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EMBEDDING MAXIMAL CLIQUES OF SETS IN MAXIMAL CLIQUES OF BIGGER SETS

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Characterizations are obtained of the maximal $(k+s)$-cliques that contain a given maximal k-clique as a substructure: (1) when $s=1$; (2) for arbitrary s when no line of the clique contains exactly one point of the subclique. These characterizations are used to obtain maximal cliques which have fewer lines (for given k) than previously known examples.

1. Introduction

An *incidence structure* $\Sigma = (S, C)$ is a set S (whose elements are called *points*) together with a collection C of distinguished subsets (called *lines*). An incidence structure with constant line size k is called a *k-clique* if every point lies on at least one line and if every pair of lines has at least one common point. One calls a k-clique *maximal* if it cannot be extended to another k-clique by adjoining another line and, possibly, additional points. Recall that a *blocking set* of Σ is a subset of S that intersects every member of C but that contains no member of C. If Σ is a k-clique, one sees that Σ is maximal if and only if Σ has no blocking set of cardinality k or less.

Meyer [10] introduced the problem of obtaining bounds on the function $m(k)$ which denotes the minimum number of lines in a maximal k-clique. Erdös and Lovász proved [7] that $m(k) \geqslant (8/3)k-3$ by demonstrating that all smaller k-cliques have blocking sets of size $k-1$ or less. In [4] Dow et al. improved the lower bound to $m(k) \geqslant 3k$ for all $k \geqslant 4$. A construction of Erdös and Lovász [7] yields the inequality

$$m(k) \leqslant k \cdot m(k-1)+1, \quad \text{for all } k. \tag{1.1}$$

Since projective planes are maximal cliques, repeated use of (1.1) yields $m(k) < k^{k-n+1}$ if $n < k$ is the order of a projective plane. It has been proved (for certain values of θ) that there is at least one prime between k and $k-k^{\theta}$ for all sufficiently large integers k: the smallest value yet announced for θ is 17/31 (see Pintz [11, p. 395]). One obtains

$$m(k) < k^{ck^{\theta}}, \quad \theta = 17/31, \quad \text{for all } k. \tag{1.2}$$

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For particular values of k, much better upper bounds for $m(k)$ are known. As indicated, all projective planes (more generally, all n-uniform projective Hjelmslev planes) are maximal cliques [6], a fact which yields the following theorem.

Theorem 1.1 (D.A. Drake and S.S. Sane). *If q is the order of a projective plane, then for every positive integer n,*

$$m(q^n + q^{n-1}) \leqslant q^{2n} + q^{2n-1} + q^{2n-2}.$$

A second class of values of k for which $m(k)$ has a small upper bound is given by the following result.

Theorem 1.2 (Z. Füredi [8, Theorem 1]). *If q is the order of a projective plane, then $m(2q) \leqslant 3q^2$.*

The Erdös–Lovász proof of (1.1) is a matter of embedding an arbitrary maximal $(k-1)$-clique in an appropriate maximal k-clique. In this paper we investigate the problem of embedding a maximal k-clique Σ in a maximal $(k+s)$-clique Σ'. In Section 2 we characterize all such embeddings with $s = 1$ and thus obtain minor improvements in the inequality (1.1).

In Section 3 we characterize embeddings for arbitrary s subject to the condition that no line of Σ' contain exactly one point of Σ. Every such Σ' is obtained by joining Σ to an 'extender' Σ^* (defined in Section 3): Σ^* is an incidence structure with 'vertical' lines of size s and 'horizontal' lines of size $k+s$.

In Section 4 we consider special extenders called 'trains'; we prove that trains and 'boxcars' can be 'stretched' and that boxcars can be 'coupled' to trains to produce larger trains. The removal of a single point from a projective plane of order q yields a train with lines of sizes q and $q+1$. By coupling boxcars to stretched versions of these projective-plane trains, one can assemble a useful collection of trains (Proposition 4.5).

In Section 5 we prove that Desarguesian affine planes of odd order are boxcars (if the lines of certain parallel classes of the plane are distinguished as vertical lines). To obtain this conclusion, we apply a theorem of Blokhuis which yields the non-existence of extraneous transversals in certain nets.

In an appendix, the trains constructed in Sections 4 and 5 are used to obtain upper bounds for $m(k)$ for values of $k < 100$. For given values of k, these bounds are far superior to those obtained by the repeated application of (1.1); unfortunately, however, the methods of this paper yield no significant improvement in the asymptotic bound (1.2).

2. Embedding maximal k-cliques in maximal $(k+1)$-cliques

A k-clique $\Sigma = (S, C)$ is said to be a *subclique* of an incidence structure $\Sigma' = (S', C')$ provided that: (1) $S \subseteq S'$, and (2) C is just the collection of all sets

$L \cap S$ of cardinality two or more such that L is in C'. A point of Σ' is called *internal* if it is in S and *external* otherwise. A line L of Σ' is called an *external line*, a *piercing line* or a *structural line* if the number of internal points in L is 0, 1 or k, respectively.

The goal of Section 2 is to characterize all maximal $(k+1)$-cliques that contain a given maximal k-clique as a subclique. In Section 3 we characterize, for arbitrary s, all maximal $(k+s)$-cliques that contain a given maximal k-clique without piercing lines.

Lemma 2.1. *Let Σ be a maximal k-clique on v points, $k \geqslant 4$. Suppose that Σ is a subclique of Σ', a maximal $(k+1)$-clique on v' points. Then $v' = v + k + 1$; all piercing lines (if any exist) are incident with the same set K of k external points; and there is a (unique) external line L.*

Proof. Let S and S' denote the point sets of Σ and Σ', L denote $S' \setminus S$. One sees that $v' \geqslant v + k + 1$, since otherwise L would be a blocking set of S' with fewer than $k+1$ points.

Assume that G and H are distinct piercing lines with $G \cap S = g$ and $H \cap S = h$. Write B to denote $\{g, h\} \cup (G \cap H)$. If I is any non-structural line, then $S \cap (G \cup H \cup I)$ contains at most three points. Since Σ has no blocking sets of size three or less, there is a line M of Σ which is disjoint from $G \cup H \cup I$. Thus, if $M \cup \{x\}$ is any extension of M to a structural line, x must lie in $G \cap H \cap I$. We have proved that all non-structural lines I contain points of B. It is clear therefore that all lines of Σ' meet B. Since $|B \cap S| = 2$, B is not a line of Σ'. Thus $|B| \geqslant k+2$. It follows that $g \neq h$ and that $G \cap H =: K$ consists of the k points of $G \setminus \{g\} = H \setminus \{h\}$.

Assume, by way of contradiction, that there is no external line. In this case, since $v' - v > k$, there is a point x in $S' \setminus S$ which does not lie on any non-structural line. Then x is incident with a structural line G, and $G \setminus \{x\}$ is a blocking set of Σ' of size k. This contradiction of the maximality of Σ' yields the existence of at least one external line L. Assume that there are two external lines L and L'. It is routine to prove that $L \cap L'$ is a blocking set of Σ' of size less than $k+1$. This contradiction proves that L is the unique external line.

Suppose $v' > v + k + 1$. Then there exists a point x in $S' \setminus (S \cup L)$. Every structural line contains only one point outside of S; since this point must belong to L, x lies only on piercing lines. Then, if P is any (piercing) line through x, $P \setminus \{x\}$ is a blocking set of size k. The contradiction implies that $v' \leqslant v + k + 1$ and, thus, completes the proof of the lemma. □

In order to state the main theorem of Section 2, we need to define the notion of a closed set in an incidence structure. Let R denote a subset either of the point set or of the set of lines of an incidence structure Σ. We then write $[R] = [R]^1$ to denote the set of all lines, respectively, all points, that are incident with all members of R. We also write $[R]^{i+1}$ to denote $[[R]^i]$. Then $[R]^{i+2}=[R]^i$ for all $i\geq 1$. In particular, $[R]^4=[R]^2$; and hence, since $R\subseteq[R]^2$ for every R, $[\cdot]^2$ is a closure operator on the collection of subsets of the point set (also on the line set) of Σ. Accordingly, we call $[R]^2$ the *closure* of R; and we say that R is a *closed set* if it is equal to its closure. The set $[R]$ is closed for arbitrary R.

If Σ is finite, we define the *closure number* $c(R)$ of a point or line set R to be $|[R]|+|[R]^2|$. For given Σ, let C_1, C_2, C_3, C_4 denote the set of closure numbers, respectively, for all point sets, all closed point sets, all line sets, and all closed line sets. Clearly $c([R])=c(R)$ for any R. It follows that $C_1\subseteq C_4\subseteq C_3\subseteq C_2\subseteq C_1$ and, hence, that all four C_i's are equal. We write $c(\Sigma)$ to denote the smallest integer in C_2 and call $c(\Sigma)$ the *closure number* of Σ.

Construction 1. Let Σ be a maximal k-clique, and let R be a closed subset of the point set S of Σ. Let $L=\{p_0,\dots,p_k\}$ be a set of points disjoint from S. Define K to be $\{p_1,\dots,p_k\}$, S' to be $S\cup L$. Define Σ' to be the incidence structure with point set S' and the following four kinds of lines:

(i) all sets $G\cup p_i$ with G a line of Σ, $1\leq i\leq k$;
(ii) all sets $G\cup p_0$ with G in $[R]$;
(iii) all sets $K\cup q$ with q in R;
(iv) L.

Theorem 2.2. *Let Σ be a maximal k-clique, $k\geq 4$. The maximal $(k+1)$-cliques which contain Σ as a subclique are precisely the structures Σ' described in Construction* 1.

Proof. Let Σ' be obtained from Σ relative to some closed point set R by the method of Construction 1. It is easy to see that Σ is a subclique of Σ' and that Σ' is a $(k+1)$-clique. Assume, by way of contradiction, that Σ' is not maximal. Then Σ' has a blocking set B of size $k+1$ or less. Since B meets the line L, $|B\cap S|\leq k$.

Suppose first that $|B\cap S|=k$. Then there is a point p_i in $K\setminus B$. Since B meets all lines of Σ' of type (i), $B\cap S$ meets all lines of Σ. The maximality of Σ implies that $B\cap S=G$ for some line G of Σ. Since B meets L but is not a line (of type (i)), $B=G\cup p_0$. Since B meets all lines of type (iii), B is a line of type (ii). This contradiction yields the conclusion that $|B\cap S|<k$.

The maximality of Σ guarantees the existence of a line G of Σ that is disjoint from B. Since B meets all lines of type (i), B contains K. Since $B\neq L$ and since B meets all lines of type (ii), $B=K\cup q$ for some point q which lies in every line of $[R]$. Since R is closed, q is in R; and therefore B is a line of type (iii). This final contradiction completes the proof of the maximality of Σ'.

Conversely, assume that Σ' is any maximal $(k+1)$-clique which contains Σ as a subclique. Denote the full point sets of Σ and Σ' by S and S', respectively. By Lemma 2.1 there is a unique external line $L=\{p_0,\dots,p_k\}=S'\setminus S$. Further, all piercing lines (if any) contain the set $K = \{p_1, \ldots, p_k\}$. The maximality of Σ' guarantees that all sets of type (i) are lines of Σ'.

Define R to be the set of all points q (if any) such that $K \cup q$ is a piercing line. (If there are no piercing lines, R is the empty set.) To complete the proof of Theorem 2.2, it suffices to prove that R is closed and that $G \cup p_0$ is a line of Σ' if and only if G is in $[R]$. If G is in $[R]$, then $G \cup p_0$ is a set of $k+1$ points which meets all lines of Σ' and, hence, is itself a line. Conversely, it is clear that $G \cup p_0$ is a line only if G is in $[R]$. If r is in $[R]^2$, then $K \cup r$ is a set of $k+1$ points that meets all lines of Σ'. It follows that $K \cup r$ is a line of Σ', hence that r is in R, and therefore that R is closed. The proof of Theorem 2.2 is complete. □

Special cases. Let Σ and Σ' satisfy the conditions of Construction 1. Write b and b' to denote the numbers of lines in Σ and Σ', respectively. If $k \neq 1$, no point of Σ is incident with all lines of Σ. It follows that the empty set E is a closed set of points of Σ and that $[E]$ is the set of all lines of Σ. Using $R = E$ in Construction 1, one obtains b lines of type (ii) and no lines of type (iii); in this case $b' = b(k+1)+1$. This special case has been described by Erdös and Lovász in [7, p. 620(c)]. At the other extreme, one may take R to be the set of all points of Σ. Then Σ' has v lines of type (iii) and no lines of type (ii), so $b' = bk + v + 1$. Other possibilities are to take R to be the points of some line (so that $b' = bk + k + 2$) or to be a single point p (so that $b' = bk + |[p]| + 2$).

Corollary 2.3. *Let Σ be a maximal k-clique with v points and b lines, $k \geq 4$. Let Σ be a subclique of a maximal $(k+1)$-clique Σ' which has b' lines. Then $bk + 4 \leq b' \leq bk + 1 + \max(v, b)$.*

Proof. It suffices to prove that the number of lines of Σ' of types (ii) and (iii) lies between 3 and $m := \max(v, b)$; i.e., to prove that $3 \leq c(\Sigma) \leq m$. If R is the empty set, $c(R) \geq b \geq 3$. If R is a single point, $c(R) \geq 3$ because every point of Σ must lie on at least two lines. If R is a closed set of two points, the points must lie on a common line, so again $c(R) \geq 3$.

Let R be a closed point set of Σ. Clearly $c(R) \leq m$ if R contains all or none of the points of Σ. If $|R| < v$, then $|R| \leq k$. If $|R| = k$, R must be a line; so $c(R) = k + 1 \leq m$. Let r denote $\max |[p]|$ as p ranges over all points of Σ. If R consists of a single point, then $c(R) \leq 1 + r < b \leq m$. Lastly, consider the case that $1 < |R| < k$. Since there are no blocking sets of size $k-1$, every point on any given line G must be incident with at least one line that meets no other point of G. Then $c(R) \leq (k-1) + (r-1) < r + (k-1) \leq b \leq m$. □

Remark. The special cases mentioned above make it clear that the upper bound of Corollary 2.3 is achieved for every choice of Σ. For every k, there is a maximal k-clique with points of valence 2 (see [7, p. 620(b)]); thus the lower bound in Corollary 2.3 is best possible for every k.

3. Subcliques without piercing lines

The goal of Section 3 is to describe all the maximal cliques which contain a given maximal clique Σ without piercing lines. Such cliques are obtained from Σ by means of 'extenders' which we proceed to define.

Let $\Sigma = (S, E)$ be an incidence structure whose line set E is the disjoint union of the non-empty sets C and D. Then (S, C, D) is called an *(m, n)-grating* if every line of C (called a *vertical line*) has m points, every line of D (called a *horizontal line*) has n points, and every horizontal line meets every vertical line. We sometimes write Σ for (S, C, D); and we denote $|C|$ by c, $|D|$ by d.

An (m, n)-grating $\Sigma = (S, C, D)$ is said to be *long* if no set of fewer than n points intersects all vertical lines, *(horizontally) braided* if each two horizontal lines intersect, *(horizontally) loose* if each vertical line is disjoint from at least one other vertical line or if there is only one vertical line; Σ is said to be *vertically loose* if (S, D, C) is (horizontally) loose. Generally we shall omit the adverb 'horizontally' while retaining the descriptor 'vertically'.

An *(m, n)-boxcar* is a vertically loose (m, n)-grating that cannot be embedded in another (m, n)-grating through the addition of a line (and the possible addition of points to the new line). An *(m, n)-extender* is a loose, braided (m, n)-grating which cannot be embedded in another braided (m, n)-grating through the addition of a line (and the possible addition of points to the new line). A long (m, n)-extender is called an *(m, n)-train*.

Construction 2. Let $\Sigma = (S, E)$ be a maximal k-clique, $\Sigma^* = (S^*, C, D)$ be an $(m, k+m)$-extender such that S and S^* are disjoint. We define an incidence structure Σ' on the point set $S' = S \cup S^*$ by taking the following subsets of S' as lines:

(i) every set $G \cup H$ with G in E and H in C;
(ii) every line K of D.

Theorem 3.1. *Let Σ be a maximal k-clique, m be a positive integer. The maximal $(k+m)$-cliques that contain Σ as a subclique without piercing lines are precisely the structures Σ' that are described in Construction 2.*

Proof. Given Σ and an $(m, k+m)$-extender Σ^*, it is clear that Construction 2 produces a $(k+m)$-clique Σ'. It is also obvious that Σ is a subclique and that there are no piercing lines. Assume, by way of contradiction, that Σ' is not maximal. Then there is a blocking set B of size $k+m$ or less.

Suppose first that $|B \cap S^*| \leq m$. Since $B \cap S^*$ intersects all lines in D, the definition of an extender guarantees that $B \cap S^*$ is a line of C. In particular, $|B \cap S^*| = m$, so $|B \cap S| \leq k$. If $B \cap S^*$ were the only vertical line of Σ^*, one could adjoint a new horizontal line (that contained $B \cap S^*$). Thus Σ^* has at least two vertical lines; and, as Σ^* is loose, there is a line H in C which is disjoint from

$B \cap S^*$. Since B intersects all lines of Σ' of type (i), $B \cap S$ must intersect all lines of Σ. The maximality of Σ thus implies that $B \cap S$ is a line of Σ, hence that B is a line of Σ'. This contradiction forces the conclusion $|B \cap S^*| > m$.

Then $|B \cap S| < k$, so there is a line G of Σ that does not intersect $B \cap S$. It follows that $B \cap S^*$ intersects every line H of C (as well as every line K of D). The definition of extender thus forces $B \cap S^* = B$ to be a line of D, because B is of size $k + m$ or less. This contradiction completes the proof that Σ' is maximal.

Conversely, assume that Σ' is a maximal $(k + m)$-clique which contains the maximal k-clique Σ as a subclique and that there are no piercing lines. Let S and S' be the respective point sets of Σ and Σ', and write S^* for $S' \backslash S$. Let C denote the collection of all subsets $L \cap S^*$ such that L is a structural line; D, the collection of all external lines. Clearly $\Sigma^* := (S^*, C, D)$ is a braided $(m, k + m)$-grating. If Σ^* were not loose, some structural line L of Σ' would intersect all other lines of Σ' in points of S^*. Then $L \cap S^*$ would be a blocking set of Σ' of size $m < k + m$. We conclude that Σ^* is loose.

Assume next that Σ^* can be extended to a braided $(m, k + m)$-grating by adding a new line G^* to C. Then, taking G to be any line of Σ, the set $G \cup G^*$ is a new set of size $k + m$ which intersects all lines of Σ'. This contradiction proves that Σ^* could only be extended by adding a new horizontal line H^* to D. The latter alternative would yield the contradiction that Σ' has a blocking set, H^*, of size $k + m$. We are compelled to conclude that Σ^* is an $(m, k + m)$-extender.

We have yet to verify that the application of Construction 2 to Σ and Σ^* yields Σ'. By intent, the lines of type (ii) generated by Construction 2 are precisely the external lines of Σ'. It is also clear that every structural line of Σ' is a line of type (i). Conversely, the maximality of Σ' guarantees that every line of type (i) is a structural line of Σ'. □

4. Couplings and stretchings of boxcars and trains

We intend to apply Construction 2 to extend maximal k-cliques to maximal $(k + m)$-cliques. To do so, we must create a supply of $(m, k + m)$-extenders. The following lemma gets us started.

Lemma 4.1. *Let q be one or the order of a finite projective plane. Then there is a $(q, q + 1)$-train $\Pi = (S, C, D)$ on $v = q^2 + q$ points where C is a parallel class with $|C| = q + 1$ and $|D| = q^2$.*

Proof. Let p be a point of a projective plane Σ of order q. Take S to be the set of all points of Σ except p. Let C consist of the lines of Σ that are incident with p, each restricted to its intersection with S; and let D consist of the remaining lines of Σ. Since Σ is a maximal clique, $\Pi = (S, C, D)$ is a train. The case $q = 1$ is left to the reader. □

A moment's reflection will convince one that the use of the trains of Lemma 4.1 as extenders in Construction 2 will produce nothing more than projective planes. Ultimately, however, the trains of Lemma 4.1 will prove useful: the goal of Section 4 is to describe methods for 'coupling' and 'stretching'.

Let $\Pi_i = (S_i, C_i, D_i)$ be (m, n_i)-gratings for $i = 1$ and 2. Write S and C, respectively, for the disjoint unions $S_1 \cup S_2$ and $C_1 \cup C_2$; write D for the collection of all sets $K_1 \cup K_2$ with K_i in D_i for each i. Then $\Pi = (S, C, D)$ is an $(m, n_1 + n_2)$-grating called the *coupling* of Π_1 and Π_2.

Lemma 4.2 (The Coupling Lemma). *Let Π_1 be an (m, n_1)-train, Π_2 be an (m, n_2)-boxcar. Suppose that $m \neq n_2$ or that Π_2 is loose. Then the coupling Π of Π_1 and Π_2 is an $(m, n_1 + n_2)$-train. If each Π_i has c_i vertical lines and d_i horizontal lines, then Π has $c_1 + c_2$ vertical lines and $d_1 d_2$ horizontal lines.*

Proof. Clearly the coupling Π is a loose, braided (m, n)-grating with $n = n_1 + n_2$. No set S of fewer than n_2 points of Π_2 intersects all vertical lines of Π_2, because any such S could be extended to a new horizontal line. Since Π_1 is long, it is clear that Π is long.

It now suffices to prove that Π cannot be extended to another braided (m, n)-grating through the addition of a line. Thus assume that G is a set of m points of Π which intersects all lines of D and that G is not in C_1. Since Π_1 cannot be extended, G is not contained in S_1. Therefore, some line K_1 of D_1 is disjoint from G. It follows that G intersects all lines K_2 from D_2, hence that G is a line of C_2. We have proved that Π cannot be extended by adjoining a vertical line.

Assume next that K is a set of n points which intersects all lines of $C \cup D$. Write K_1 for $K \cap S_1$, K_2 for $K \cap S_2$. Since Π_1 is long, $|K_1| \geq n_1$. Since Π_2 cannot be extended, $|K_2| \geq n_2$. Therefore $|K_i| = n_i$ for $i = 1$ and 2, and K_2 is a line of Π_2. If K_2 were in C_2, then m would be equal to n_2; and Π_2 could not be loose. We conclude that K_2 is in D_2. Since Π_2 is vertically loose, there is a line in D_2 that is disjoint from K_2. Since K meets all lines of $C \cup D$, K_1 must meet all lines of $C_1 \cup D_1$. Then K_1 must be in D_1, else Π_1 could be extended. Thus K is in D. We have proved that Π cannot be extended, so Π is a train. □

Construction 3a (Uniform stretching). Let $\Pi = (S, C, D)$ be an (m, n)-grating, k be a positive integer. To each point p of Π, associate a set (p) of k points, so chosen that the sets (p) are disjoint. Define an incidence structure Π' on the union S' of the (p) by taking as lines all point sets of the following two types:

(i) to each line $G = \{p_1, \ldots, p_m\}$ of C, all sets $\{x_1, \ldots, x_m\}$ with x_i in (p_i) for each i;

(ii) to each line $H = \{p_1, \ldots, p_n\}$ of D, the single set $(p_1) \cup \cdots \cup (p_n)$.

Take C' to be the collection of all lines of type (i), D' to be the collection of all lines of type (ii). The (m, nk)-grating $\Pi' = (S', C', D')$ is said to arise from Π by *uniform horizontal stretching*. It is clear how one obtains an (mk, n)-grating $\Pi'' = (S'', C'', D'')$ from Π by means of *uniform vertical stretching*.

Construction 3b (Non-uniform stretching). Let $\Pi = (S, C, D)$ be an (m, n)-grating which has a *vertical parallel class*; i.e., a subset $C_0 = \{G_1, \ldots, G_n\}$ of n lines of C which partitions the points of Π. Let $k_1, \ldots, k_n$ be fixed positive integers. For each $i \leq n$ and each point p on G_i, one takes (p) to be a set of cardinality k_i, choosing the sets (p) to be disjoint. One continues as in Construction 3a to obtain a *non-uniform horizontal stretching* of Π to an $(m, k_1 + \cdots + k_n)$-grating $\Pi' = (S', C', D')$. The existence of a *horizontal parallel class* $D_0 = \{H_1, \ldots, H_m\}$ in D allows one to obtain *non-uniform vertical stretchings to* $(k_1 + \cdots + k_m, n)$-gratings $\Pi'' = (S'', C'', D'')$.

Lemma 4.3 (Stretching Lemma). *Every stretching (horizontal or vertical, uniform or not) of a loose boxcar is a loose boxcar. If Π is an (m, n)-train with $m < n$, every horizontal stretching of Π is a train.*

Proof. Clearly, all stretchings preserve looseness (horizontal and vertical); and all horizontal stretchings preserve horizontal braidings. Since the definition of loose boxcars is symmetric in the treatment of vertical and horizontal, it suffices to consider horizontal stretchings. Thus, let Π be a loose (m, n)-boxcar or an (m, n)-train, and let Π' be a horizontal stretching of Π. If Π is a train, we assume $m < n$. We observe that Π' is an (m, N)-grating where N is either nk or $k_1 + \cdots + k_n$.

Let G' be any set of m or fewer points of Π' which intersects all horizontal lines of Π'. Then G' induces a set G of at most m points of Π, and G intersects all horizontal lines of Π. It follows that G is a line of Π. If Π is a boxcar, G cannot be a horizontal line, since boxcars are vertically loose. If Π is a train, G cannot be a horizontal line, since one has $m < n$ in this case. Therefore G is a vertical line of Π, so G' is a vertical line of Π'. Then Π' cannot be extended by the addition of a new vertical line.

Let H' be a set of N or fewer points of Π' which intersects all vertical lines of Π'. We treat the case that Π' is obtained from Π by uniform stretching and leave the non-uniform case to the reader. The k^m lines of Π' obtained from a given vertical line G of Π can all be covered by the points of H' only if H' contains a set (p) for some p in G. Since Π is long (even when Π is a boxcar!), at least n sets (p) are needed. Then H' has cardinality $nk = N$, and H' is the union of n sets (p). We have proved that Π' is long.

If Π is a train, we may now assume that H' also intersects all horizontal lines of Π' (because it suffices to prove that Π' cannnot be extended *as a braided grating*). The set H' induces a set H of n points of Π which intersects all vertical lines of Π; if Π is a train, H also intersects all horizontal lines of Π. Thus H is a line of Π which intersects all vertical lines of Π. Since Π is (horizontally) loose, either H is a horizontal line or else H is the sole vertical line of Π. If H is a vertical line, $n = m$; so Π is a boxcar. Then each point p of H is on a horizontal line of Π, else a new vertical line could be added to Π; namely, $(H \setminus \{p\}) \cup \{q\}$, where q is a new point. If G is a horizontal line, then $G \neq H$; so G contains a point x not in H, and x must lie on some vertical line. This contradiction (that Π has two vertical lines) yields the conclusion that H is a horizontal line of Π. Then H' is a horizontal line of Π'. We have proved that Π' cannot be extended by the addition of a new horizontal line. Then Π' is a loose boxcar or a train, as required. □

Corollary 4.4. *Let k_1, k_2, m be positive integers with $m \geq 2$. Then, for each of the following sets of parameters, there is a loose boxcar $\Pi = (S, C, D)$ with a horizontal parallel class and a vertical parallel class:*
(i) *a $(2, k_1 + k_2)$-boxcar with $|C| = k_1^2 + k_2^2$ and $|D| = 4$;*
(ii) *an $(m, 2)$-boxcar with $|C| = 2$ and $|D| = m^2$;*
(iii) *a $(k_1 + k_2, 2)$-boxcar with $|C| = 4$ and $|D| = k_1^2 + k_2^2$;*
(iv) *a $(2, m)$-boxcar with $|C| = m^2$ and $|D| = 2$.*

Proof. The complete graph of four points may be regarded as a loose $(2,2)$-boxcar Π with $|C| = 2$ and $|D| = 4$. Stretching Π horizontally yields the examples of type (i) while vertical stretchings of Π yield the examples of type (ii). Interchanging the vertical and horizontal sets will turn the examples of types (i) and (ii) into the examples of types (iii) and (iv). □

Proposition 4.5. *Let m be the order of a projective plane, x and y be integers with $x \geq m$ and $y \geq 0$. Let $A_0, \ldots, A_x$ and $B_1, \ldots, B_y$ be positive integers. Then there is an (m, n)-train Σ^* with*

$$n = \sum_{i=0}^{x} A_i + 2 \sum_{i=1}^{y} B_i,$$

$$c = \sum_{i=0}^{x} A_i^m + 4 \sum_{i=1}^{y} B_i^m,$$

$$d = m^{x-m+2}[(m^2+1)/2]^y,$$

where $[\cdot]$ denotes the greatest integer function.

Proof. By Lemma 4.1 there is an $(m, m+1)$-train with m^2 horizontal lines and $m+1$ vertical lines that are a parallel class. The Stretching Lemma yields an (m, n_0)-train Σ_0 with $n_0 = A_0 + \cdots + A_m$, $c_0 = A_0^m + \cdots + A_m^m$ and $d_0 = m^2$. The unique $(m, 1)$-boxcar is loose and hence may be stretched into a loose (m, A_i)-boxcar Σ_i with $c_i = A_i^m$ and $d_i = m$ for $m < i \leq x$. Next we apply Corollary 4.4(iii) with $k_1 = k_2 = m/2$ if m is even, with $k_1 = (m+1)/2$ and $k_2 = (m-1)/2$ if m is odd. In both cases we obtain a loose $(m, 2)$-boxcar with four vertical lines and $[(m^2+1)/2] =: m'$ horizontal lines. Uniform stretching yields loose $(m, 2B_i)$-boxcars Π_i with $4B_i^m$ vertical lines and m' horizontal lines for $1 \leq i \leq y$. Coupling the boxcars $\Sigma_{m+1}, \ldots, \Sigma_x$ and $\Pi_1, \ldots, \Pi_y$ to the train Σ_0 yields the desired train Σ^*. □

Remark. At this point the reader may wish to skip ahead to the appendix to see examples of the use of Proposition 4.5 in obtaining upper bounds for $m(k)$ for selected values of k.

5. Nets and boxcars

In Section 5 we use nets (see [2] or [3]) to construct additional boxcars. A *net* of *degree r, deficiency d* and *order g* is an incidence structure $\Pi = (S, C)$ with the following properties:

(1) $|S| = q^2$;

(2) C is the disjoint union of r parallel classes of lines where each class is a set of q disjoint lines of size q;

(3) two non-parallel lines have a (unique) point of intersection;

(4) $r + d = q + 1$.

A net Π of degree r is often called an *r-net.* A *transversal* to Π is a set of q points that contains (exactly) one point from each line of Π. An *affine plane* of *order q* is a $(q+1)$-net of order q. Two nets $\Pi_i = (S, C_i)$, $i = 1$ and 2, are said to be *complementary nets* if C_1 and C_2 are disjoint and if $\Pi := (S, C_1 \cup C_2)$ is an affine plane. The easy proof of the following lemma is left to the reader.

Lemma 5.1. *Let $\Pi_1 = (S, C)$ and $\Pi_2 = (S, D)$ be complementary nets of order q. Suppose also that the only transversals of Π_1 and Π_2 are the lines of Π_2 and Π_1, respectively. Then (S, C, D) is a loose (q, q)-boxcar.*

In order to apply Lemma 5.1, we need the following result.

Theorem 5.2 (A. Blokhuis [1]). *Let F be a subfield of order q of $K = \mathrm{GF}(q^2)$ with q odd. Let T be a subset of q elements of K with the property that all or none of the differences of pairs of elements of T are squares. Then T is a coset of a subspace of K (regarded as a vector space over F).*

Corollary 5.3. *For each odd prime power q, there is a loose (q, q)-boxcar (K, C, D) on q^2 points with the following property: each of C and D is the disjoint union of $(q+1)/2$ parallel classes of lines.*

Proof. Let E be the collection of all cosets of 1-dimensional subspaces of K regarded as a vector space over F. Then $\Pi = (K, E)$ is an affine plane of order q. Half of the lines through the origin consist of points that are squares of elements of K. Let C denote the collection of all cosets of these lines, D denote $E \setminus C$. Then two points of Π are joined by a line of C if their difference is a square and by a line of D if their difference is a non-square. By Theorem 5.2, the complementary nets (K, C) and (K, D) have no extra transversals, so Corollary 5.3 follows from Lemma 5.1. □

Remarks. By the theorem of Blokhuis, one may partition the parallel classes of any finite Desarguesian affine plane of odd order q into two nets of degree $(q+1)/2$ in such a manner that the only transversals of each net are the lines of the complementary net. In general the choice of partition is critical: it is well known, for example, that some nets of degree as large as $q - q^{1/2}$ have 'extra' transversals. On the other hand, if q is a prime, the following theorem shows that the choice of partition is completely arbitrary: for every non-linear function f from GF(p) into itself, p a prime, the difference quotient $(f(y)-f(x))/(y-x)$ assumes at least $(p+3)/2$ distinct values. This theorem is due to Rédei with an assist from Megyesi (see Theorem 24′ and the preceding discussion on page 226 of [12]). A recent brief proof of this theorem has been given by Lovász and Schrijver [9].

Proposition 5.4. *Let m be an odd prime power; x and z be integers with $x \geqslant m$ and $z \geqslant 0$. Let $A_0, \ldots, A_x$ and $C_1, \ldots, C_z$ be positive integers. Then there is an (m, n)-train Σ^* with*

$$n = \sum_{i=0}^{x} A_i + m \sum_{i=1}^{z} C_i,$$

$$c = \sum_{i=0}^{x} A_i^m + ((m^2+m)/2) \sum_{i=1}^{z} C_i^m,$$

$$d = m^{x-m+2}((m^2+m)/2)^z.$$

Proof. Corollary 5.3 guarantees the existence of loose (m, m)-boxcars which may be stretched into loose (m, mC_i)-boxcars Π_i with $(m^2+m)C_i^m/2$ vertical lines and $(m^2+m)/2$ horizontal lines. The desired trains Σ^* are obtained by successively coupling boxcars $\Pi_1, \ldots, \Pi_z$ to those trains of Proposition 4.5 which have y equal to zero. □

Appendix

Call a value of k a 'good' value if k is of the form $q^n + q^{n-1}$ or $2q$, where q is a prime power and n is a positive integer. Theorems 1.1 and 1.2 give $m(k) < k^2$ for all good values of k. Thus (1.1) yields $m(k+s) < (k+s)^{(s+2)}$ if k is good and s is a positive integer. Improvements in this bound for $k+s < 100$, $s = 1$ were obtained in [5]. In this appendix we apply Propositions 4.5 and 5.5 to improve known upper bounds for $k+s < 100$, with $s \geqslant 2$. The bounds and sketches of the proofs are indicated in Table 1.

As an example, consider the case $k = 40$. Using $2q = 38$, Theorem 1.2 gives $m(38) \leqslant 1083$. Application of either of the Propositions 4.5, 5.5 (with $m = 2$, $x = 13$, $A_0 = A_1 = 2$ and $A_2 = A_3 = \cdots = A_{13} = 3$) assures the existence of a (2,

Table 1. Upper bounds for $m(k)$

k	upper bound for $m(k)$	proof
40	133,820	$(2q=38)$, $m=2$, $A_0=A_1=2$, $A_2=\cdots=A_{13}=3$
41	426,291	$(2q=38)$, $m=3$, $A_0=2$, $A_1=\cdots=A_3=3$, $C_1=\cdots=C_5=2$
52	355,268	$(2q=50)$, $m=2$, $A_0=\cdots=A_3=4$, $A_4=\cdots=A_{15}=3$
53	1502,484	$(2q=50)$, $m=3$, $A_0=A_1=4$, $A_2=A_3=C_1=\cdots=C_3=3$, $C_4=C_5=2$
67	1168,201	$(q+1=65)$, $m=2$, $A_0=3$, $A_1=\cdots=A_{16}=4$
70	1379,690	$(q+1=68)$, $m=2$, $A_0=A_1=3$, $A_2=\cdots=A_{17}=4$
71	6293,877	$(q+1=68)$, $m=3$, $A_0=5$, $A_1=\cdots=A_3=4$, $C_1=\cdots=C_6=3$
76	1461,740	$(2q=74)$, $m=2$, $A_0=\cdots=A_3=5$, $A_4=\cdots=A_{17}=4$
77	7377,162	$(2q=74)$, $m=3$, $A_0=\cdots=A_3=5$, $C_1=4$, $C_2=\cdots=C_6=3$
78	56041,072	$(2q=74)$, $m=4$, $A_0=\cdots=A_4=6$, $B_1=\cdots=B_6=4$
79	64034,058	$(2q=74)$, $m=5$, $A_0=4$, $A_1=\cdots=A_5=C_1=C_2=3$, $C_3=\cdots=C_5=2$
88	2527,376	$(2q=86)$, $m=2$, $A_0=A_1=4$, $A_2=\cdots=A_{17}=5$
89	14046,090	$(2q=86)$, $m=3$, $A_0=\cdots=A_3=5$, $C_1=C_2=4$, $C_3=\cdots=C_7=3$
92	3851,072	$(q+1=90)$, $m=2$, $A_0=\cdots=A_2=4$, $A_3=\cdots=A_{18}=5$.
93	21267,725	$(q+1=90)$, $m=3$, $A_0=6$, $A_1=\cdots=A_3=5$, $C_1=\cdots=C_3=4$, $C_4=\cdots=C_7=3$

40)-train with $c=116$ and $d=8192$. Thus Theorem 3.1 yields $m(40)\leq c\cdot m(38)+d\leq 133{,}820$.

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