Differential Geometry and its Applications 29 (2011) 685–698

Contents lists available at ScienceDirect

Differential Geometry and its Applications

www.elsevier.com/locate/difgeo

Improved Chen–Ricci inequality for curvature-like tensors and its applications

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ARTICLE INFO

Article history:
Received 20 April 2011
Available online 29 July 2011
Communicated by O. Kowalski

MSC:
53C40
53C42
53B25
53C15
53C25

Keywords:
Curvature like tensor
Riemannian vector bundle
Chen–Ricci inequality
Improved Chen–Ricci inequality
Lagrangian submanifold
Kaehlerian slant submanifold
C-totally real submanifold
Complex space form
Sasakian space form

ABSTRACT

We present Chen–Ricci inequality and improved Chen–Ricci inequality for curvature like tensors. Applying our improved Chen–Ricci inequality we study Lagrangian and Kaehlerian slant submanifolds of complex space forms, and C-totally real submanifolds of Sasakian space forms.

1. Introduction

Since the celebrated theory of J.F. Nash of isometric immersion of a Riemannian manifold into a suitable Euclidean space gives very important and effective motivation to view each Riemannian manifold as a submanifold in a Euclidean space, the problem of discovering simple sharp relationships between intrinsic and extrinsic invariants of a Riemannian submanifold becomes one of the most fundamental problems in submanifold theory. The main extrinsic invariant is the squared mean curvature and the main intrinsic invariants include the classical curvature invariants namely the Ricci curvature and the scalar curvature. There are also many other important modern intrinsic invariants of (sub)manifolds introduced by B.-Y. Chen (cf. [9,11,16]).

In 1999, B.-Y. Chen [13] proved a basic inequality involving the Ricci curvature Ric and the squared mean curvature $\|H\|^2$ of submanifolds in a real space form as follows.

Theorem 1.1. *(See [13, Theorem 4].) Let M be an n-dimensional submanifold of a real space form $R^m(c)$. Then the following statements are true.*

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0926-2245/$ – see front matter
doi:10.1016/j.difgeo.2011.07.008

(a) *For each unit vector $X \in T_pM$, we have*

$$\|H\|^2 \geqslant \frac{4}{n^2}\{\mathrm{Ric}(X) - (n-1)c\}. \tag{1.1}$$

(b) *If $H(p) = 0$, then a unit vector $X \in T_pM$ satisfies the equality case of (1.1) if and only if X belongs to the relative null space $\mathcal{N}(p)$ given by*

$$\mathcal{N}(p) = \{X \in T_pM\colon \sigma(X,Y) = 0 \text{ for all } Y \in T_pM\}.$$

(c) *The equality case of (1.1) holds for all unit vectors $X \in T_pM$ if and only if either p is a geodesic point or $n = 2$ and p is an umbilical point.*

The inequality (1.1) drew attention of several authors and they established similar inequalities for different kind of submanifolds in ambient manifolds possessing different kind of structures. The submanifolds included mainly invariant, anti-invariant and slant submanifolds, while ambient manifolds included mainly real space forms, complex space forms and Sasakian space forms. Thus, after putting an extra condition on the Riemann curvature tensor of the ambient manifold, like its constancy in the case of real space forms, the constancy of holomorphic sectional curvature in the case of complex space forms and the constancy of φ-holomorphic sectional curvature in the case of Sasakian space forms; one proves the results similar to that of [13, Theorem 4] or [15, Theorem 1].

Motivated by the result of B.-Y. Chen [13, Theorem 4], in [25] and [24], the authors presented a general theory for a submanifold of Riemannian manifolds by proving a basic inequality, now called Chen–Ricci inequality [37], involving the Ricci curvature and the squared mean curvature of the submanifold. The goal was achieved by use of the concept of k-Ricci curvature $(2 \leqslant k \leqslant n)$ in an n-dimensional Riemannian manifold [13]. It can be noted that a k-Ricci curvature is a $(k-1)$-Ricci curvature in the sense of H. Wu [38]. In fact, without assuming any further condition on the Riemann curvature tensor of the ambient manifold $\widetilde{M}$, we established a Chen–Ricci inequality involving Ricci curvature and the squared mean curvature for a submanifold M of $\widetilde{M}$ as follows.

Theorem 1.2. *Let M be an n-dimensional submanifold of a Riemannian manifold. Then, the following statements are true.*

(a) *For $X \in T_p^1M$, it follows that*

$$\mathrm{Ric}(X) \leqslant \frac{1}{4}n^2\|H\|^2 + \widetilde{\mathrm{Ric}}_{(T_pM)}(X), \tag{1.2}$$

where $\widetilde{\mathrm{Ric}}_{(T_pM)}(X)$ is the n-Ricci curvature of T_pM at $X \in T_p^1M$ with respect to the ambient manifold $\widetilde{M}$.

(b) *The equality case of (1.2) is satisfied by $X \in T_p^1M$ if and only if*

$$\begin{cases} \sigma(X,Y) = 0, & \text{for all } Y \in T_pM \text{ orthogonal to } X, \\ 2\sigma(X,X) = nH(p). \end{cases} \tag{1.3}$$

If $H(p) = 0$, then $X \in T_p^1M$ satisfies the equality case of (1.1) if and only if $X \in \mathcal{N}(p)$.

(c) *The equality case of (1.2) holds for all $X \in T_p^1M$ if and only if either p is a geodesic point or $n = 2$ and p is an umbilical point.*

Continuing the study of [25] and [24] we also studied Chen–Ricci inequality for submanifolds in contact metric manifolds and obtained many interesting results (see [26,27,37] and references cited therein).

In 2005, Oprea [33] (see also [34]) proved Chen–Ricci inequality by using optimization techniques applied in the setup of Riemannian geometry. He also improved Chen–Ricci inequality for Lagrangian submanifolds of complex space forms. Later, Deng [23] proved the improved Chen–Ricci inequality for Lagrangian submanifolds of complex space forms just by using some crucial algebraic inequalities and also discussed the equality case.

However, improved Chen–Ricci inequalities for Kaehlerian slant submanifolds of complex space forms and C-totally real submanifolds of Sasakian space forms are not known so far. Even improved Chen–Ricci inequalities in these two cases cannot be obtained directly from the results of Oprea [33] and Deng [23].

Under these circumstances it becomes necessary to give a general theory, which could be applied to obtain Chen–Ricci inequality and improved Chen–Ricci (in)equality in different situations. Motivated by [3], we present Chen–Ricci inequality and improved Chen–Ricci inequality for curvature like tensors (see Theorems 2.1 and 3.3). Then we apply our improved Chen–Ricci inequality for curvature like tensors in study of Kaehlerian slant submanifolds of complex space forms and C-totally real submanifolds of Sasakian space forms. In the process of this study, we come across several natural problems, which can be studied in future.

The paper is organized as follows. In Section 2, first we give concepts related with curvature like tensors. Next, given an n-dimensional Riemannian manifold (M,g), a Riemannian vector bundle (B,g_B) over M, a B-valued symmetric $(1,2)$-tensor field ζ and a (curvature-like) tensor field T satisfying the algebraic Gauss equation (2.1), we establish Chen–Ricci inequality (2.2) involving T-Ricci curvature Ric_T and $\|\operatorname{trace}\zeta\|$. In Section 3, we improve Chen–Ricci inequality (2.2) under certain restrictions on ζ and obtain improved Chen–Ricci inequality (3.2) (cf. Theorem 3.3). In Section 4, applying our main Theorem 3.3 we obtain improved Chen–Ricci inequality for Lagrangian submanifolds of a complex space form in Theorem 4.1 [23, Theorem 3.1], which is an improvement of [13, Theorem 4]. It is known that [23, Example 3.1] the Whitney 2-sphere in $\mathbb{C}^2$ satisfies the equality case of the improved Chen–Ricci inequality (4.4). In Section 5, we next apply our Theorem 3.3 and obtain improved Chen–Ricci inequality (5.1) for Kaehlerian slant submanifolds in complex space forms. This inequality is an improvement of Chen–Ricci inequality for Kaehlerian slant submanifolds in complex space forms given in [30, inequality (2.1) of Theorem 2.1] and [32, inequality (2.1) of Theorem 2.1]. We also note that totally umbilical Lagrangian submanifolds, of dimension $n \geqslant 2$, in a complex space form must be totally geodesic [21, Theorem 1]. As an improvement of this result, in Theorem 5.3, we prove that if M is a totally umbilical Lagrangian submanifold of a Kaehler manifold then either $\dim(M) = 1$ or M is totally geodesic. Next, combining Theorem 5.3 with the result that every totally umbilical proper slant submanifold of a Kaehler manifold is totally geodesic [35, Theorem 3.1], we also conclude that each n-dimensional ($n \geqslant 2$) totally umbilical non-invariant slant submanifold of a $2n$-dimensional Kaehler manifold is always totally geodesic (cf. Theorem 5.4). We also discover that proper slumbilical surfaces [18] cannot satisfy the equality case of the improved Chen–Ricci inequality (5.1); thus we propose the definition and classification of H-slumbilical submanifolds (in particular, H-slumbilical surfaces) in complex space forms (cf. Problem 5.5). Finally, in Section 6, applying Theorem 3.3, we obtain improved Chen–Ricci inequality (6.8) for C-totally real submanifolds of a Sasakian space form (cf. Theorem 6.1), which is an improvement of Chen–Ricci inequality [31, inequality (2.1) of Theorem 2.1]. Like the concept of H-umbilical Lagrangian submanifolds [12], we propose the definition and classification of H-umbilical C-totally real submanifolds in Sasakian space forms (cf. Problem 6.3).

2. Chen–Ricci inequality

Let (M, g) be an n-dimensional Riemannian manifold. Let T be a curvature-like tensor field so that it satisfies the following symmetry properties

$$T(X,Y,Z,W) = -T(Y,X,Z,W),$$
$$T(X,Y,Z,W) = -T(X,Y,W,Z),$$
$$T(X,Y,Z,W) + T(X,Z,W,Y) + T(X,W,Y,Z) = 0$$

for all vector fields X, Y, Z and W on M. For a curvature-like tensor field T, the *T-sectional curvature* associated with a 2-plane section Π_2 spanned by orthonormal vectors X and Y at $p \in M$, is given by [3]

$$K_T(\Pi_2) = K_T(X \wedge Y) = T(X,Y,Y,X).$$

Let $\{e_1, e_2, \ldots, e_n\}$ be any orthonormal basis of T_pM. The *T-Ricci tensor* S_T is defined by

$$S_T(X,Y) = \sum_{j=1}^{n} T(e_j, X, Y, e_j), \quad X, Y \in T_pM.$$

The *T-Ricci curvature* is given by

$$\mathrm{Ric}_T(X) = S_T(X,X), \quad X \in T_pM.$$

We denote the set of unit vectors in T_pM by T_p^1M; thus

$$T_p^1M = \{X \in T_pM \mid g(X,X) = 1\}.$$

If T is replaced by the Riemann curvature tensor R, then T-sectional curvature K_T, T-Ricci tensor S_T and T-Ricci curvature Ric_T, become sectional curvature K, Ricci tensor S and Ricci curvature Ric, respectively.

Let (M, g) be an n-dimensional Riemannian manifold and (B, g_B) a Riemannian vector bundle over M. If ζ is a B-valued symmetric $(1,2)$-tensor field and T a $(0,4)$-tensor field on M such that

$$T(X,Y,Z,W) = g_B\big(\zeta(X,W), \zeta(Y,Z)\big) - g_B\big(\zeta(X,Z), \zeta(Y,W)\big) \tag{2.1}$$

for all vector fields X, Y, Z, W on M, then Eq. (2.1) is said to be an algebraic Gauss equation [20]. If T is a $(0,4)$-tensor field on M which satisfies (2.1) then T becomes a curvature-like tensor. A typical example of an algebraic Gauss equation is given for a submanifold M of a Euclidean space, if B is the normal bundle, ζ the second fundamental form and T the curvature tensor. Some nice situations, in which such T and ζ satisfying an algebraic Gauss equation exist, are Lagrangian and Kaehlerian slant submanifolds of complex space forms and C-totally real submanifolds of Sasakian space forms.

Now, let $\{e_1, \ldots, e_n\}$ be an orthonormal basis of the tangent space T_pM and e_r belong to an orthonormal basis $\{e_{n+1}, \ldots, e_m\}$ of the Riemannian vector bundle (B, g_B) over M at p. We put

$$\zeta_{ij}^r = g_B\big(\zeta(e_i, e_j), e_r\big), \qquad \|\zeta\|^2 = \sum_{i,j=1}^n g_B\big(\zeta(e_i, e_j), \zeta(e_i, e_j)\big),$$

$$\operatorname{trace}\zeta = \sum_{i=1}^n \zeta(e_i, e_i), \qquad \|\operatorname{trace}\zeta\|^2 = g_B(\operatorname{trace}\zeta, \operatorname{trace}\zeta),$$

$$\mathcal{N}_\zeta(p) = \big\{X \in T_pM\colon\ \zeta(X, Y) = 0 \text{ for all } Y \in T_pM\big\}.$$

Theorem 2.1. *Let (M, g) be an n-dimensional Riemannian manifold, (B, g_B) a Riemannian vector bundle over M and ζ a B-valued symmetric $(1, 2)$-tensor field. Let T be a curvature-like tensor field satisfying the algebraic Gauss equation (2.1). Then, the following statements are true*:

(a) *For $X \in T_p^1M$, it follows that*

$$\operatorname{Ric}_T(X) \leqslant \frac{1}{4}\|\operatorname{trace}\zeta\|^2. \tag{2.2}$$

(b) *The equality case of (2.2) is satisfied by $X \in T_p^1M$ if and only if*

$$\begin{cases} \zeta(X, Y) = 0, & \text{for all } Y \in T_pM \text{ such that } g(X, Y) = 0, \\ \zeta(X, X) = \frac{1}{2}\operatorname{trace}\zeta. \end{cases} \tag{2.3}$$

(c) *The equality case of the inequality (2.2) is true for all $X \in T_p^1M$ if and only if either $\zeta = 0$ or $n = 2$ and*

$$\zeta_{11}^r = \zeta_{22}^r = \frac{1}{2}\big(\zeta_{11}^r + \zeta_{22}^r\big). \tag{2.4}$$

Proof. First, we note that

$$\begin{aligned} \|\zeta\|^2 = {} & \frac{1}{2}\|\operatorname{trace}\zeta\|^2 + \frac{1}{2}\sum_{r=n+1}^m \big(\zeta_{11}^r - \zeta_{22}^r - \cdots - \zeta_{nn}^r\big)^2 \\ & + 2\sum_{r=n+1}^m \sum_{j=2}^n \big(\zeta_{1j}^r\big)^2 - 2\sum_{r=n+1}^m \sum_{2\leqslant i<j\leqslant n} \big(\zeta_{ii}^r\zeta_{jj}^r - \big(\zeta_{ij}^r\big)^2\big). \end{aligned} \tag{2.5}$$

From (2.1), we get

$$(K_T)_{ij} = \sum_{r=n+1}^m \big(\zeta_{ii}^r\zeta_{jj}^r - \big(\zeta_{ij}^r\big)^2\big), \tag{2.6}$$

which implies that

$$\tau_T(p) = \frac{1}{2}\|\operatorname{trace}\zeta\|^2 - \frac{1}{2}\|\zeta\|^2. \tag{2.7}$$

From (2.7) and (2.5) we get

$$\begin{aligned} \tau_T(p) = {} & \frac{1}{4}\|\operatorname{trace}\zeta\|^2 - \frac{1}{4}\sum_{r=n+1}^m \big(\zeta_{11}^r - \zeta_{22}^r - \cdots - \zeta_{nn}^r\big)^2 \\ & - \sum_{r=n+1}^m \sum_{j=2}^n \big(\zeta_{1j}^r\big)^2 + \sum_{r=n+1}^m \sum_{2\leqslant i<j\leqslant n} \big(\zeta_{ii}^r\zeta_{jj}^r - \big(\zeta_{ij}^r\big)^2\big). \end{aligned} \tag{2.8}$$

From (2.6) we also have

$$\sum_{2\leqslant i<j\leqslant n} (K_T)_{ij} = \sum_{r=n+1}^m \sum_{2\leqslant i<j\leqslant n} \big(\zeta_{ii}^r\zeta_{jj}^r - \big(\zeta_{ij}^r\big)^2\big). \tag{2.9}$$

From (2.8) and (2.9), we obtain

$$\mathrm{Ric}_T(e_1) = \frac{1}{4}\|\mathrm{trace}\,\zeta\|^2 - \sum_{r=n+1}^{m}\sum_{j=2}^{n}\left(\zeta_{1j}^r\right)^2 - \frac{1}{4}\sum_{r=n+1}^{m}\left(\zeta_{11}^r - \zeta_{22}^r - \cdots - \zeta_{nn}^r\right)^2. \tag{2.10}$$

Since, we can choose $e_1 = X$ as any unit vector in $T_p^1 M$, therefore (2.10) implies (2.2).

To prove the statement (b), assuming $X = e_1$, from (2.10), the equality in (2.2) is valid if and only if

$$\zeta_{12}^r = \cdots = \zeta_{1n}^r = 0 \quad \text{and} \quad \zeta_{11}^r = \zeta_{22}^r + \cdots + \zeta_{nn}^r, \quad r \in \{n+1, \ldots, m\}, \tag{2.11}$$

which is equivalent to (2.3).

Now we prove the statement (c). Assuming the equality case of (2.2) for all unit vectors $X \in T_p^1 M$, in view of (2.11), for each $r \in \{n+1, \ldots, m\}$ it follows that

$$\zeta_{ij}^r = 0, \quad i \neq j, \tag{2.12}$$

$$2\zeta_{ii}^r = \zeta_{11}^r + \zeta_{22}^r + \cdots + \zeta_{nn}^r, \quad i \in \{1, \ldots, n\}. \tag{2.13}$$

From (2.13), we have

$$2\zeta_{11}^r = 2\zeta_{22}^r = \cdots = 2\zeta_{nn}^r = \zeta_{11}^r + \zeta_{22}^r + \cdots + \zeta_{nn}^r,$$

which implies that

$$(n-2)\left(\zeta_{11}^r + \zeta_{22}^r + \cdots + \zeta_{nn}^r\right) = 0.$$

Thus, either $\zeta_{11}^r + \zeta_{22}^r + \cdots + \zeta_{nn}^r = 0$ or $n = 2$. If $\zeta_{11}^r + \zeta_{22}^r + \cdots + \zeta_{nn}^r = 0$, then in view of (2.13), we get

$$\zeta_{ii}^r = 0, \quad i \in \{1, \ldots, n\}.$$

This together with (2.12) gives $\zeta_{ij}^r = 0$ for all $i, j \in \{1, \ldots, n\}$ and $r \in \{n+1, \ldots, m\}$, that is, $\zeta = 0$. If $n = 2$, then from (2.13) we get (2.4). The proof of the converse part is straightforward. □

We immediately have the following

Corollary 2.2. *Let (M, g) be an n-dimensional Riemannian manifold, (B, g_B) a Riemannian vector bundle over M and ζ a B-valued symmetric $(1, 2)$-tensor field. Let T be a curvature-like tensor field defined by (2.1). Then for $X \in T_p^1 M$ any two of the following three statements imply the remaining one.*

(a) *X satisfies the equality case of (2.2),*
(b) $\mathrm{trace}\,\zeta(p) = 0$,
(c) $X \in \mathcal{N}_\zeta(p)$.

Several results in form of Chen–Ricci inequalities in [24–27] and papers cited therein are among others, which can be proved by suitable applications of Theorem 2.1. However, in this paper we are concerned with improved Chen–Ricci inequalities in different situations. Now, in the following section, we improve the Chen–Ricci inequality for curvature like tensors satisfying an algebraic Gauss equation (2.1) under certain restrictions on the tensor field ζ.

3. Improved Chen–Ricci inequality

First, we state following two lemmas for later use:

Lemma 3.1. *(See [23, Lemma 2.2].) Let $f_1 : \mathbb{R}^n \to \mathbb{R}$ be a function defined by*

$$f_1\left(a^1, \ldots, a^n\right) = a^1 \sum_{j=2}^{n} a^j - \sum_{j=2}^{n}\left(a^j\right)^2.$$

If $a^1 + \cdots + a^n = 2na$, we have

$$f_1\left(a^1, \ldots, a^n\right) \leqslant \frac{n-1}{4n}\left(a^1 + \cdots + a^n\right)^2.$$

The equality sign holds if and only if

$$\frac{1}{n+1}a^1 = a^2 = \cdots = a^n = a.$$

Lemma 3.2. *(See [23, Lemma 2.3].) Let $f_2 : \mathbb{R}^n \to \mathbb{R}$ be a function defined by*

$$f_2(a^1, \dots, a^n) = a^1 \sum_{j=2}^{n} a^j - (a^1)^2.$$

If $a^1 + \dots + a^n = 4a$, we have

$$f_2(a^1, \dots, a^n) \leqslant \frac{1}{8}(a^1 + \dots + a^n)^2.$$

The equality sign holds if and only if

$$a^1 = a, \quad a^2 + \dots + a^n = 3a.$$

Now, we obtain an improved Chen–Ricci inequality in the following

Theorem 3.3. *Let (M, g) be a Riemannian manifold of dimension n $(n \geqslant 2)$, (B, g_B) a Riemannian vector bundle of dimension $(n+s)$ over M and ζ a B-valued symmetric $(1,2)$-tensor field. Suppose that for any orthonormal basis $\{e_1, \dots, e_n\}$ of the tangent space T_pM there is an orthonormal basis $\{e_{n+1}, \dots, e_{2n+s}\}$ of the Riemannian vector bundle (B, g_B) over M at p, such that*

$$\begin{aligned} &\zeta_{jk}^{n+i} = \zeta_{ki}^{n+j} = \zeta_{ij}^{n+k}, \quad && i, j, k \in \{1, \dots, n\}, \\ &\zeta_{ij}^{r} = 0, && r \in \{2n+1, \dots, 2n+s\}. \end{aligned} \tag{3.1}$$

Let T be a curvature-like tensor satisfying the algebraic Gauss equation (2.1). Then for any unit vector $X \in T_p^1M$, we have

$$\mathrm{Ric}_T(X) \leqslant \frac{n-1}{4n} \|\mathrm{trace}\,\zeta\|^2. \tag{3.2}$$

The equality sign holds for any unit tangent vector at p if and only if either $\zeta = 0$ at p or $n = 2$ and

$$\zeta(e_1, e_1) = 3\mu e_{n+1}, \qquad \zeta(e_2, e_2) = \mu e_{n+1}, \qquad \zeta(e_1, e_2) = \mu e_{n+2}$$

for some suitable function μ with respect to some suitable orthonormal local frame field.

Proof. Take a point $p \in M$ and an orthonormal frame $\{e_1, \dots, e_n\}$ in T_pM such that $X = e_1$. Then from (2.1) we have

$$\mathrm{Ric}_T(X) = \sum_{\ell=1}^{n} \sum_{j=2}^{n} \left(\zeta_{11}^{n+\ell} \zeta_{jj}^{n+\ell} - (\zeta_{1j}^{n+\ell})^2\right). \tag{3.3}$$

Since

$$\sum_{\ell=1}^{n} \sum_{j=2}^{n} (\zeta_{1j}^{n+\ell})^2 \geqslant \sum_{j=2}^{n} (\zeta_{1j}^{n+1})^2 + \sum_{j=2}^{n} (\zeta_{1j}^{n+j})^2,$$

therefore (3.3) gives

$$\mathrm{Ric}_T(X) \leqslant \sum_{\ell=1}^{n} \sum_{j=2}^{n} \zeta_{11}^{n+\ell} \zeta_{jj}^{n+\ell} - \sum_{j=2}^{n} (\zeta_{1j}^{n+1})^2 - \sum_{j=2}^{n} (\zeta_{1j}^{n+j})^2. \tag{3.4}$$

Using (3.1) in (3.4) we get

$$\mathrm{Ric}_T(X) \leqslant \sum_{\ell=1}^{n} \sum_{j=2}^{n} \zeta_{11}^{n+\ell} \zeta_{jj}^{n+\ell} - \sum_{j=2}^{n} (\zeta_{11}^{n+j})^2 - \sum_{j=2}^{n} (\zeta_{jj}^{n+1})^2. \tag{3.5}$$

Now, suppose that

$$\begin{aligned} f_1(\zeta_{11}^{n+1}, \dots, \zeta_{nn}^{n+1}) &= \zeta_{11}^{n+1} \sum_{j=2}^{n} \zeta_{jj}^{n+1} - \sum_{j=2}^{n} (\zeta_{jj}^{n+1})^2, \\ f_\ell(\zeta_{11}^{n+\ell}, \dots, \zeta_{nn}^{n+\ell}) &= \zeta_{11}^{n+\ell} \sum_{j=2}^{n} \zeta_{jj}^{n+\ell} - (\zeta_{11}^{n+\ell})^2, \quad \ell \in \{2, \dots, n\}. \end{aligned}$$

Since the first component of trace ζ is

$$(\operatorname{trace}\zeta)^1 = \zeta_{11}^{n+1} + \cdots + \zeta_{nn}^{n+1},$$

by using Lemma 3.1, we have

$$f_1\big(\zeta_{11}^{n+1}, \ldots, \zeta_{nn}^{n+1}\big) \leqslant \frac{n-1}{4n}\big((\operatorname{trace}\zeta)^1\big)^2. \tag{3.6}$$

Similarly, by Lemma 3.2, for $2 \leqslant \ell \leqslant n$, in view of $n \geqslant 2$, we have

$$f_\ell\big(\zeta_{11}^{n+\ell}, \ldots, \zeta_{nn}^{n+\ell}\big) \leqslant \frac{1}{8}\big((\operatorname{trace}\zeta)^\ell\big)^2 \leqslant \frac{n-1}{4n}\big((\operatorname{trace}\zeta)^\ell\big)^2. \tag{3.7}$$

Now, in view of (3.4), (3.6) and (3.7) we get

$$\operatorname{Ric}_T(X) \leqslant \frac{n-1}{4n}\sum_{\ell=1}^{n}\big((\operatorname{trace}\zeta)^\ell\big)^2 = \frac{n-1}{4n}\|\operatorname{trace}\zeta\|^2,$$

which gives (3.2).

Now we assume that $n \geqslant 2$ and the equality sign of (3.2) is true for all unit vectors $X \in T_p^1 M$. From (3.7), it follows that $(\operatorname{trace}\zeta)^\ell = 0$ for $\ell \geqslant 2$ (or simply choose e_{n+1} parallel to trace ζ). Combining this and Lemma 3.2 we have

$$\zeta_{1j}^{n+1} = \zeta_{11}^{n+j} = \frac{(\operatorname{trace}\zeta)^j}{4} = 0, \quad j \geqslant 2.$$

From (3.4), we get

$$\zeta_{jk}^{n+1} = 0, \quad j, k \geqslant 2,\ j \neq k.$$

From Lemma 3.1, the matrix (ζ_{jk}^{n+1}) must be diagonal with

$$\zeta_{11}^{n+1} = (n+1)\frac{(\operatorname{trace}\zeta)^1}{2n}, \qquad \zeta_{jj}^{n+1} = \frac{(\operatorname{trace}\zeta)^1}{2n}, \quad j \geqslant 2.$$

Now if we compute $\operatorname{Ric}_T(e_2)$ as we do for $\operatorname{Ric}_T(X) = \operatorname{Ric}_T(e_1)$ in (3.4), from the equality we get

$$\zeta_{2j}^{n+\ell} = \zeta_{j\ell}^{n+2} = 0, \quad \ell \neq 2,\ j \neq 2,\ \ell \neq j.$$

From the equality and Lemma 3.1, we obtain

$$\frac{1}{n+1}\zeta_{11}^{n+2} = \zeta_{22}^{n+2} = \cdots = \zeta_{nn}^{n+2} = \frac{(\operatorname{trace}\zeta)^2}{2n} = 0.$$

Since the equality holds for all unit tangent vectors, the argument is also true for matrices $(\zeta_{jk}^{n+\ell})$. Thus, finally we have

$$\zeta_{2\ell}^{n+2} = \zeta_{22}^{n+\ell} = \frac{(\operatorname{trace}\zeta)^\ell}{2n} = 0, \quad \ell \geqslant 3.$$

Therefore the matrix (ζ_{jk}^{n+2}) has only two possible nonzero entries, that is,

$$\zeta_{12}^{n+2} = \zeta_{21}^{n+2} = \zeta_{22}^{n+1} = \frac{(\operatorname{trace}\zeta)^1}{2n}.$$

Similarly the matrix $(\zeta_{jk}^{n+\ell})$ has only two possible nonzero entries

$$\zeta_{1\ell}^{n+\ell} = \zeta_{\ell 1}^{n+\ell} = \zeta_{\ell\ell}^{n+1} = \frac{(\operatorname{trace}\zeta)^1}{2n}, \quad \ell \geqslant 3.$$

Now, we compute $\operatorname{Ric}_T(e_2)$ as follows. From (2.1), we get

$$T(e_j, e_2, e_2, e_j) = g_B\big(\zeta(e_j, e_j), \zeta(e_2, e_2)\big) - g_B\big(\zeta(e_2, e_j), \zeta(e_j, e_2)\big)$$

so we have

$$T(e_j, e_2, e_2, e_j) = \left(\frac{(\operatorname{trace}\zeta)^1}{2n}\right)^2, \quad j \geqslant 3. \tag{3.8}$$

From

$$T(e_1, e_2, e_2, e_1) = g_B\big(\zeta(e_1, e_1), \zeta(e_2, e_2)\big) - g_B\big(\zeta(e_2, e_1), \zeta(e_1, e_2)\big)$$

we get

$$T(e_1, e_2, e_2, e_1) = (n+1)\left(\frac{(\operatorname{trace}\zeta)^1}{2n}\right)^2 - \left(\frac{(\operatorname{trace}\zeta)^1}{2n}\right)^2. \tag{3.9}$$

By combining (3.8) and (3.9), we get

$$\operatorname{Ric}_T(e_2) = (n+1)\left(\frac{(\operatorname{trace}\zeta)^1}{2n}\right)^2 - \left(\frac{(\operatorname{trace}\zeta)^1}{2n}\right)^2 + (n-2)\left(\frac{(\operatorname{trace}\zeta)^1}{2n}\right)^2,$$

which gives

$$\operatorname{Ric}_T(e_2) = \frac{n-1}{2n^2}\big((\operatorname{trace}\zeta)^1\big)^2. \tag{3.10}$$

On the other hand from the equality assumption, we have

$$\operatorname{Ric}_T(e_2) = \frac{n-1}{4n}\|\operatorname{trace}\zeta\|^2 = \frac{n-1}{4n}\big((\operatorname{trace}\zeta)^1\big)^2. \tag{3.11}$$

From (3.10) and (3.11), it is clear that

$$\frac{1}{4n^2}(n-1)(n-2)\big((\operatorname{trace}\zeta)^1\big)^2 = 0.$$

Since $n \neq 1$, we have either $(\operatorname{trace}\zeta)^1 = 0$ or $n = 2$. If $(\operatorname{trace}\zeta)^1 = 0$, then all $\zeta_{jk}^{n+\ell}$ are zero and hence $\zeta = 0$. If $n = 2$, then we have

$$\zeta(e_1, e_1) = \lambda e_{n+1}, \qquad \zeta(e_2, e_2) = \mu e_{n+1}, \qquad \zeta(e_1, e_2) = \mu e_{n+2},$$

with

$$\lambda = 3\mu = \frac{3(\operatorname{trace}\zeta)^1}{2n}.$$

The converse is easy to prove by simple computation. □

4. Lagrangian submanifolds

Let M be an n-dimensional submanifold of a Riemannian manifold $(\widetilde{M}, g)$. Then the second fundamental form σ of the immersion is related to the shape operator A by

$$g\big(\sigma(X, Y), N\big) = g(A_N X, Y),$$

and the equation of Gauss is given by

$$R(X, Y, Z, W) = \widetilde{R}(X, Y, Z, W) + g\big(\sigma(X, W), \sigma(Y, Z)\big) - g\big(\sigma(X, Z), \sigma(Y, W)\big) \tag{4.1}$$

for all $X, Y, Z, W \in TM$, where $\widetilde{R}$ and R are the curvature tensors of $\widetilde{M}$ and M respectively.

A point $p \in M$ is called a geodesic point if the second fundamental form σ vanishes at p. The submanifold is said to be totally geodesic if every point of M is a geodesic point. A Riemannian submanifold M is a totally geodesic submanifold of $\widetilde{M}$ if and only if every geodesic of M is a geodesic of $\widetilde{M}$. The submanifold M is minimal if the mean curvature vector $H = \frac{1}{n}\operatorname{trace}(\sigma)$ vanishes identically. A point $p \in M$ is called an umbilical point if $\sigma = g \otimes H$ at p, that is, the shape operator A_N is proportional to the identity transformation for all $N \in T_p^{\perp}M$. The submanifold is said to be totally umbilical if every point of the submanifold is an umbilical point. If the shape operator A_H at the mean curvature vector H satisfies $A_H X = g(H, H)X$ for every $X \in TM$, then M is said to be pseudo-umbilical. Totally umbilical submanifolds are the simplest submanifolds, which are pseudo-umbilical. Thus for a totally umbilical submanifold the shape operator A_H at H has exactly one eigenvalue $g(H, H)$; moreover, $A_N = 0$ for each normal vector N orthogonal to H.

Let $(\widetilde{M}, J, g)$ be a $2m$-dimensional almost Hermitian manifold. If $\widetilde{M}$ is a Kaehler manifold with constant holomorphic sectional curvature c, then it is called a complex space form, denoted by $\widetilde{M}(c)$. In this case, the almost complex structure J is parallel, and the Riemann curvature tensor $\widetilde{R}$ is given by

$$\widetilde{R}(X, Y)Z = \frac{c}{4}\big(g(X, Z)Y - g(Y, Z)X + g(JX, Z)JY - g(JY, Z)JX + 2g(JX, Y)JZ\big)$$

for all vector fields X, Y, Z on $\widetilde{M}$. The model spaces for complex space forms are the complex Euclidean spaces $\mathbb{C}^n$ $(c = 0)$, the complex projective spaces $\mathbb{C}P^n$ $(c > 0)$ and the complex hyperbolic spaces $\mathbb{C}H^n$ $(c < 0)$.

An n-dimensional submanifold M of $(\widetilde{M}, J, g)$ is called a Lagrangian submanifold (see for more details [17]) if the almost complex structure J of $\widetilde{M}$ carries each tangent space of M onto its corresponding normal space, that is, $J(T_pM) = T_p^{\perp}M$ for every $p \in M$.

It is well known from the work of Cartan [5, p. 231] that an n-dimensional totally umbilical submanifold of a Euclidean m-space is always an open portion of either an n-plane or an n-sphere. Totally umbilical submanifolds, if they exist, are the simplest submanifolds next to totally geodesic submanifolds in a Riemannian manifold from extrinsic point of views. However, from Theorem 1 of [21], it follows that there exist no totally umbilical Lagrangian submanifolds, of dimension $n \geqslant 2$, in a complex space form except the totally geodesic ones.

Because of nonexistence of totally umbilical Lagrangian submanifolds, B.-Y. Chen [12] introduced the concept of H-umbilical Lagrangian submanifolds, which are the simplest Lagrangian submanifolds next to the totally geodesic ones in complex space forms $\widetilde{M}(c)$. By an H-umbilical Lagrangian submanifold of a Kaehler manifold $\widetilde{M}$ we mean a Lagrangian submanifold whose second fundamental form σ assumes the following simple form:

$$\sigma(e_1, e_1) = \lambda J e_1, \qquad \sigma(e_2, e_2) = \cdots = \sigma(e_n, e_n) = \mu J e_1, \tag{4.2}$$

$$\sigma(e_1, e_j) = \mu J e_j, \qquad \sigma(e_j, e_k) = 0, \quad j \neq k, \; j, k = 2, \ldots, n \tag{4.3}$$

for some suitable functions λ and μ with respect to some suitable orthonormal local frame field.

Now, we apply Theorem 3.3 and obtain an improved Chen–Ricci inequality for Lagrangian submanifolds of a complex space form in the following

Theorem 4.1. *(See [23, Theorem 3.1].) Let M be a Lagrangian submanifold of real dimension n $(n \geqslant 2)$ in a complex space form $\widetilde{M}(c)$ and X be a unit tangent vector in T_p^1M. Then we have*

$$\operatorname{Ric}(X) \leqslant \frac{n-1}{4}\left(c + n\|H(p)\|^2\right), \tag{4.4}$$

where H is the mean curvature vector of M in $\widetilde{M}(c)$ and $\operatorname{Ric}(X)$ is the Ricci curvature of M at X. The equality sign holds for any unit tangent vector at p if and only if either p is a geodesic point or $n = 2$ and p is an H-umbilical point with $\lambda = 3\mu$, that is

$$\sigma(e_1, e_1) = 3\mu J e_1, \qquad \sigma(e_2, e_2) = \mu J e_1, \qquad \sigma(e_1, e_2) = \mu J e_2$$

for some suitable function μ.

Proof. In (2.1), we set

$$T(X, Y, Z, W) = R(X, Y, Z, W) + \frac{c}{4}\big(g(Y, Z)g(X, W) - g(X, Z)g(Y, W)\big)$$

with R the Riemannian curvature tensor on M, and $\zeta = \sigma$ with σ the second fundamental form of the immersion of M into $\widetilde{M}(c)$. Then we see that

$$\operatorname{Ric}_T(X) = \operatorname{Ric}(X) - \frac{1}{4}(n-1)c$$

for all unit vectors X. Using this in (3.2), we can complete the proof. $\square$

From Theorem 4.1, we have the following

Corollary 4.2. *(See [23, Corollary 3.2].) Let M be an n-dimensional $(n \geqslant 2)$ Lagrangian submanifold of a complex space form $\widetilde{M}(c)$. If*

$$\operatorname{Ric}(X) = \frac{n-1}{4}\left(c + n\|H\|^2\right)$$

for all unit tangent vector X of M, then either M is a totally geodesic submanifold in $\widetilde{M}(c)$ or $n = 2$ and M is a Lagrangian H-umbilical surface of $\widetilde{M}(c)$ with $\lambda = 3\mu$.

Example 4.3. (See [23, Example 3.1].) The Whitney 2-sphere in $\mathbb{C}^2$ satisfies the improved Chen–Ricci equality.

Remark 4.4. In [33, Theorem 3.2], Oprea proved the improved Chen–Ricci inequality (4.4) for Lagrangian submanifolds of complex space forms using optimization techniques on Riemannian submanifolds. Later, Deng [23, Theorem 3.1] proved the improved Chen–Ricci inequality (4.4) by algebraic techniques.

Remark 4.5. H-umbilical Lagrangian submanifolds in complex space forms satisfying the condition $\lambda = 3\mu$ have been classified completely [12]. For more details about Lagrangian submanifolds, we refer to [4], [6], [7], [10], [16, pp. 331–332], [19] and [22].

Problem 4.6. To extend Theorem 4.1 to obtain an improved Chen–Ricci inequality for Lagrangian submanifolds in Quaternion projective spaces, which will improve Theorem 3.1 of [29].

5. Kaehlerian slant submanifolds

Let M be a submanifold of an almost Hermitian manifold $(\widetilde{M}, J, g)$. We write

$$JX = PX + FX, \quad X \in TM,$$

where PX and FX are the tangential and the normal components of JX, respectively. Then, P is an endomorphism of the tangent bundle TM and F is a normal bundle valued 1-form on TM. For any nonzero vector X tangent to M at a point $p \in M$, the Wirtinger angle of X, denoted by $\theta(X)$ is the angle between JX and the tangent space T_pM. The submanifold M is called a slant submanifold if $\theta(X)$ is independent of the choice of $p \in M$ and of $X \in T_pM$. The Wirtinger angle of a slant submanifold is called the slant angle of the slant submanifold. For slant submanifolds, $P^2 = tI$, for some $t \in [-1, 0]$, where I is the identity transformation of TM. Moreover, if M is a slant submanifold and θ is the slant angle of M, then $t = -\cos^2\theta$. Hence, for a slant submanifold, we have

$$g(PX, PY) = \cos^2\theta\, g(X, Y),$$
$$g(FX, FY) = \sin^2\theta\, g(X, Y)$$

for X, Y tangent to M.

We note that a slant submanifold M is J-invariant, anti-J-invariant, non-invariant slant or proper slant according as $\theta = 0$ $(t = -1)$, $\theta = \pi/2$ $(t = 0)$, $\theta \neq 0$ $(t \neq -1)$ or $0 \neq \theta \neq \pi/2$ $(-1 < t = -\cos^2\theta < 0)$, respectively.

A proper slant submanifold is said to be Kaehlerian slant if the endomorphism P is parallel. A Kaehlerian slant submanifold is a Kaehler manifold with respect to the induced metric and the almost complex structure $J' = (\sec\theta)J$, where θ is the slant angle. Examples of proper slant submanifolds and Kaehlerian slant submanifolds are given in [8].

For Kaehlerian slant submanifolds in $2n$-dimensional complex space form $\widetilde{M}(c)$ we prove the following improved Chen–Ricci inequality.

Theorem 5.1. *Let M be an n-dimensional Kaehlerian slant submanifold of a $2n$-dimensional complex space form $\widetilde{M}(c)$, and X a unit tangent vector in T_p^1M, $p \in M$. Then*

$$\mathrm{Ric}(X) \leqslant \frac{1}{4}\big((n-1)n\|H\|^2 + (n-1)c + 3c\cos^2\theta\big), \tag{5.1}$$

where H is the mean curvature vector of M in $\widetilde{M}(c)$ and Ric(X) *is the Ricci curvature of M at X. The equality sign holds for any unit tangent vector at p if and only if either*

(a) *p is a geodesic point or*
(b) *$n = 2$ and*

$$\begin{cases} \sigma(e_1, e_1) = 3\mu \dfrac{Fe_1}{\|Fe_1\|} = 3\mu \csc\theta Fe_1, \\ \sigma(e_2, e_2) = \mu \dfrac{Fe_1}{\|Fe_1\|} = \mu \csc\theta Fe_1, \\ \sigma(e_1, e_2) = \mu \dfrac{Fe_2}{\|Fe_2\|} = \mu \csc\theta Fe_2 \end{cases} \tag{5.2}$$

for some suitable function μ, where σ is the second fundamental form of the immersion of M into $\widetilde{M}(c)$.

Proof. In (2.1), we set

$$T(X, Y, Z, W) = R(X, Y, Z, W) + \frac{c}{4}\big\{g(Y, Z)g(X, W) - g(X, Z)g(Y, W) + g(PX, Z)g(PY, W) \\ - g(PY, Z)g(PX, W) + 2g(PX, Y)g(PZ, W)\big\}$$

with R the Riemannian curvature tensor on M, and $\zeta = \sigma$. Then it can be shown that

$$\mathrm{Ric}_T(X) = \mathrm{Ric}(X) - \frac{1}{4}(n-1)c - \frac{3}{4}c\cos^2\theta \tag{5.3}$$

for all unit vectors X. Since the shape operator A for each Kaehlerian slant submanifold M of a Kaehler manifold satisfies [8]

$$A_{FX}Y = A_{FY}X$$

for any X, Y tangent to M, therefore (3.1) is satisfied for $\zeta = \sigma$. Now, using (5.3) in (3.2), we can complete the proof. □

Remark 5.2. The inequality (5.1) is an improvement of Chen–Ricci inequality [25, inequality **(13)** of Theorem 5.2] or [30, inequality (2.1) of Theorem 2.1]. If M is a slant surface, the inequality (5.1) becomes the inequality (3.1) of [14].

We recall that totally umbilical submanifolds, if they exist, are the simplest submanifolds next to totally geodesic submanifolds in a Riemannian manifold. From Theorem 1 of [21], it follows that there do not exist totally umbilical Lagrangian submanifolds, of dimension $n \geqslant 2$, in a complex space form except the totally geodesic ones. We improve this result in the following

Theorem 5.3. *If M is a totally umbilical Lagrangian submanifold of a Kaehler manifold then either* $\dim(M) = 1$ *or M is totally geodesic.*

Proof. If $\dim(M) > 1$, let $X, Y \in T_pM$ such that $g(X,Y) = 0$ and $g(X,X) = 1$. Then

$$g(H, FY) = g\big(\sigma(X,X), FY\big) = g(A_{FY}X, X) = g(A_{FX}Y, X) = g\big(\sigma(X,Y), FX\big) = 0,$$

which shows that $H = 0$, and consequently M is totally geodesic. □

Recently, Sahin [35, Theorem 3.1] proved that every totally umbilical proper slant submanifold of a Kaehler manifold is totally geodesic. Combining the result of Sahin with Theorem 5.3, we get the following

Theorem 5.4. *Every n-dimensional $(n \geqslant 2)$ totally umbilical non-invariant slant submanifold of a $2n$-dimensional Kaehler manifold is totally geodesic.*

However, since the shape operator of every proper slant surface (which is always Kaehlerian slant) and also every Kaehlerian slant submanifold of a Kaehler manifold must satisfy another condition

$$A_{FX}Y = A_{FY}X$$

for any X, Y tangent to M, there do not exist totally umbilical Kaehlerian slant submanifold in a Kaehlerian manifold. For these reasons, B.-Y. Chen [18] studied the simplest slant submanifolds which satisfy the pseudo-umbilical condition $A_HX = g(H,H)X$ and $A_{FX}Y = A_{FY}X$, and defined such submanifolds to be slant umbilical submanifolds, or simply slumbilical submanifolds (although slant pseudo-umbilical submanifold could be a more correct name). In some sense, slumbilical submanifolds play the role of totally umbilical submanifolds of Euclidean space in the family of slant submanifolds. An n-dimensional slant submanifold in a Kaehlerian manifold is a slumbilical submanifold with slant angle $\theta \in (0, \pi/2)$ if its second fundamental form satisfies [18]

$$\sigma(e_1,e_1) = \cdots = \sigma(e_n,e_n) = \lambda\csc\theta Fe_1, \tag{5.4}$$

$$\sigma(e_1,e_j) = \lambda\csc\theta Fe_j, \qquad \sigma(e_j,e_k) = 0, \quad j \neq k,\ j,k = 2,\dots,n \tag{5.5}$$

for some suitable function λ with respect to some orthonormal frame field $\{e_1,\dots,e_n\}$. In [18], Chen obtained a complete classification of slumbilical submanifolds in complex space forms. In fact, there exist twelve families of slumbilical submanifolds in complex space forms with slant angle $\theta \in (0,\pi/2)$. Conversely, every slumbilical submanifold in a complex space form is given by one of these twelve families.

Now we return to Theorem 5.1, and in view of (5.2) we observe that a Kaehlerian slant surface, which is not totally geodesic, satisfying the improved Chen–Ricci equality (5.1) is different from slumbilical surfaces. A proper slumbilical surface cannot satisfy the improved Chen–Ricci equality (5.1). Thus, we propose the following

Problem 5.5. A Kaehlerian slant submanifold M^n of a complex space form $\widetilde{M}(c)$ will be called an H-slumbilical submanifold if its second fundamental form σ assumes the following simple form:

$$\sigma(e_1,e_1) = \lambda\csc\theta Fe_1, \qquad \sigma(e_2,e_2) = \cdots = \sigma(e_n,e_n) = \mu\csc\theta Fe_1, \tag{5.6}$$

$$\sigma(e_1,e_j) = \mu\csc\theta Fe_j, \qquad \sigma(e_j,e_k) = 0, \quad j \neq k,\ j,k = 2,\dots,n \tag{5.7}$$

for some suitable functions λ and μ with respect to some suitable orthonormal local frame field $\{e_1,\dots,e_n\}$. The problem is to obtain a complete classification of H-slumbilical submanifolds in complex space forms.

6. C-totally real submanifolds

A differentiable 1-form η on a $(2m+1)$-dimensional differentiable manifold $\widetilde{M}$ is called a contact form if $\eta \wedge (d\eta)^m \neq 0$ everywhere on $\widetilde{M}$, and $\widetilde{M}$ equipped with a contact form is a contact manifold. Since rank of $d\eta$ is $2m$, there exists a unique global vector field ξ, called the characteristic vector field, such that

$$\eta(\xi) = 1, \qquad \pounds_\xi \eta = 0, \tag{6.1}$$

where $\pounds_\xi$ denotes the Lie differentiation by ξ. Moreover, it is well known that there exist a Riemannian metric g and a $(1,1)$-tensor field φ such that

$$\varphi\xi = 0, \qquad \eta \circ \varphi = 0, \qquad \eta(X) = g(X, \xi), \tag{6.2}$$

$$\varphi^2 = -I + \eta \otimes \xi, \qquad d\eta(X, Y) = g(X, \varphi Y), \tag{6.3}$$

$$g(X, Y) = g(\varphi X, \varphi Y) + \eta(X)\eta(Y) \tag{6.4}$$

for $X, Y \in T\widetilde{M}$. The structure (η, ξ, φ, g) is called a contact metric structure and the manifold $\widetilde{M}$ endowed with such a structure is said to be a contact metric manifold.

The contact metric structure (η, ξ, φ, g) on $\widetilde{M}$ gives rise to a natural almost Hermitian structure on the product manifold $\widetilde{M} \times \mathbb{R}$. If this structure is integrable, then $\widetilde{M}$ is said to be a Sasakian manifold. A Sasakian manifold is characterized by the condition

$$(\widetilde{\nabla}_X \varphi)Y = g(X, Y)\xi - \eta(Y)X, \quad X, Y \in T\widetilde{M}, \tag{6.5}$$

where $\widetilde{\nabla}$ is Levi-Civita connection. Also, a contact metric manifold $\widetilde{M}$ is Sasakian if and only if the curvature tensor $\widetilde{R}$ satisfies

$$\widetilde{R}(X, Y)\xi = \eta(Y)X - \eta(X)Y, \quad X, Y \in T\widetilde{M}. \tag{6.6}$$

A plane section in $T_p\widetilde{M}$ is called a φ-section if there exists a vector $X \in T_p\widetilde{M}$ orthogonal to ξ such that $\{X, \varphi X\}$ span the section. The sectional curvature is called φ-sectional curvature. Just as the sectional curvatures of a Riemannian manifold determine the curvature completely and the holomorphic sectional curvatures of a Kaehler manifold determine the curvature completely, on a Sasakian manifold the φ-sectional curvatures determine the curvature completely. Moreover on a Sasakian manifold of dimension $\geqslant 5$ if at each point the φ-sectional curvature is independent of the choice of φ-section at the point, it is constant on the manifold and the curvature tensor is given by

$$\begin{aligned}\widetilde{R}(X, Y)Z = {} & \frac{c+3}{4}\big\{g(Y, Z)X - g(X, Z)Y\big\} + \frac{c-1}{4}\big\{g(X, \varphi Z)\varphi Y - g(Y, \varphi Z)\varphi X \\ & + 2g(X, \varphi Y)\varphi Z + \eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X, Z)\eta(Y)\xi - (Y, Z)\eta(X)\xi\big\}\end{aligned} \tag{6.7}$$

for all $X, Y, Z \in T\widetilde{M}$. A Sasakian manifold of constant φ-sectional curvature c is called a Sasakian space form $\widetilde{M}(c)$.

A well-known result of Tanno [36] is that a complete simply connected Sasakian manifold of constant φ-sectional curvature c is isometric to one of certain model spaces depending on whether $c > -3$, $c = -3$ or $c < -3$. The model space for $c > -3$ is a sphere with a D-homothetic deformation of the standard structure. For $c = -3$ the model space is $\mathbb{R}^{2n+1}$ with the contact form $\eta = \frac{1}{2}(dz - \sum_{i=1}^n y^i\, dx^i)$ together with the metric $ds^2 = \eta \otimes \eta + \frac{1}{4}\sum_{i=1}^n((dx^i)^2 + (dy^i)^2)$. For $c < -3$ one has a canonically defined contact metric structure on the product $B^n \times \mathbb{R}$ where B^n is a simply connected bounded domain in C^n with a Kaehler structure of constant negative holomorphic curvature. In particular, Sasakian space forms exist for all values of c. For more details we refer to [2].

A submanifold M in a contact manifold is called a C-totally real submanifold [39] if every tangent vector of M belongs to the contact distribution. Thus, a submanifold M in a contact metric manifold is a C-totally real submanifold if ξ is normal to M. A submanifold M in an almost contact metric manifold is called anti-invariant [40] if $\varphi(TM) \subset T^\perp M$. If a submanifold M in a contact metric manifold is normal to the structure vector field ξ, then it is anti-invariant. Thus C-totally real submanifolds in a contact metric manifold are anti-invariant, as they are normal to ξ.

Now, we apply Theorem 3.3 to get an improved Chen–Ricci inequality for C-totally real submanifolds of a Sasakian space form.

Theorem 6.1. *Let M be a C-totally real submanifold of real dimension n $(n \geqslant 2)$ in a Sasakian space form $\widetilde{M}(c)$ of dimension $2n+1$, and X a unit tangent vector in T_p^1M. Then we have*

$$\mathrm{Ric}(X) \leqslant \frac{n-1}{4}\big(c + 3 + n\|H\|^2\big), \tag{6.8}$$

where H is the mean curvature vector of M in $\widetilde{M}(c)$ and $\mathrm{Ric}(X)$ is the Ricci curvature of M^n at X. The equality sign holds for any unit tangent vector at p if and only if either p is a geodesic point or $n = 2$ and

$$\sigma(e_1, e_1) = 3\mu\varphi e_1, \qquad \sigma(e_2, e_2) = \mu\varphi e_1, \qquad \sigma(e_1, e_2) = \mu\varphi e_2$$

for some suitable function μ, where σ is the second fundamental form of the immersion of M into $\widetilde{M}(c)$.

Proof. In (2.1), we set $\zeta = \sigma$ and

$$T(X, Y, Z, W) = R(X, Y, Z, W) + \frac{c+3}{4}\big(g(Y, Z)g(X, W) - g(X, Z)g(Y, W)\big).$$

Then we see that

$$\mathrm{Ric}_T(X) = \mathrm{Ric}(X) - \frac{1}{4}(n-1)(c+3) \tag{6.9}$$

for all unit vectors X. Also, for C-totally real submanifold of a Sasakian manifold of dimension $2n+1$, Eq. (3.1) is satisfied for $\zeta = \sigma$ with $s = 1$. Now, the proof follows by using (6.9) in (3.2). □

Remark 6.2. The improved Chen–Ricci inequality (6.8) is an improvement of Chen–Ricci inequality [31, Inequality (2.1) of Theorem 2.1].

Problem 6.3. Like the concept of H-umbilical Lagrangian submanifolds [12], we can define an H-umbilical C-totally real submanifold of a Sasakian space form $\widetilde{M}(c)$. By an H-umbilical C-totally real submanifold of a Sasakian manifold $\widetilde{M}$ we mean a C-totally real submanifold whose second fundamental form σ assumes the following simple form:

$$\sigma(e_1, e_1) = \lambda\varphi e_1, \qquad \sigma(e_2, e_2) = \cdots = \sigma(e_n, e_n) = \mu\varphi e_1, \tag{6.10}$$

$$\sigma(e_1, e_j) = \mu\varphi e_j, \qquad \sigma(e_j, e_k) = 0, \quad j \neq k,\ j, k = 2, \ldots, n \tag{6.11}$$

for some suitable functions λ and μ with respect to some suitable orthonormal local frame field $\{e_1, \ldots, e_n\}$. The problem is to obtain a complete classification of H-umbilical C-totally real submanifolds, or at least H-umbilical C-totally real surfaces in Sasakian space forms.

Problem 6.4. To extend Theorem 6.1 for integral submanifolds of S-space forms (cf. [1,28]) and to obtain a complete classification of H-umbilical integral submanifolds of S-space forms.

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