JOURNAL OF COMBINATORIAL THEORY, Series B 50, 168–178 (1990)

Counting 1-Factors in Infinite Graphs

RON AHARONI

Department of Mathematics, The University of Calgary, Calgary, Alta, Canada T2N 1N4

AND

MAO LIN ZHENG

Department of Mathematics, Xinjiang University, Urumqi, People's Republic of China

Communicated by the Managing Editors

Received April 6, 1987

F. Bry (*J. Combin. Theory Ser. B* **34** (1983), 48–57) proved that a locally finite infinite n-connected factorizable graph has at least $(n-1)!$ 1-factors and showed that for $n=2$ this lower bound is sharp. We prove that for $n \geqslant 3$ any infinite n-connected factorizable graph has at least $n!$ 1-factors (which is a sharp lower bound). © 1990 Academic Press, Inc.

I. INTRODUCTION

"Quantitative matching theory" (counting matchings, as opposed to finding conditions for their existence) probably starts with M. Hall's paper [6]. It was proved there that if in a bipartite graph there is a perfect matching and the degree of every vertex in one of the sides is at least n, then there are at least $n!$ perfect matchings in the graph. (Counting perfect matchings in bipartite graphs is related to the Van der Waerden problem.) For general graphs, it was realized that a condition should be posed on the degree of connectedness of the graph, rather than on the degrees of the vertices: it is easy to construct for every k a graph in which the degree of every vertex is larger than k, but the graph has a unique perfect matching (see, e.g., [7, p. 183]). We denote by $f(G)$ the number of 1-factors in a graph G. Beineke and Plummer [3] proved that $f(G) \geqslant 2$ for a factorizable 2-connected graph. Zaks [10] proved that $f(G) \geqslant n!! = n(n-2)(n-4)\cdots 1$ for any factorizable n-connected graph. Lovász [7] improved this to $f(G) \geqslant n!$ for any factorizable n-connected non-bicritical graph (a graph G

0095-8956/90 $3.00

is *bicritical* if $G-\{x, y\}$ is factorizable for any $x, y \in V(G)$). Mader [8] proved that within the set of $(2k+1)$-connected factorizable graphs $f(G)$ attains its minimum uniquely at K_{2k+2}, and among $2k$-connected factorizable graphs $f(G)$ is minimal at K_{2k+2} from which a perfect matching is removed. (All this relates, so far, to *finite* graphs.)

In [4] Bry proved that $f(G) \geqslant (n-1)!$ for any n-connected factorizable locally finite infinite graph. For $n=2$, he showed that this bound is sharp: there is such a graph with a unique 1-factor. For $n \geqslant 3$, however, he could only find examples of such graphs G with $f(G)=n!$. In this paper we extend Bry's result to general infinite graphs and prove the sharp lower bound $n!$ on $f(G)$ for any infinite n-connected factorizable graph, where $n \geqslant 3$. Counting finitely many matchings in an infinite graph may, indeed, not be of great importance by itself. But it gives us an opportunity to examine in depth the structure of matchings in infinite graphs.

II. Definitions and Notation

For any graph $G=(V, E)$ we write $V(G)=V$, $E(G)=E$. The letters V and E will always be associated with a graph denoted by G. If $S \subseteq V$ we write $G[S]$ for the subgraph of G spanned by S, and $G-S$ for $G[V \backslash S]$. If $S=\{x\}$ we write $G-x$ for $G-S$.

The degree of a vertex v in G is denoted by $d(v)=d_G(v)$.

A bipartite graph Γ with bipartition (M, W) and edge set K will be denoted by $(M, W, K)=(M_\Gamma, W_\Gamma, K_\Gamma)$. When the identity of a bipartite graph Π is clear from the context, we omit the subscript Π in K_Π and simply write K. The letter Γ is always associated with the bipartite graph (M, W, K).

Let $G=(V, E)$ by any graph, and $F \subseteq E$. For any $v \in V$ we write $F\langle v\rangle = \{u \in V: (v, u) \in F\}$. If $|F\langle v\rangle|=1$ we denote by $F(v)$ the single element of $F\langle v\rangle$. For $S \subseteq V$ we write $F[S]=\bigcup \{F\langle v\rangle : v \in S\}$.

A *matching* in G is a subset F of E such that $|F\langle v\rangle| \leqslant 1$ for any $v \in V$. It is called a *perfect matching*, or a *1-factor*, if $|F\langle v\rangle|=1$ for any $v \in V$. By a matching *of* G we mean a perfect matching. If G has a perfect matching we say that G is *matchable* (or *factorizable*). The number of matchings of G is denoted by $f(G)$.

A matching F in a bipartite graph $\Gamma=(M, W, K)$ is called an *espousal* if $|F\langle m\rangle|=1$ for every $m \in M$. If Γ has an espousal it is called *espousable*. The number of espousals in Γ is denoted by $e(\Gamma)$.

If Γ is espousable we denote by $\sigma(\Gamma)$ the cardinal $\sup\{Z \subseteq W: \Gamma - Z$ is espousable$\}$ (in [2] it is proved that the supremum in this definition is, in fact, attained; i.e., it is a maximum). A subset N of M is called *matchable* if $\Gamma[N \cup W]$ is espousable. An espousal of $\Gamma[N \cup W]$ is then called a *matching of N*.

A graph P is called *factor-critical* if it is unmatchable, but $P-x$ is matchable for every $x \in V(P)$. For $S \subseteq V$ we denote by $\mathscr{P}(S)=\mathscr{P}(G, S)$ the set of factor-critical connected components of $G-S$. Write $P(S)= \bigcup \{V(P): P \in \mathscr{P}(S)\}$, $T(S)=P(S) \cup S$, and $C(S)=V \backslash T(S)$. We form a bipartite graph $\Pi(S)=\Pi(G, S)=(\mathscr{P}(S), S, K)$, where $(P, s) \in K$ if $(t, s) \in E$ for some $t \in V(P)$. A graph G is called *bicritical* if it is matchable and $G-\{x, y\}$ is matchable for any pair of vertices x, y in $V(G)$.

A subset C of M in a bipartite graph $\Gamma=(M, W, K)$ is called *critical* if it is matchable (i.e., $\Gamma[C \cup W]$ is espousable), but for every matching F of C (i.e., an espousal of $\Gamma[C \cup W]$) there holds $F[C]=K[C]$ (in other words, $\sigma(\Gamma[C \cup K[C]])=0$). If M is critical we say that Γ is *pressed*. If Γ has no non-empty critical set we say that Γ is *loose*. A subset S of V in a graph $G=(V, E)$ is called *pressing* if $\Pi(G, S)$ is pressed.

Let $F, H \subseteq E$. A path P is called $F-H$-alternating if for any two consecutive edges in P one edge is in F and one is in H. A path P is called *F-alternating* if it is $(E \backslash F)-F$-alternating. An F-alternating path is called an *F-walk* if its first edge is not in F. Given two matchings F and H and a vertex v, when we refer to "the $F-H$-alternating path containing v" we mean the *maximal* such path, i.e., the connected component in the graph $(V, F \cup H)$ containing v. Similarly the phrase "the $H-F$-alternating path starting with the edge (v, u)" refers to the maximal such path.

Given two paths $P=(x_1, ..., x_m)$ and $Q=(y_1, ..., y_n)$, where $x_m=y_1$ or $(x_m, y_1) \in E$ and $V(P) \cap V(Q)=\{x_m\}=\{y_1\}$, we denote by $P * Q$ the concatenation of P and Q. If $(P_j: j<\omega)$ is a sequence of paths whose concatenation is well defined and yields a path, we write $*_{j<\omega} P_j$ for this concatenation.

III. Some Properties of Critical Sets

Lemma 3.1. *If C is critical, $B \subseteq C$, F is a matching of C, and $F[B]=K[B]$, then B is critical.*

Lemma 3.2 [9]. *Let C be a matchable subset of M and let F be a matching of C. Then C is critical if and only if*

(a) *$K[C]=F[C]$ and*

(b) *there does not exist an infinite F-walk.*

Lemma 3.3. *Every critical set contains a minimal non-empty critical set.*

Proof. Let C be a critical set and let F be a matching of C. For a, b in

C say that b is *reachable from* a if there exists an F-walk from a to b. Choose any $a_1 \in C$, and let C_1 be the set of vertices in M reachable from a_1. Clearly $F[C_1] = K[C_1]$, and hence C_1 is critical, by Lemma 3.1. It is also easy to prove that any subset D of M containing a_1 and satisfying $F[D] = K[D]$ must contain C_1. Thus, by Lemma 3.1, C_1 is minimal among the critical sets containing a_1. Hence if C_1 is not minimal critical, there exists a non-empty critical subset C_2 of C_1 such that $a_1 \notin C_2$. Choose any $a_2 \in C_2$. Then a_2 is reachable from a_1, while a_1 is not reachable from a_2. If C_2 is not minimal critical then there exists a_3 which is reachable from a_2, but from which a_2 cannot be reached. Assuming that the lemma fails we can construct in this way an infinite sequence $(a_i : i < \omega)$ of vertices in C, such that a_j is reachable from a_i if and only if $i < j$. But this clearly implies the existence of an infinite F-walk, in contradiction to Lemma 3.2. ∎

LEMMA 3.4. *If $\Gamma - w$ is pressed for some $w \in W$ then $\sigma(\Gamma) = 1$.*

Proof. This is a special case of a more general result: if $\Gamma - S$ is pressed ($S \subseteq W$) then $\sigma(\Gamma) = |S|$. (This result appears in [2].) The proof: let H be any espousal of Γ. For any $v \in W \backslash H[M]$ let $A(v)$ be the $H-F$-alternating path starting at v. By Lemma 3.2, $A(v)$ is finite, and hence ends at some vertex $s = s(v) \in S$. The correspondence $v \to s(v)$ is an injection from $W \backslash H[M]$ into S. ∎

LEMMA 3.5. *Let B, C be subsets of M, $z \in W$, and suppose that the following conditions hold:*

(i) *C is critical in $\Gamma - z$.*

(ii) *B is critical in $\Gamma - (K[C] \backslash \{z\})$.*

(iii) *$z \in K[B]$.*

Then $D = B \cup C$ is critical in Γ.

Proof. By (i) and (ii), D is matchable. Let F be any matching of D. By Lemma 3.4, $|(K[C] \cup \{z\}) \backslash F[C]| \leqslant 1$. In fact, $|(K[C] \cup \{z\}) \backslash F[C]| = 1$, since otherwise $F[B] \subseteq K[B] \backslash \{z\}$, which is impossible by (ii). Let $\{t\} = (K[C] \cup \{z\}) \backslash F[C]$. Then, writing $\tilde{\Gamma} = \Gamma[B \cup (K[B] \backslash (K[C] \backslash \{t\}))]$, we have that $\tilde{\Gamma} - t$ is pressed (by (ii)), and, hence, by Lemma 3.4, $\sigma(\tilde{\Gamma}) = 1$. Since $z \in W_{\tilde{F}} \backslash F[M_F]$, it follows that $W_{\tilde{\Gamma}} \backslash F[M_{\tilde{\Gamma}}] = \{z\}$. This means that $F[D] = F[B] \cup F[C] = K[B] \cup K[C] = K[D]$, which shows that D is critical. ∎

LEMMA 3.6 [9]. *If Γ is espousable, $w \in W$, and $\Gamma - w$ is inespousable, then there exists a critical set C such that $w \in K[C]$ (sketch of proof: let F be an espousal of Γ, and take C to be the set of all vertices in M which are reachable by an F-walk starting at $F(w)$).*

COROLLARY 3.6a. *If Γ is pressed but contains no non-empty critical proper subset of M (i.e., M is minimal critical in Γ) and $a \in M$, then for every $w \in K\langle a \rangle$ the edge (a, w) can be extended to an espousal of Γ.*

Proof. Rephrased, the assertion is that $\Gamma - \{a, w\}$ is espousable for any $w \in K\langle a \rangle$. If this fails, then, by Lemma 3.6, $\Gamma - a$ contains a non-empty critical set, contradicting the assumption on Γ. ▮

COROLLARY 3.6b. *If Γ is as in Corollary 3.6a and F is an espousal of Γ, then every edge $(a, w) \in K \backslash F$ belongs to an F-alternating circuit.*

Proof. By Corollary 3.6a, (a, w) belongs to an espousal H of Γ. Let T be the connected component in the graph $(V(\Gamma), F \cup H)$ containing a. Then T is an F-alternating path or circuit. Since both F and H are perfect matchings, if T is a path then it is two-way infinite ($d_T(x) = 1$ for a vertex x means that x is not covered by either F or H). But by Lemma 3.2 there is no infinite F-walk. Hence T must be a circuit. ▮

LEMMA 3.7 [9]. (i) *If C is critical then $D \backslash C$ is critical in $\Gamma - K[C]$ if and only if D is critical in Γ.*

(ii) *If $(C_\alpha : \alpha < \zeta)$ is an ascending chain of critical sets then $\bigcup \{C_\alpha : \alpha < \zeta\}$ is critical.*

IV. PASSING FROM GENERAL GRAPHS TO BIPARTITE GRAPHS

In this section we prove a few lemmas on the bipartite graphs $\Pi(S)$ ($S \subseteq V$). The technique of considering espousals in $\Pi(S)$ in order to study 1-factors in G lies at the base of a well-known approach to Tutte's theorem (the author of this approach is probably Gallai [5]). It is also a main constituent in Bry's proof [4].

The following is a generalization of Tutte's theorem to the infinite case:

THEOREM 4.1 [1]. *If G is not factorizable then $\Pi(G, S)$ is inespousable for some subset S of V.*

Note that the converse of this theorem is obvious.

Let S be a pressing set and let F be a matching of G. Since every $P \in \mathscr{P}(S)$ is an unmatchable connected component in $G - S$, at least one vertex from P must be matched with a vertex $v(P)$ from S. Since $\Gamma(S)$ is pressed there is precisely one such vertex for every $P \in \mathscr{P}(S)$ and $\{v(P) : P \in \mathscr{P}(S)\} = S$. The set $(P, v(P))$ is therefore a matching of $\Pi(S)$, which will be denoted by $\pi(F) = \pi(F, S)$.

We have shown above that $F[S] \subseteq P(S)$, and since by the definition of

$P(S)$ as the union of connected components in $G-S$ there holds $F[P(S)] \subseteq P(S) \cup S$, there follows $F[T(S)] = T(S)$. There immediately follows:

LEMMA 4.2. *If G is matchable and S is a pressing subset of V, then $G[T(S)]$ and $G[C(S)]$ are both matchable.*

Suppose that H is an espousal of $\Pi(S)$. For any $P \in \mathscr{P}(S)$ choose a vertex $v = v(P) \in V(P)$ which is connected in G to $H(P)$. Since P is factor-critical there exists a matching $M(P)$ of $P - v$. Then $\bigcup \{(v(P), H(P)) \cup M(P): P \in \mathscr{P}(S)\}$ is a matching of $G[T(S)]$. Assuming that G is matchable, by Lemma 4.2 so is $G[C(S)]$, and hence the above matching of $G[T(S)]$ can be completed to a matching of G. Denote this matching by $\zeta(H) = \zeta(H, S)$. Note that the definition of $\zeta(H)$ is not unique, since it depends on the choice of $v(P)$ and $M(P)$ for $P \in \mathscr{P}(S)$, as well as the choice of the matching of $G[C(S)]$. But $\zeta(\pi(H)) = H$ for any espousal H of $\Pi(S)$; i.e., ζ is onto. Hence we have:

LEMMA 4.3. *If G is matchable and S is pressing then $f(G) \geqslant e(\Pi(S))$.*

LEMMA 4.4. *If G is matchable but $G-x$ is not, then there exists a pressing set containing x.*

Proof. By Theorem 4.1, $\Pi(G-x, S')$ is inespousable for some subset S' of $V \setminus \{x\}$. Let $S = S' \cup \{x\}$. Since G is matchable, $\Pi(G, S)$ is espousable. Note that $\mathscr{P}(G, S) = \mathscr{P}(G-x, S')$, and hence $\Pi(G, S) - x = \Pi(G-x, S')$, which is inespousable. By Lemma 3.6 it follows that there exists a critical subset $\mathscr{C}$ of $\mathscr{P}(G, S)$ such that $x \in K[\mathscr{C}]$. Since $\mathscr{C}$ is critical, $K[\mathscr{C}]$ is pressing, and this is the set desired in the lemma. ∎

LEMMA 4.5. *Let S be pressing. A subset R of $C(S)$ is pressing in $G[C(S)]$ if and only if $S \cup R$ is pressing in G.*

Proof. The lemma follows from the fact that $\mathscr{P}(G, S') = \mathscr{P}(G, S) \cup \mathscr{P}(G[C(S)], R)$ and Lemma 3.7(i). ∎

LEMMA 4.6. *Let $z \in V$ and suppose that S is pressing in $G-z$ and R is a subset of $C = C(S \cup \{z\}) \cup \{z\}$ such that $z \in R$ and R is pressing in $G[C]$. Then $S \cup R$ is pressing in G.*

Proof. Note that $\mathscr{P}(G, S \cup R) = \mathscr{P}(G-z, S) \cup \mathscr{P}(G[C], R)$. The lemma now follows by an application of Lemma 3.5, with Γ replaced by $\Pi(G, S \cup R)$, B replaced by $\mathscr{P}(G[C], R)$, and C replaced by $\mathscr{P}(G-z, S)$. ∎

We define a partial order $\succ$ on the set of pressing sets by $S_1 \succ S_2$ if $S_1 \supseteq S_2$ and $S_1 \backslash S_2 \subseteq C(S_2)$. By Lemma 4.6 it follows that then $S_1 \backslash S_2$ is pressed in $G[S_1]$.

LEMMA 4.7. *If $(S_\alpha : \alpha < \zeta)$ is a $(\prec)$-ascending sequence of pressing sets then $S = \bigcup_{\alpha < \zeta} S_\alpha$ is pressing and $S \succ S_\alpha$ for any $\alpha < \zeta$.*

Proof. Write $U_\alpha = S_{\alpha+1} \backslash S_\alpha$ and $\mathscr{P}_\alpha = \mathscr{P}(G[C(S_\alpha)], U_\alpha)$ $(\alpha < \zeta)$. We are assuming that $S_0 = \varnothing$. Then $\mathscr{P}(G, S) = \bigcup_{\alpha < \zeta} \mathscr{P}_\alpha$ and for every $\beta < \mathscr{C}$ there holds $\mathscr{P}^\beta = \mathscr{P}(G, S_\beta) = \bigcup_{\alpha < \beta} \mathscr{P}_\alpha$. Since in $\Pi(G, S)$ there holds $K[\mathscr{P}^\beta] \subseteq S_\beta$ (this follows from the definition of $\mathscr{P}^\beta$ as a set of connected components in $G - S_\beta$), each $\mathscr{P}^\beta$ is critical in $\Pi(G, S)$. The lemma now follows by Lemma 3.7(ii) from the fact that $\mathscr{P}(G, S) = \bigcup_{\beta < \zeta} \mathscr{P}^\beta$, which was shown above. ∎

By Zorn's lemma there follows:

COROLLARY 4.7a. *Every pressed set is contained in a $(\prec)$-maximal pressed set.*

V. A LOWER BOUND FOR THE NUMBER OF 1-FACTORS

As already mentioned, the main aim of this paper is to prove:

THEOREM 5.1. *If $n \geqslant 3$ and G is infinite, n-connected, and matchable, then $f(G) \geqslant n!$*

The proof will require the following preliminary results:

THEOREM 5.2 [11]. *If G is an infinite matchable and bicritical graph then $f(G)$ is infinite.*

The following is a generalization of a result in [6]:

THEOREM 5.3. *If Γ is espousable and $d(a) \geqslant n$ for every $a \in M$, then $e(\Gamma) \geqslant n!$*

Proof. By induction on n. Assume first that Γ is loose. Let $a \in M$. By Lemma 3.6 for every $w \in K\langle a \rangle$ the graph $\Gamma - w$ is espousable, and hence so is $\Gamma_w = \Gamma - \{a, w\}$. By the induction assumption $e(\Gamma_w) \geqslant (n-1)!$. Since $|K\langle a \rangle| \geqslant n$, and since every edge (a, w), where $w \in K\langle a \rangle$, can be extended to a matching of Γ by any matching in Γ_w, it follows that $e(\Gamma) \geqslant n!$

Assume now that Γ contains a non-empty critical set C. By Lemma 3.3 we may assume that C is minimal. Write $\tilde{\Gamma} = \Gamma[C \cup K(C)]$. Let F be an espousal of Γ. Then $F[C] = K[C]$, hence $F[M \backslash C] \subseteq W \backslash K[C]$, and hence

$\Gamma - C - K[C]$ is espousable. Hence the theorem will follow if we prove that $e(\tilde{\Gamma}) \geqslant n!$. Let $a \in C$ and $w \in K\langle a \rangle$. By Lemma 3.6, $\tilde{\Gamma} - \{a, w\}$ is espousable, since otherwise there would exist a critical set in $\tilde{\Gamma} - a$, contradicting the minimality of C. Thus every edge incident with a can be extended to an espousal of $\tilde{\Gamma}$. Since there are at least n such edges it follows by the induction hypothesis that $e(\tilde{\Gamma}) \geqslant n!$ ▮

THEOREM 5.4. *Let $n \geqslant 3$. If Γ is matchable and $\sigma(\Gamma) = 1$ and $d(m) \geqslant n$ for any $m \in M$, then $f(\Gamma) \geqslant n!$*

Proof.

Case a. There exists a non-empty critical subset C of M. Then clearly every espousal of $\Gamma[C, K[C]])$ can be extended to a matching of Γ. Since by Theorem 5.3 $e(\Gamma[C, K[C]]) \geqslant n!$, it follows that $f(\Gamma) \geqslant n!$.

Case b. Γ is loose.

Let F be a matching of Γ and H an espousal of Γ such that $W \backslash H[M] = \{z\}$ for some $z \in W$. Let $A = (w_0 = z, m_1, w_1, m_2, \ldots)$ be the infinite $F-H$-alternating path starting at z and let $L = M \cap V(A)$, $N = M \backslash L$, $X = W \cap V(A)$, $Y = W \backslash X$. By changing H, if necessary, we may assume that $H \restriction N = F \restriction N$ (here "$\restriction$" means "restricted to"). Write then $I = H \restriction N = H \cap F$. Note that, by the definition of σ, the graph $\Gamma - z$ is pressed. If there exist infinitely many F-alternating circuits, then $f(G)$ is infinite. Hence we may assume that there exist only finitely many F-alternating circuits, or, equivalently, that the set B of vertices lying on such circuits is finite. Let k be largest such that $m_k \in B$ (if $B \cap L = \varnothing$ let $k = 0$).

Write $J = K \backslash (F \cup H)$. Since $|K\langle m \rangle| \geqslant 3$ for each $m \in M$, we have $|J\langle m \rangle| \geqslant 1$ for every $m \in L$ and $|J\langle m \rangle| \geqslant 2$ for every $m \in N$.

For each $q < \omega$ let C_q be the set of vertices in N lying on an I-walk starting at m_q.

ASSERTION 1. *If $q > k$ then there does not exist an F-walk from m_q to w_i for any $i < q$ (in particular, $(m_q, w_i) \notin K$ for $i < q - 1$).*

Proof. Suppose that such a walk T exists. We may clearly assume that w_i is the first vertex on T of the form w_j, $j < q$. Then the circuit $T * (w_i, m_{i+1}, w_{i+1}, \ldots, w_{q-1}, m_q)$ is F-alternating and contains m_q. Thus $m_q \in B$, contradicting the definition of k. ▮

ASSERTION 2. *Let $q > k$. If Q is an I-alternating path from m_q to w_i and $p > i$, then $C_p \cap V(Q) = \varnothing$.*

Proof. If $C_p \cap V(Q) \neq \varnothing$, then there exists an I-walk T from m_p to w_i, and then $T * (w_i, m_i + 1, w_{i+1}, \ldots, m_p)$ is an F-alternating circuit, contradicting the definition of k. ▮

ASSERTION 3. *Let $q > k$. At least one of the following cases holds*:

1. *There exists $w_i \in J\langle m_q \rangle$ for some $i > q$.*

2. *There exists an I-alternating path $Q = Q_q$ from m_q to w_i for some $i \geqslant q$, such that, denoting by d the vertex preceding w_i on Q, there holds either*:

(2A) *d lies on some $I \cap E(Q)$-alternating circuit or*

(2B) *$w_j \in J\langle d \rangle$ for some $q \leqslant j < i$.*

Proof. Since $J\langle m_q \rangle \neq \varnothing$ it follows by Assertion 1 that if Case 1 does not hold then there exists $t \in J\langle m_q \rangle \cap Y$, implying that $C_1 \neq \varnothing$. By Lemma 3.1, $\Gamma_q = \Gamma[C_q, I[C_q]]$ is pressed, and hence, by Lemma 3.3, it contains a minimal non-empty critical set D_q. Suppose that Case 2B does not hold. Then, by Assertion 1, $|K\langle d \rangle \cap X| \leqslant 1$ for any $d \in D_q$ and hence $|K\langle d \rangle \cap I[D_q]| \geqslant 2$ for any such d. Applying Corollary 3.6b to Γ_q we have that every vertex in D_q participates in an I-alternating circuit. Since, by our assumption, Γ is loose, D_q cannot be critical in Γ, which means that $K\langle d \rangle \cap X \neq \varnothing$ for some $d \in D_q$. This means that Case 2A holds. ▌

We can now conclude the proof of Theorem 5.4. We construct a sequence (l_j) of elements of L in the following way. Let $l_1 = m_{k+1}$. Assume that l is already chosen and that $l_j = m_q$ for some $q > k$. If Case 1 or Case 2B occurs for m_q, let $l_{j+1} = m_i$ (here and below we use the notation of Assertion 3). If neither Case 1 nor Case 2B holds for m_q, and m_i is as in Case 2A, let $l_{j+1} = m_{i+1}$. Note now that there must be infinitely many j's at which Case 2A occurs. If not, then from a certain j_0 only Cases 1 and 2B occur. For each $j > j_0$ let $P_j = Q_q$ in Case 2B, and $P_j = (m_q, w_i)$ if Case 1 occurs (here m_q is l_j). Then, by Assertion 2, $*_{j > j_0} P_j$ is an infinite H-alternating path, contradicting, by Lemma 3.2, the tightness of $\Gamma - z$. By Assertion 2 every j at which Case 2A occurs gives rise to a distinct I-alternating circuit. Since there are infinitely many j's at which Case 2B occurs, this implies the existence of infinitely many different F-alternating circuits, which means that $f(\Gamma)$ is infinite. ▌

Remarks. The assumption "$\sigma(\Gamma) = 1$" could be replaced by "$1 \leqslant \sigma(\Gamma) \leqslant n - 2$," but this is not necessary in the proof of Theorem 5.1.

Theorem 5.4 fails for $n = 2$, and this is the only point at which the assumption that $n \geqslant 3$ is used.

LEMMA 5.5. *Let G be matchable and n-connected. Suppose that G contains a pressing set S such that either*

(i) *$|S| > 1$ or*

(ii) *$C(S) \neq \varnothing$.*

Then $f(G) \geqslant n!$.

Proof. The removal of S separates any $P \in \mathscr{P}(S)$ from any other component in $\mathscr{P}(S)$, as well as from $C(S)$. Hence the degree of P in $\Pi(S)$ is at least n. Thus $f(G) \geqslant e(\Pi) \geqslant n!$, by Theorem 5.3 and Lemma 4.3. ∎

Proof of Theorem 5.1. By Theorem 5.2 we may assume that G is not bicritical. Let x, y be vertices such that $G - \{x, y\}$ is unmatchable. By Theorem 4.1 there exists $S' \subseteq V \backslash \{x, y\}$ such that $\Pi(G - \{x, y\}, S')$ is inespousable. Let $S = S' \cup \{x, y\}$ and write $\Pi = \Pi(S)$, $\mathscr{P} = \mathscr{P}(S)$, $T = T(S)$, $C = C(S)$. Then $\Pi - \{x, y\} = \Pi(G - \{x, y\}, S')$ is inespousable, while Π itself is espousable, by the converse of Theorem 4.1.

Let $z \in S$. If $\Pi - z$ is inespousable then, by Lemma 3.6, there exists a pressing subset U of S, containing z. Since $|S| \geqslant 2$, either $|U| \geqslant 2$ or $C(U) \neq \varnothing$ (note that $S \backslash U \subseteq C(U)$.) Hence the theorem follows in this case, by Lemma 5.5. Thus we may assume that

(A) $\Pi - z$ is espousable for every $z \in S$.

Let $\Pi' = \Pi - x$. The graph Π' is espousable, by (A), whereas $\Pi' - y$ is not. Hence, by Lemma 3.6, there exists a subset R' of $S \backslash \{x\}$ such that $y \in R'$ and $R' = K_{\Pi'}[\mathscr{C}]$ for some critical set $\mathscr{C}$ in Π'. Let $R = R' \cup \{x\}$. Clearly then, $\mathscr{C} \subseteq \mathscr{P}(R)$. On the other hand, if $\mathscr{P}(R) \backslash \mathscr{C} \neq \varnothing$ then it is easy to check that Π' cannot be espousable, contrary to our assumption. Thus $\mathscr{P}(R) = \mathscr{C}$. By Lemma 3.4 we deduce that $\sigma(\Pi(R)) = 1$. Replacing, if necessary, S by R, we may assume

(B) $\sigma(\Pi) = 1$,

which, by Lemma 3.4 and (A) (it is easy to see that (A) still holds after replacing S by R), implies

(C) $\Pi - z$ is pressed for every $z \in S$.

Let F be a matching of G and let $H = \pi(F, S)$. By (B) there holds $|S \backslash H[\mathscr{P}]| \leqslant 1$. Assume first that $H[\mathscr{P}] = S$. Since G is n-connected and $|S| \geqslant 2$, there holds $d_{\Pi}(P) \geqslant n$ for every $P \in \mathscr{P}$. By Theorem 5.3 this implies that $f(\Pi) \geqslant n!$. But the fact that $H[\mathscr{P}] = S$ means that $F[C(S)] = C(S)$; i.e., $F' = F \upharpoonright C(S)$ is a matching of $G[C(S)]$. Hence every matching of $G[T(S)]$ can be extended by F' to a matching of G, and hence $f(G) \geqslant n!$.

Next consider the case that $|S \backslash H[\mathscr{P}]| = 1$. Let $\{z\} = S \backslash H[\mathscr{P}]$. By (C), $\tilde{S} = S \backslash \{z\}$ is pressing in $\tilde{G} = G - z$. By Corollary 4.7a there exists a pressing set $\hat{S} \succcurlyeq \tilde{S}$ such that $\hat{S} \succcurlyeq \tilde{S}$ and $\hat{S}$ is $(\succ)$-maximal (all in $\tilde{G}$). Replacing, if necessary, $\tilde{S}$ by $\hat{S}$, we may assume that $\tilde{S}$ itself is $(\succ)$-maximal.

Write $D = G[C(S) \cup \{z\}]$. Clearly, $F \upharpoonright V(D)$ is a matching of D, and hence D is matchable. Note that this implies that $C(S) \neq \varnothing$. Suppose, if possible, that $\tilde{D} = D - z$ is unmatchable. Then, by Theorem 5.1, there exists a subset R of $V(\tilde{D}) = C(S)$ such that $\Pi(\tilde{D}, R)$ is inespousable. Since

$\Pi(D, R \cup \{z\})$ is espousable, it follows by Lemma 3.6 that there exists a critical subset $\mathscr{B}$ of $\mathscr{P}(D, R \cup \{z\})$ such that $z \in K[\mathscr{B}]$. Then $S \cup K[\mathscr{B}]$ is easily seen to be pressing in G, and the theorem follows by Lemma 5.5.

Assume next that $\tilde{D} - u$ is unmatchable for some $u \in V(\tilde{D})$. Then, by Lemma 3.6, there exists a pressing subset Q of $V(\tilde{D})$, containing u. By Lemma 4.5, $Q \cup \tilde{S}$ is pressing in $\tilde{G}$, contradicting the maximality of $\tilde{S}$. Hence we may also assume that

(D) $\tilde{D} - u$ is matchable for every $u \in V(\tilde{D})$.

Since S separates $C(S)$ from $P(S)$ there holds

(E) $|E[C(S)] \cap S| \geqslant n$.

Let (u, t) be an edge such that $u \in C(S)$ and $t \in S$. By (A) and (B), $\Pi - t$ is matchable. Since in Π there holds $|K\langle P \rangle| \geqslant n$ for every $P \in \mathscr{P}$, in $\Pi - t$ there holds $|K\langle P \rangle| \geqslant n - 1$. By Theorem 5.3 it follows that $e(\Pi - t) \geqslant (n-1)!$, which by (B) means that $f(\Pi - t) \geqslant (n-1)!$. Applying the operation ζ to these matchings, we see that $f(G[T(S) \backslash t]) \geqslant (n-1)!$. By (D), $D - u$ is matchable. Thus each edge (u, t) $(u \in C(S), t \in S)$ can be extended in at least $(n-1)!$ ways to perfect matchings of G, each containing only one edge joining $C(S)$ to S. By (E) it follows that $f(G) \geqslant n!$. ∎

REFERENCES

1. R. AHARONI, Matchings in infinite graphs, *J. Combin. Theory Ser. B*, in press.
2. R. AHARONI, C. ST. J. A. NATH-WILLIAMS, AND S. SHELAH, A general criterion for the existence of transversals, *Proc. London Math. Soc.* **47** (1983), 43–68.
3. L. W. BEINEKE AND M. D. PLUMMER, On the 1-factors of non-separable graphs, *J. Combin. Theory* **2** (1967), 285–289.
4. F. BRY, On the number of 1-factors of locally finite graphs, *J. Combin. Theory Ser. B* **34** (1983), 48–57.
5. T. GALLAI, On factorization of graphs, *Acta Math. Acad. Sci. Hungar.* **1** (1950), 133–153.
6. M. HALL, Distinct representatives of substants, *Bull. Amer. Math. Soc.* **54** (1948), 922–926.
7. L. LOVÁSZ, On the structure of factorizable graphs, *Acta Math. Acad. Sci. Hungar.* **23** (1972), 179–185.
8. W. MADER, 1-Faktoren von graphen, *Math. Ann.* **201** (1973), 269–282.
9. K. P. PODEWSKI AND K. STEFFENS, Injective choice functions for countable families, *J. Combin. Theory Ser. B* **21** (1976), 40–46.
10. J. ZAKS, On the 1-factors of n-connected graphs, *J. Combin. Theory Ser. B* **11** (1971), 169–180.
11. MAO LIN ZHENG, A theorem on the 1-factors of locally finite 1-critical graphs, *Acta Math. Appl. Sinica* **10**, No. 2 (1987), 211–214.