, n$$

$$b_i = \left(\frac{1}{2}\right)^i \sum \left(\frac{1}{2}\right)^i \qquad i = 1, \cdots, n$$

In order to be able to compare our results with Rinks's "point-valued" fuzzy set approach, we have used the same weights. Naturally, other weighting schemes are just as plausible; and in fact they should be specified by the manager (expert) in charge of planning or the one who provides the rules for the rule base. The ultimate choice will depend on the cost structure of the firm. It is to be noted, however, that in choosing a particular weighting scheme, the decision-maker is implicitly making trade-offs among the various costs.

Membership Values

The linguistic variables high, **H**; low, **L**; and average **A**; etc., as shown in Table 1 for the aggregate production planning variables—i.e., the independent base variables S_t, I_{t-1}, W_{t-1} and the dependent base variables P_t and ΔW_t—are defined by the exponential functions shown in Table 3 as suggested by Ostergaard [23], as found experimentally and validated by regression analysis by Zysno [24], and as used by Rinks [22]. Table 3 contains all the membership functions of the allowable linguistic terms used in the prototype study. An important feature on the exponential functions used in practice is that all base variables are scaled to be placed in the interval $[-1, 1]$. When the same linguistic variable is used to describe several base variables, this proves to be extremely advantageous for the adoption of a linguistic variable to the particular context of the base variable. For example, we show in Table 5 the upper and lower bounds of the base variables for the paint factory case to be discussed in the sequel. It needs to be pointed out that although the membership functions are defined analytically for continuous values of the base variables, the linguistic

Table 5. Bounds on the Base Variables of Paint Factory Case

Variable	Lower Bound	Upper Bound
W_{t-1}	60.	115.
ΔW_t	−10.	10.
P_t	250.	750.
I_{t-1}	150.	490.
FS_t	250.	750.

variables are actually represented by finite support sets for computational efficiency. Furthermore, they need not be defined analytically. They could be taken as given by experts or as found by a measurement experiment (Norwich and Turksen [25], Zysno [24]).

Observed System

Descriptions of the observed system characteristics are given with statements of the form

$$X_1 \text{ is } A_1^* \text{ and } X_2 \text{ is } A_2^* \text{ and } X_3 \text{ is } A_3^*$$

where X_1, the sales forecast, is either S_t^P or S_t^W for the production rate or work force level decision, respectively; X_2 is the inventory level I_{t-1}; and X_3 is the work force level W_{t-1}. Furthermore, the $\mathbf{A}_i^*$ are the linguistic variables describing the observed base variables with the allowable linguistic variables given in Table 3. If the user would like to use a linguistic variable that is not in Table 3, then the system should ask the user for its membership description.

APPLICATION TO A PAINT FACTORY

In order to test the performance of the Method 1 version of this approximate reasoning approach to aggregate production planning, I used the paint factory data depicted graphically in Figure 1 and compared it with the classic Holt–Modigliani–Muth–Simon (HMMS) [14–16] analysis. The reasons for choosing the paint factory data for comparative purposes are twofold: (1) the case uses real-world data and is sufficiently well documented for such a comparative analysis; and (2) the HMMS paint factory linear decision rule solutions have become the standard by which other aggregate planning models are compared (Gordon [26], Jones [17], Taubert [27], Lee and Khumawala [19]).

Important assumptions of the comparative analysis are (1) the quadratic cost function derived for the paint factory is the "true" cost function (2) the

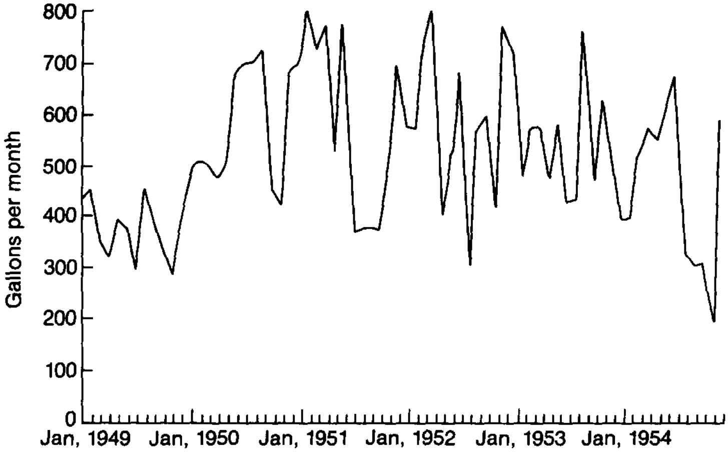

Figure 1. Perfect sales forecast—actual orders.

decisions computed from the derived linear rules are optimal, and (3) perfect sales forecasts, i.e., actual orders (see Figure 1), are available. Other studies, in particular Jones [17], Taubert [27], and Rinks [22], have also made these same assumptions and compared their results with those of the linear decision rules.

The Paint Factory

According to the HMMS formulation, the objective of the production planner is to minimize the total cost of regular payroll, overtime, hirings and layoffs, inventories, and shortages during a given planning interval of N periods. After making the necessary assumptions and statistically estimating the cost coefficients, the mathematical formulation of the problem for the HMMS paint factory was stated as (Holt et al. [16]):

$$\text{Minimize} \quad C_N = \sum_{t=1} C_t \qquad \text{(Total cost)} \tag{14}$$

$$\begin{aligned} \text{where } C_t = {} & 340W_t && \text{(Regular payroll costs)} \\ & + 64.3(W_t - W_{t-1})^2 && \text{(Hiring and layoff costs)} \\ & + 0.20(P_t - 5.67W_t)^2 && \\ & + 51.2P_t - 28W_t && \text{(Overtime costs)} \\ & + 0.0825(I_t - 320)^2 && \text{(Inventory costs)} \end{aligned}$$

subject to the material balance equations

$$I_t = I_{t-1} + P_t - S_t; \qquad t = 1, \cdots, N$$

The objective function, as can be seen, is a quadratic function in the decision variables. With the assumption of quadratic costs, the HMMS solution methodology forms a set of linear equations by taking partial derivatives with respect to each period work force and each period inventory and setting them equal to zero. Then, solving this set of linear equations, two *linear decision rules* (LDR) are obtained that indicate for any planning period the production level and size of work force that yield the lowest cost (Holt et al. [14–16]).

The actual order pattern for the six-year period that HMMS analyzed, 1949–1954, was extremely variable, including both the 1949 recession and the Korean war. In this respect, the paint factory data provide a rather rigorous test environment. The order pattern is graphically depicted in Figure 1.

An application of HMMS LDR gives, using 12-month perfect sales forecasts, a total cost of \$2,053,196 for 1949–1954. Since it is assumed that the quadratic cost structure is the "true" cost structure, the total cost of \$2,053,196 is the base against which our approximate reasoning approach is compared.

Parametrization

In order to implement the approximate reasoning approach, it is necessary that the model variables first be parametrized. In actuality, this step describes the universes over which the values (fuzzy subsets) of the linguistic variables are defined. Two parameters are required for each variable in the model—the lower bound and the upper bound that the variable is expected to take on. In effect, the lower bound corresponds to the scaled value of -1, and the upper bound corresponds to the scaled value of $+1$ on the scale $[-1, +1]$ used in the definition of the basic fuzzy subsets in Table 3.

Table 5 lists the values used to parametrize the approximate reasoning model for the paint factory data. it should be noted that the information required for determination of the parameter values is usually available to managers making aggregate planning decisions. In the absence of historical data, a manager would use his judgment to make these determinations.

COMPARISON OF APPROACHES

In Table 6 for the years 1949–1954, we show the comparison of the HMMS LDR model of aggregate production planning against the Rinks "point-valued fuzzy set" (PVFS) model and my "interval-valued fuzzy set" (IVFS) model. This comparison is made on the basis of the 40 rules provided by Rinks [22]. It should be observed that the IVFS approach gives a total cost of $\$2109 \times 10^3$ compared to the PVFS approach of $\$2156 \times 10^3$. Thus the IVFS approach is closer to the LDR result of $\$2053 \times 10^3$, and in fact it is within 2.7% of the LDR result. Table 7, for the years 1949–1950, shows the comparison of the

Table 6. Total Operating Costs 1949–1954, Paint Factory Case (40 Rules, Max-Min Operators)

Costs ($000)	LDR		Rink's Point-Valued Fuzzy Set Model		Turksen's Interval-Valued Fuzzy Set Model	
Regular payroll	1879		1857		1812	
Hiring and layoff	20		42		42	
Overtime	129		198		201	
Personnel		2028		2097		2055
Inventory		25		59		54
Total cost		2053		2156		2109

HMMS LDR model with my IVFS approach. This comparison is made on the basis of the 27 anchor rules that were hypothesized in the previous section. It should be observed that IVFS produced a cost value of $\$721 \times 10^3$, which is less than the cost value produced by the LDR model ($\$734 \times 10^3$), i.e., the IVFS result is 1.36% less than that of the LDR result. In this case, it was not possible to compare it against the Rinks [22] PVFS case since there were no results available from Rinks's PVFS model for the case of 27 rules and for the years 1949–1950.

In these comparisons, total costs achieved by the approximate reasoning models were based on (1) 12-period weighted forecasts for S_t^P and S_t^W, and (2) the value of the base variable corresponding to the membership value of 1 for each of the decision variables and then using the quadratic cost function of the paint factory. (An average was taken over the base variable values whenever there were more than one value of the base variable corresponding to the membership value of 1.)

Based on the comparative results shown in Tables 6 and 7, it is reasonable to say that approximate reasoning based on the interval-valued fuzzy set approach

Table 7. Total Operating Costs 1949–1950, Paint Factory Case (27 Rules, Max-Min Operators)

Costs ($000)	HMMS-LDR Model	Turksen's IVFS Model
Total costs	734	721

provides reasonably "good" and "robust" results in addition to providing a communication link between managerial experts and expert systems based on linguistic expressions of the natural languages. Furthermore, it does so in a much more "user friendly" manner.

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