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Some asymptotics for Sobolev orthogonal polynomials involving Gegenbauer weights[☆]

Cleonice F. Bracciali^{a,*}, Laura Castaño-García^b, Juan J. Moreno-Balcázar^{b,c}^a DCCE, IBILCE, UNESP - Univ Estadual Paulista, 15054-000 São José do Rio Preto, SP, Brazil^b Dpto de Estadística y Matemática Aplicada, Universidad de Almería, La Cañada de San Urbano s/n, 04120, Almería, Spain^c Instituto Carlos I de Física Teórica y Computacional, Spain

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ABSTRACT

We consider the Sobolev inner product

$$(f, g) = \int_{-1}^1 f(x)g(x)(1-x^2)^{\alpha-\frac{1}{2}} dx + \int f'(x)g'(x)d\psi(x), \quad \alpha > -\frac{1}{2},$$

where $d\psi$ is a measure involving a Gegenbauer weight and with mass points outside the interval $(-1, 1)$. We study the asymptotic behaviour of the polynomials which are orthogonal with respect to this inner product. We obtain the asymptotics of the largest zeros of these polynomials via a Mehler–Heine type formula. These results are illustrated with some numerical experiments.

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1. Introduction

Let $\{P_n^\psi\}_{n=0}^\infty$ be a sequence of monic orthogonal polynomials with respect to the inner product

$$(f, g)_\psi = \int_{-\xi}^{\xi} f(x)g(x)d\psi(x),$$

where $d\psi$ is a symmetric positive measure with bounded support $I = [-\xi, \xi]$, $\xi > 0$.

We consider the Sobolev inner product

$$(f, g)_S = \int_{-\xi_0}^{\xi_0} f(x)g(x)d\psi_0(x) + \int_{-\xi_1}^{\xi_1} f'(x)g'(x)d\psi_1(x), \quad (1)$$

where $d\psi_i$, $i = 0, 1$, are symmetric positive measures with bounded support $I_i = [-\xi_i, \xi_i]$, respectively. We denote by $\{S_n\}_{n=0}^\infty$ the sequence of monic orthogonal polynomials with respect to the inner product (1), which are known as Sobolev orthogonal polynomials. We denote $\tilde{k}_n = \langle S_n, S_n \rangle_S$.

The literature on Sobolev orthogonal polynomials is very wide and different surveys have been published along the last two decades, see for example, [1,2]. Thus, we only highlight some facts of this history useful to support our contribution.

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* Corresponding author. Tel.: +55 17 32212213; fax: +55 17 32212203.

E-mail addresses: cleonice@ibilce.unesp.br (C.F. Bracciali), lcastano@ual.es (L. Castaño-García), balcazar@ual.es (J.J. Moreno-Balcázar).

At the end of the 80's (published later) Iserles et al. [3] introduced the concepts of coherent pairs and symmetrically coherent pairs of measures. Thus, they could consider Sobolev inner products where the polynomials orthogonal with respect to these nonstandard inner products satisfied nice properties from the numerical and algebraic point of view. Later, other authors showed that these properties are very adequate to obtain the asymptotics of Sobolev polynomials (see the surveys mentioned above). In 1997 Meijer gave in [4] a complete classification of all coherent pairs and symmetrically coherent pairs. However, later it was observed that coherence was not necessary to have good algebraic relations between Sobolev orthogonal polynomials and the standard ones. In this context we can consider the articles in [5,6]. Here, we consider the following Sobolev inner product

$$\langle f, g \rangle = \int_{-1}^1 f(x)g(x)(1-x^2)^{\alpha-\frac{1}{2}} dx + \int f'(x)g'(x)d\psi(x), \quad \alpha > -\frac{1}{2}, \quad (2)$$

where $d\psi$ is a measure involving a Gegenbauer weight and with mass points outside the interval $(-1, 1)$. This nonstandard inner product was introduced in [5] and some asymptotic properties were obtained in [7]. Now, we complete this study. Using Mehler–Heine type formulas, we can describe their largest zeros asymptotically (or smallest ones due to the symmetry of the polynomials). Furthermore, we will show that in some sense the inner product (2) is balanced, which we will explain in Section 3. This type of balance is different from the one considered in [8,9]. The structure of the paper is the following: in Section 2 we give some properties of the classical Gegenbauer polynomials that we will use along the paper. In Section 3 we introduce in detail the inner product (2) and some properties of the corresponding polynomials. Finally, in Section 4 we introduce our results: Mehler–Heine type formulas for Sobolev polynomials including a special case that occurs when we put masses on the extremes of the interval $[-1, 1]$. The asymptotic behaviour of the zeros of these polynomials is described in terms of Bessel functions of the first kind and illustrated with numerical experiments.

2. Gegenbauer polynomials: basic facts and asymptotics

We denote by $\{G_n^{(\alpha)}\}_{n=0}^{\infty}$ the sequence of monic Gegenbauer polynomials that are orthogonal in $[-1, 1]$ with respect to the measure

$$d\psi^{(\alpha)}(x) = (1-x^2)^{\alpha-\frac{1}{2}} dx,$$

with $\alpha > -1/2$. It is known that they satisfy $\frac{d}{dx}G_n^{(\alpha)}(x) = nG_{n-1}^{(\alpha+1)}(x)$. Let $k_n^{(\alpha)} = \langle G_n^{(\alpha)}, G_n^{(\alpha)} \rangle_{\psi^{(\alpha)}}$ be the square of the norm.

Monic Gegenbauer polynomials can be written as (see [10, p. 80] where another normalization is used):

$$2^n G_n^{(\alpha)}(x) = \frac{\sqrt{\pi}}{2^{2\alpha-1}} \frac{\Gamma(n+1)\Gamma(n+2\alpha)}{\Gamma(n+\alpha)\Gamma(n+\alpha+1/2)} P_n^{(\alpha-\frac{1}{2}, \alpha-\frac{1}{2})}(x), \quad n \geq 1, \quad (3)$$

$$G_0^{(\alpha)}(x) = 1$$

where $\alpha > -1/2$, Γ is the Gamma function and $P_n^{(\alpha, \beta)}$ are Jacobi orthogonal polynomials with the standard normalization appearing, for example, in Szegő's book (see [10, p. 58]). We also use the following notation throughout this paper

$$G_n^{(\alpha)}(x) = \frac{\sqrt{\pi}}{2^{2\alpha-1}} r_n^{(\alpha)} P_n^{(\alpha-\frac{1}{2}, \alpha-\frac{1}{2})}(x), \quad \alpha > -\frac{1}{2}, \quad n \geq 1, \quad (4)$$

where $r_n^{(\alpha)} = \frac{\Gamma(n+1)\Gamma(n+2\alpha)}{2^n \Gamma(n+\alpha)\Gamma(n+\alpha+1/2)}$.

In this work we use two well-known algebraic relations for monic Gegenbauer polynomials, one of them is the three-term recurrence relation:

$$G_{n+1}^{(\alpha)}(x) = xG_n^{(\alpha)}(x) - \frac{n(n+2\alpha-1)}{4(n+\alpha)(n+\alpha-1)} G_{n-1}^{(\alpha)}(x), \quad n \geq 1, \quad (5)$$

$$G_{n+1}^{(\alpha-1)}(x) = G_{n+1}^{(\alpha)}(x) - \frac{n(n+1)}{4(n+\alpha)(n+\alpha-1)} G_{n-1}^{(\alpha)}(x), \quad n \geq 1. \quad (6)$$

For our purposes we need to know some asymptotic properties of Gegenbauer polynomials. We begin with the asymptotic behaviour of their zeros. One way to derive their asymptotics is through Mehler–Heine formula which, for Jacobi orthogonal polynomials, is

$$\lim_{n \rightarrow \infty} \frac{P_n^{(\alpha, \beta)}(\cos(x/n))}{n^\alpha} = \left(\frac{x}{2}\right)^{-\alpha} J_\alpha(x),$$

where J_α is the Bessel function of the first kind and α and β are arbitrary real numbers (see [10, Th. 8.1.1]). It holds uniformly on every bounded region of the complex plane \mathbb{C} . In fact, we have for a fixed $j \in \mathbb{Z}$

$$\lim_{n \rightarrow \infty} \frac{P_n^{(\alpha, \beta)}(\cos(x/(n+j)))}{n^\alpha} = \left(\frac{x}{2}\right)^{-\alpha} J_\alpha(x), \quad (7)$$

uniformly on compact subsets of \mathbb{C} . Thus, taking into account the limit relation

$$\lim_{n \rightarrow \infty} \frac{n^{b-a} \Gamma(n+a)}{\Gamma(n+b)} = 1, \tag{8}$$

and the formulas (4) and (7), we deduce the Mehler–Heine formula for monic Gegenbauer orthogonal polynomials, that is,

$$\lim_{n \rightarrow \infty} \frac{2^n G_n^{(\alpha)}(\cos(x/(n+j)))}{n^\alpha} = \sqrt{\pi} (2x)^{\frac{1}{2}-\alpha} J_{\alpha-\frac{1}{2}}(x), \tag{9}$$

uniformly on compact subsets of \mathbb{C} and uniformly for $j \in \mathbb{Z}$. This result has important consequences in the asymptotic behaviour of the zeros of Gegenbauer polynomials. Since these polynomials are symmetric, that is, $G_n^{(\alpha)}(-x) = (-1)^n G_n^{(\alpha)}(x)$, we only need to pay attention to the positive zeros located in $[0, 1]$. So, from (9) and applying Hurwitz’s Theorem we have for a fixed $i \in \mathbb{N}$,

$$\lim_{n \rightarrow \infty} n \arccos(x_{n,i}^{(\alpha)}) = j_i^{(\alpha-\frac{1}{2})},$$

where $j_i^{(\alpha-\frac{1}{2})}$ is the i th positive zero of $J_{\alpha-\frac{1}{2}}$ and $\arccos(x_{n,i}^{(\alpha)}) \in (0, \pi)$ being $x_{n,i}^{(\alpha)}$ the $([n/2] - i + 1)$ th positive zero of Gegenbauer polynomial $G_n^{(\alpha)}(x)$, i.e., let $m = [n/2]$ be the number of positive zeros of $G_n^{(\alpha)}(x)$ then $x_{n,m}^{(\alpha)} < x_{n,m-1}^{(\alpha)} < \dots < x_{n,1}^{(\alpha)}$, where $[r]$ means the largest integer less than or equal to r . Observe that formula (9) describes asymptotically the largest zeros of Gegenbauer polynomials quite well.

Finally, we introduce another family of orthogonal polynomials related to Gegenbauer polynomials that we will need later. In fact, we consider the measure

$$d\psi^{(\alpha,q)}(x) = \frac{(1-x^2)^{\alpha+\frac{1}{2}}}{1+qx^2} dx + M^{(q)} \left(\delta \left(\frac{-1}{\sqrt{-q}} \right) + \delta \left(\frac{1}{\sqrt{-q}} \right) \right),$$

where $\alpha > -1/2, q \geq -1$, and

$$M^{(q)} \begin{cases} =0, & \text{if } q \geq 0, \\ \geq 0, & \text{if } -1 \leq q < 0. \end{cases} \tag{10}$$

We denote by $\{G_n^{(\alpha,q)}\}_{n=0}^\infty$ the sequence of monic polynomials orthogonal with respect to the inner product

$$\begin{aligned} \langle f, g \rangle_{\psi^{(\alpha,q)}} &= \int_{-1}^1 f(x)g(x) d\psi^{(\alpha,q)}(x) = \int_{-1}^1 f(x)g(x) \frac{(1-x^2)^{\alpha+\frac{1}{2}}}{1+qx^2} dx \\ &\quad + M^{(q)} \left(f \left(\frac{-1}{\sqrt{-q}} \right) g \left(\frac{-1}{\sqrt{-q}} \right) + f \left(\frac{1}{\sqrt{-q}} \right) g \left(\frac{1}{\sqrt{-q}} \right) \right), \end{aligned}$$

and $k_n^{(\alpha,q)} = \langle G_n^{(\alpha,q)}, G_n^{(\alpha,q)} \rangle_{\psi^{(\alpha,q)}}$.

The polynomials $G_n^{(\alpha,q)}$ are related to Gegenbauer polynomials through

$$G_n^{(\alpha,q)}(x) = G_n^{(\alpha+1)}(x) + d_{n-2}^{(\alpha,q)} G_{n-2}^{(\alpha+1)}(x), \quad n \geq 2, \tag{11}$$

where $G_0^{(\alpha,q)}(x) = 1, G_1^{(\alpha,q)}(x) = x$, and $d_{n-2}^{(\alpha,q)} = \frac{qk_n^{(\alpha,q)}}{k_{n-2}^{(\alpha+1)}}$, see [11].

The zeros of $G_n^{(\alpha,q)}$ are real, simple and lie symmetrically within the interval $(-\xi^{(q)}, \xi^{(q)})$, where

$$\xi^{(q)} = \begin{cases} 1/\sqrt{-q}, & \text{if } -1 \leq q < 0 \text{ and } M^{(q)} > 0, \\ 1, & \text{otherwise.} \end{cases}$$

More details about the zeros of these polynomials can be found in [11].

3. Gegenbauer–Sobolev orthogonal polynomials: the known facts

We consider the Sobolev inner product introduced in [5]

$$\begin{aligned} \langle f, g \rangle_S &= \langle f, g \rangle_{\psi^{(\alpha)}} + \kappa_1 \langle f', g' \rangle_{\psi^{(\alpha+1)}} + \kappa_2 \langle f', g' \rangle_{\psi^{(\alpha,q)}} \\ &= \int_{-1}^1 f(x)g(x)(1-x^2)^{\alpha-\frac{1}{2}} dx + \kappa_1 \int_{-1}^1 f'(x)g'(x)(1-x^2)^{\alpha+\frac{1}{2}} dx \\ &\quad + \kappa_2 \left[\int_{-1}^1 f'(x)g'(x) \frac{(1-x^2)^{\alpha+\frac{1}{2}}}{1+qx^2} dx + M^{(q)} \left(f' \left(\frac{-1}{\sqrt{-q}} \right) g' \left(\frac{-1}{\sqrt{-q}} \right) + f' \left(\frac{1}{\sqrt{-q}} \right) g' \left(\frac{1}{\sqrt{-q}} \right) \right) \right], \end{aligned} \tag{12}$$

where $\alpha > -1/2, q \geq -1, \kappa_1 \geq 0, \kappa_2 > 0$ and $M^{(q)}$ as defined in (10).

Observe that if $\kappa_1 \neq 0$, the pair of measures $\{d\psi^{(\alpha)}, \kappa_1 d\psi^{(\alpha+1)} + \kappa_2 d\psi^{(\alpha,q)}\}$ do not form a symmetrically coherent pair according to Meijer’s classification given in [4]. Moreover, when $\kappa_2 = 0$ we have a trivial situation, being Sobolev polynomials the Gegenbauer orthogonal polynomials $G_n^{(\alpha)}$.

As we have written in the introduction, the study of these orthogonal polynomials is motivated by two facts. First, they are more general than the symmetrically coherent pairs of Gegenbauer type II. Furthermore, we think that the main reason to pay attention to this nonstandard inner product is to balance (12) in some sense. That is, if we take $\kappa_1 = 0$ in (12) we know that the measure playing the main role in the asymptotic behaviour is $d\psi^{(\alpha,q)}$ because we have a quadratic factor n^2 out of the derivatives when we use monic polynomials (see, for example, [2] for more details). However, when $\kappa_1 \neq 0$ we can observe that Gegenbauer polynomials $G_n^{(\alpha)}$ are orthogonal with respect to $\langle f, g \rangle_{\psi^{(\alpha)}} + \kappa_1 \langle f', g' \rangle_{\psi^{(\alpha+1)}}$, and now we also have a quadratic factor n^2 in the “first part” of (12). Therefore, the two parts of (12), $\langle f, g \rangle_{\psi^{(\alpha)}} + \kappa_1 \langle f', g' \rangle_{\psi^{(\alpha+1)}}$ and $\kappa_2 \langle f', g' \rangle_{\psi^{(\alpha,q)}}$, play a similar role and, in this way, the inner product (12) is now balanced. This way of balancing Sobolev inner product is different from the one considered in [8,9], where the authors considered varying Sobolev orthogonal polynomials.

We denote by $S_n = S_n^{(\alpha,q,\kappa_1,\kappa_2)}$, $n \geq 0$, the Gegenbauer–Sobolev polynomials orthogonal with respect to (12). In [5] the authors established the relation

$$S_{n+1}(x) + a_{n-1}S_{n-1}(x) = G_{n+1}^{(\alpha)}(x) + b_{n-1}^{(\alpha,q)}G_{n-1}^{(\alpha)}(x), \quad n \geq 2, \tag{13}$$

where $S_0(x) = 1$, $S_1(x) = G_1^{(\alpha)}(x) = x$, $S_2(x) = G_2^{(\alpha)}(x)$,

$$a_n = a_n^{(\alpha,q,\kappa_1,\kappa_2)} = b_n^{(\alpha,q)} \frac{k_n^{(\alpha)} + \kappa_1 n^2 k_{n-1}^{(\alpha+1)}}{\tilde{k}_n}, \quad n \geq 1,$$

and

$$b_n^{(\alpha,q)} = \frac{n+2}{n} d_{n-1}^{(\alpha,q)} = \frac{n+2}{n} \frac{qk_{n+1}^{(\alpha,q)}}{k_{n-1}^{(\alpha+1)}}, \quad n \geq 1. \tag{14}$$

Later in [7] the asymptotic behaviour of the coefficients in (13) has been established,

$$\lim_{n \rightarrow \infty} b_n^{(\alpha,q)} = \lim_{n \rightarrow \infty} d_n^{(\alpha,q)} = b^{(q)} = \begin{cases} \frac{1}{4} \Psi(q), & \text{if } q \geq -1 \text{ and } M^{(q)} = 0, \\ \frac{1}{4\Psi(q)}, & \text{if } -1 \leq q < 0 \text{ and } M^{(q)} > 0, \end{cases} \tag{15}$$

and

$$\lim_{n \rightarrow \infty} a_n^{(\alpha,q,\kappa_1,\kappa_2)} = a^{(q,\kappa_1,\kappa_2)} = \frac{1}{4} \Psi\left(\frac{q\kappa_1}{\kappa_1 + \kappa_2}\right), \tag{16}$$

where Ψ is a real function defined by $\Psi(x) = x/(1 + \sqrt{1+x})^2$, for $x \geq -1$. For $x > -1$, $|\Psi(x)| < 1$. Observe that $\text{sgn}(a^{(q,\kappa_1,\kappa_2)}) = \text{sgn}(q)$, and since $\frac{q\kappa_1}{\kappa_1 + \kappa_2} > -1$, then $|4a^{(q,\kappa_1,\kappa_2)}| < 1$.

The exterior asymptotics of these Sobolev polynomials have been also obtained in [7],

$$\lim_{n \rightarrow \infty} \frac{S_n(x)}{G_n^{(\alpha)}(x)} = \begin{cases} \frac{\Phi^2(x) + \Psi(q)}{\Phi^2(x) + \Psi(q\kappa_1/(\kappa_1 + \kappa_2))}, & \text{if } q \geq -1 \text{ and } M^{(q)} = 0, \\ \frac{\Phi^2(x) + 1/\Psi(q)}{\Phi^2(x) + \Psi(q\kappa_1/(\kappa_1 + \kappa_2))}, & \text{if } -1 \leq q < 0 \text{ and } M^{(q)} > 0, \end{cases}$$

uniformly on compact subsets of $\mathbb{C} \setminus [-1, 1]$,

$$\lim_{n \rightarrow \infty} \frac{S_n(x)}{G_n^{(\alpha,q)}(x)} = \left(\Phi^2(x) + \Psi\left(\frac{q\kappa_1}{\kappa_1 + \kappa_2}\right) \right)^{-1} \frac{2\Phi^2(x)}{\Phi'(x)} \tag{17}$$

and

$$\lim_{n \rightarrow \infty} \frac{S_n(x)}{S_{n+2}(x)} = \frac{4}{\Phi^2(x)},$$

uniformly on compact subsets of $\mathbb{C} \setminus [-\xi_q, \xi_q]$, where Φ is the complex function defined by $\Phi(z) = z + \sqrt{z^2 - 1}$, for $z \in \mathbb{C} \setminus [-1, 1]$. The square root in Φ is such that $\sqrt{z^2 - 1} > 0$ when $z > 1$. It is easy to show that $\Phi^2(\sqrt{x}) = -1/\Psi(-1/x)$.

Remark 1. It is important to observe that there are qualitative differences between the asymptotics appearing in (17) and the corresponding result in [12]. In any case, when $\kappa_1 = 0$, we have $\Psi(0) = 0$ and we recover the results for symmetrically coherent pairs of Gegenbauer type II.

Recently in [13] some results for the zeros of these Sobolev polynomials have been obtained under the assumptions

$$\alpha \geq 0, \quad \kappa_2 \geq \left(\frac{2\alpha + 3}{2\alpha + 2} + 2(1 + q) \right) \kappa_1 > 0. \tag{18}$$

When $-1 \leq q < 0$, S_n has n different real zeros and at least $n - 2$ of them lie inside the interval $(-1, 1)$. If $q > 0$, all the $2m + 1$ zeros of S_{2m+1} are real, simple and within the interval $(-1, 1)$, and S_{2m} has at least $2m - 2$ distinct real zeros in $(-1, 1)$. In fact, S_{2m} can have two complex zeros or $2m$ distinct real zeros in $(-1, 1)$. Since the polynomials S_n are symmetric, the two possible complex zeros would lie on the imaginary axis. From the above asymptotic results we can deduce that the zeros of S_n accumulate on \mathbb{R} .

Numerical experiments lead the authors to think that these results are true even if we remove the assumptions (18).

If we denote the m positive zeros of S_n by $s_{n,i}$, $i = 1, 2, \dots, m$, in decreasing order, $s_{n,m} < s_{n,m-1} < \dots < s_{n,2} < s_{n,1}$, applying again the above results when $-1 \leq q < 0$ and $M^{(q)} > 0$, and taking into account $\Phi^2(1/\sqrt{-q}) + 1/\Psi(q) = 0$, we obtain

$$\lim_{n \rightarrow \infty} s_{n,1} = \frac{1}{\sqrt{-q}} = \xi^{(q)} = \begin{cases} =1, & \text{if } q = -1, \\ >1, & \text{if } -1 < q < 0. \end{cases}$$

4. Asymptotics for Gegenbauer–Sobolev orthogonal polynomials: the new facts

In this section we want to describe the asymptotic behaviour of these Sobolev polynomials on compact subsets of the complex plane via Mehler–Heine type formulas which allow us to know the zeros of this family of polynomials asymptotically.

4.1. Mehler–Heine type formula for Gegenbauer–Sobolev orthogonal polynomials: general case

We need a slight modification of Lemma 3.2 in [14] to establish our result. We include here a proof of this result to clarify some technical steps in our case and to make the article more readable. We state this result in a more general form.

Lemma 1. *Let $\{c_n\}_{n=0}^\infty$ be a sequence of real numbers such that $\lim_{n \rightarrow \infty} c_n = c$ and $|c| < 1$. For $n \geq 0$, and $i = 1, 2, \dots, [n/2]$, let $t_i^{(n)} = \prod_{j=1}^i c_{n-2j}$ and $t_0^{(n)} = 1$ be. Then, there exist constants P and r where $P > 1$ and $0 < r < 1$ such that $|t_i^{(n)}| < Pr^i$ for all $n \geq 0$ and $0 \leq i \leq [n/2]$.*

Proof. Since $|c| < 1$, there exist $2n_0 \in \mathbb{N}$ and $r \in (|c|, 1)$ such that $|c_n| < r < 1$, for all $n \geq 2n_0$. Let $M = \max\{1, |c_0|, |c_1|, \dots, |c_{2n_0-1}|\}$.

(i) If $n \geq 2n_0$, then $[n/2] \geq n_0$. We have two possible cases:

- If $1 \leq i \leq [n/2] - n_0$, it means that $n - 2i \geq 2n_0$, and $|t_i^{(n)}| = \prod_{j=1}^i |c_{n-2j}| < r^i$.
- If $[n/2] - n_0 < i \leq [n/2]$, we can write

$$\begin{aligned} |t_i^{(n)}| &= \prod_{j=1}^{\lfloor \frac{n}{2} \rfloor - n_0} |c_{n-2j}| \prod_{j=\lfloor \frac{n}{2} \rfloor - n_0 + 1}^i |c_{n-2j}| < r^{\lfloor \frac{n}{2} \rfloor - n_0} M^{i - \lfloor \frac{n}{2} \rfloor + n_0} \\ &\leq r^{\lfloor \frac{n}{2} \rfloor - n_0} M^{n_0} \leq r^i \left(\frac{M}{r} \right)^{n_0}. \end{aligned}$$

(ii) If $n < 2n_0$, then $1 \leq i \leq [n/2] < n_0$ (it means that $n - 2i < 2n_0$) and

$$|t_i^{(n)}| = \prod_{j=1}^i |c_{n-2j}| < M^i = r^i \left(\frac{M}{r} \right)^i \leq r^i \left(\frac{M}{r} \right)^{n_0}.$$

Setting $P = (M/r)^{n_0}$ and since $|t_0^{(n)}| = 1$, we have proved the result. \square

Now, we define a new family of polynomials $\{R_n^{(q)}\}_{n=0}^\infty$ useful for our aim,

$$R_n^{(q)}(x) := G_n^{(\alpha)}(x) + b_{n-2}^{(\alpha,q)} G_{n-2}^{(\alpha)}(x), \tag{19}$$

with $b_0^{(\alpha,q)} = b_{-1}^{(\alpha,q)} = b_{-2}^{(\alpha,q)} = 0$. According to relation (13) we have

$$R_n^{(q)}(x) = S_n(x) + a_{n-2} S_{n-2}(x), \quad n \geq 0, \tag{20}$$

with $a_0 = a_{-1} = a_{-2} = 0$.

We can easily obtain a Mehler–Heine type formula for $R_n^{(q)}$.

Lemma 2. We have for a fixed $j \in \mathbb{Z}$ and $\alpha > -1/2$,

$$\lim_{n \rightarrow \infty} \frac{2^n R_n^{(q)}(\cos(x/(n+j)))}{n^\alpha} = (1 + 4b^{(q)})\sqrt{\pi} (2x)^{\frac{1}{2}-\alpha} J_{\alpha-\frac{1}{2}}(x),$$

uniformly on compact subsets of the complex plane.

Proof. From (19) we get

$$\frac{2^n R_n^{(q)}(\cos(x/(n+j)))}{n^\alpha} = \frac{2^n G_n^{(\alpha)}(\cos(x/(n+j)))}{n^\alpha} + 4b_{n-2}^{(\alpha,q)} \frac{(n-2)^\alpha}{n^\alpha} \frac{2^{n-2} G_{n-2}^{(\alpha)}(\cos(x/(n+j)))}{(n-2)^\alpha}.$$

The result follows from (9) and (15). \square

Now, we are ready to establish a Mehler–Heine type formula for Gegenbauer–Sobolev orthogonal polynomials.

Theorem 1. We have for $\alpha > -1/2$,

$$\lim_{n \rightarrow \infty} \frac{2^n S_n(\cos(x/n))}{n^\alpha} = \frac{1 + 4b^{(q)}}{1 + 4a^{(q,\kappa_1,\kappa_2)}} \sqrt{\pi} (2x)^{\frac{1}{2}-\alpha} J_{\alpha-\frac{1}{2}}(x), \tag{21}$$

uniformly on compact subsets of the complex plane.

Proof. Applying (20) recursively we obtain

$$S_n(x) = \sum_{i=0}^{\lfloor n/2 \rfloor} (-1)^i u_i^{(n)} R_{n-2i}^{(q)}(x), \quad n \geq 0, \tag{22}$$

where $u_i^{(n)} = \prod_{j=1}^i a_{n-2j}$, for $i = 1, 2, \dots, \lfloor n/2 \rfloor$, and $u_0^{(n)} = 1$.

From (22) we can write

$$\frac{2^n S_n(\cos(x/n))}{n^\alpha} = \sum_{i=0}^{\lfloor n/2 \rfloor} (-1)^i 4^i u_i^{(n)} \frac{(n-2i)^\alpha}{n^\alpha} \frac{2^{n-2i} R_{n-2i}^{(q)}(\cos(x/n))}{(n-2i)^\alpha}.$$

From (16) and Lemma 1 with $t_i^{(n)} = 4^i u_i^{(n)}$ we get

$$|4^i u_i^{(n)}| = \left| 4^i \prod_{j=1}^i a_{n-2j} \right| = |4a_{n-2} 4a_{n-4} \cdots 4a_{n-2i}| < Pr^i,$$

where $0 < r < 1$ and $P > 1$.

This bound and Lemma 2 allow us to use Lebesgue’s dominated convergence theorem, and we get

$$\lim_{n \rightarrow \infty} \frac{2^n S_n(\cos(x/n))}{n^\alpha} = (1 + 4b^{(q)})\sqrt{\pi} (2x)^{\frac{1}{2}-\alpha} J_{\alpha-\frac{1}{2}}(x) \sum_{i=0}^{\infty} (-4a^{(q,\kappa_1,\kappa_2)})^i,$$

from where the result follows. \square

Remark 2. When $q = -1$ we have a special situation: the previous theorem does not provide any asymptotic information since the value of the limit in (21) is 0. That is verified straightforwardly because, according to (15), $b^{(-1)} = -1/4$. Then, we need to face this particular case with another technique.

When $q > -1$ formula (21) is useful to obtain the asymptotic behaviour of the largest zeros of these polynomials. Applying Hurwitz’s Theorem to (21) we obtain the following result.

Corollary 1. Let α , κ_1 , and κ_2 satisfy the assumptions given in (18). Let

$$m = \begin{cases} \lfloor n/2 \rfloor - 1, & \text{if } S_n \text{ has 2 complex zeros or 2 zeros outside } (-1, 1), \\ \lfloor n/2 \rfloor, & \text{otherwise.} \end{cases}$$

We denote the m positive zeros of S_n within $(0, 1)$ by $s_{n,i}$, $i = 1, 2, \dots, m$, in decreasing order, i.e., $s_{n,m} < s_{n,m-1} < \cdots < s_{n,2} < s_{n,1}$.

Then, for $q > -1$,

$$\lim_{n \rightarrow \infty} n \arccos(s_{n,i}) = j_i^{(\alpha-\frac{1}{2})}, \quad i = 1, 2, \dots, m,$$

where $0 < j_1^{(\alpha-\frac{1}{2})} < j_2^{(\alpha-\frac{1}{2})} < \cdots < j_m^{(\alpha-\frac{1}{2})}$ denote the first m positive zeros of the Bessel function of the first kind $J_{\alpha-\frac{1}{2}}$.

Table 1

$n \arccos(s_{n,i})$, for $i = 1, 2, 3, \alpha = 3, q = -0.4, M^{(q)} = 3, \kappa_1 = 2, \kappa_2 = 20$. Notice that S_n has 2 zeros outside $(-1, 1)$.

$n \arccos(s_{n,i})$	$i = 1$	$i = 2$	$i = 3$	Zero > 1
$n = 25$	5.608397640	8.844163518	11.97246488	1.635434428
$n = 100$	5.724091837	9.032761282	12.23836022	1.593700525
$n = 200$	5.743679248	9.063781296	12.28059651	1.587340423
$n = 350$	5.752129190	9.077130391	12.29870740	1.584663432
$J_i^{(\alpha-1/2)}$	5.763459197	9.095011330	12.32294097	$\xi_q = 1.581138830$

Table 2

Largest zeros inside $(0, 1)$ with $\alpha = 3, q = -0.4, M^{(q)} = 3, \kappa_1 = 2, \kappa_2 = 20, e_{n,i} = |s_{n,i} - \cos(J_i^{(\alpha-1/2)}/n)|$.

n	$s_{n,1}$	$\cos(J_1^{(\alpha-1/2)}/n)$	$e_{n,1}$	$s_{n,2}$	$\cos(J_2^{(\alpha-1/2)}/n)$	$e_{n,2}$
25	0.9749420558	0.9735435181	0.139853e-2	0.9380745140	0.9345512664	0.352324e-2
100	0.9983621859	0.9983395866	0.225993e-4	0.9959232342	0.9958668887	0.563455e-4
200	0.9995876552	0.9995848105	0.284472e-5	0.9989732741	0.9989661878	0.708634e-5
350	0.9998649541	0.9998644216	0.532476e-6	0.9996637156	0.9996623895	0.132613e-5

Table 3

$n \arccos(s_{n,i})$, for $i = 1, 2, 3, \alpha = 3, q = -0.4, M^{(q)} = 0, \kappa_1 = 2, \kappa_2 = 20$.

$n \arccos(s_{n,i})$	$i = 1$	$i = 2$	$i = 3$
$n = 25$	5.101386830	8.055176038	10.92266038
$n = 100$	5.581087566	8.807331628	11.93336942
$n = 200$	5.670736310	8.948705320	12.12473715
$n = 350$	5.710092090	9.010797474	12.20884374
$J_i^{(\alpha-1/2)}$	5.763459197	9.095011330	12.32294097

Table 4

$n \arccos(s_{n,i})$ for $i = 1, 2, 3, \alpha = -0.2, q = -0.6, M^{(q)} = 3, \kappa_1 = 2, \kappa_2 = 20$. Notice that S_n has 2 zeros outside $(-1, 1)$.

$n \arccos(s_{n,i})$	$i = 1$	$i = 2$	$i = 3$	Zero > 1
$n = 25$	1.316464608	4.907267870	8.429230860	1.325988983
$n = 100$	1.205145459	4.497044361	7.740031149	1.299298735
$n = 200$	1.188102753	4.433509322	7.630890368	1.295111498
$n = 350$	1.180917365	4.406713972	7.584794584	1.293338547
$J_i^{(\alpha-1/2)}$	1.171454673	4.371391586	7.524002697	$\xi_q = 1.290994449$

Table 5

$n \arccos(s_{n,i})$, for $i = 1, 2, 3, \alpha = 3, q = 500, M^{(q)} = 0, \kappa_1 = 2, \kappa_2 = 10\,000$. Notice that S_{2n} has complex zeros for $n \leq 10$.

$n \arccos(s_{n,i})$	$i = 1$	$i = 2$	$i = 3$	Complex zeros
$n = 20$	5.239076892	8.268803396	11.20587846	$\pm 0.0064623301i$
$n = 25$	5.332779348	8.416009852	11.40411497	-
$n = 100$	5.646439388	8.910356058	12.07276354	-
$n = 200$	5.704086730	9.001319672	12.19599863	-
$n = 350$	5.729317016	9.041131446	12.24993922	-
$J_i^{(\alpha-1/2)}$	5.763459197	9.095011330	12.32294097	-

In Tables 1–5 we give some numerical examples which illustrate Corollary 1. The computation was done with 200 digits of precision. Observe that in Table 4 the parameter α does not satisfy the assumptions (18) but it is a good example to show that perhaps the restrictions in (18) could be relaxed as we commented previously.

4.2. Mehler–Heine type formula for Gegenbauer–Sobolev orthogonal polynomials: case $q = -1$

As we have just observed, we need to find another Mehler–Heine type formula for S_n when $q = -1$. Notice that when $q = -1$ and $\alpha > -1/2$ the polynomials $G_n^{(\alpha,-1)}$ are orthogonal with respect to the inner product

$$(f, g)_{\psi, (\alpha,-1)} = \int_{-1}^1 f(x)g(x)(1-x^2)^{\alpha-1/2} dx + M^{(-1)}(f(1)g(1) + f(-1)g(-1)).$$

First, we obtain a relation between $G_n^{(\alpha,-1)}$ and Gegenbauer polynomials with a technique inspired in [15].

In [10, p. 67] we find Rodrigues' formula for Jacobi polynomials

$$P_n^{(\alpha, \beta)}(x) = \frac{(-1)^n}{2^n n!} (1-x)^{-\alpha} (1+x)^{-\beta} \frac{d^n}{dx^n} [(1-x)^{n+\alpha} (1+x)^{n+\beta}].$$

Then, from (3) we have for $\alpha > -1/2$ and $n \geq 1$,

$$\begin{aligned} G_n^{(\alpha)}(x) &= \frac{(-1)^n \sqrt{\pi} \Gamma(n+2\alpha)}{2^{2n+2\alpha-1} \Gamma(n+\alpha) \Gamma(n+\alpha+\frac{1}{2})} (1-x^2)^{\frac{1}{2}-\alpha} \frac{d^n}{dx^n} [(1-x^2)^{n+\alpha-\frac{1}{2}}] \\ &= h_n^{(\alpha)} (1-x^2)^{\frac{1}{2}-\alpha} \frac{d^n}{dx^n} [(1-x^2)^{n+\alpha-\frac{1}{2}}], \end{aligned}$$

where

$$h_n^{(\alpha)} = \frac{(-1)^n \sqrt{\pi} \Gamma(n+2\alpha)}{2^{2n+2\alpha-1} \Gamma(n+\alpha) \Gamma(n+\alpha+1/2)}.$$

We define the integral

$$\gamma_n^{(\alpha)} := \int_{-1}^1 (1+x)(G_n^{(\alpha)}(x))'(1-x^2)^{\alpha-\frac{1}{2}} dx, \quad n \geq 2.$$

Thus,

$$\begin{aligned} \gamma_n^{(\alpha)} &= \int_{-1}^1 (1+x)nG_{n-1}^{(\alpha+1)}(x)(1-x^2)^{\alpha-\frac{1}{2}} dx \\ &= nh_{n-1}^{(\alpha+1)} \int_{-1}^1 \frac{1}{1-x} \frac{d^{n-1}}{dx^{n-1}} [(1-x^2)^{n+\alpha-\frac{1}{2}}] dx \\ &= \frac{\sqrt{\pi} \Gamma(n+1) \Gamma(\alpha+1/2)}{2^{n-1} \Gamma(n+\alpha)}. \end{aligned} \tag{23}$$

Lemma 3. It holds for $n \geq 1$,

$$\begin{aligned} \gamma_{2n}^{(\alpha)} &= \int_{-1}^1 x^{2k+1} (G_{2n}^{(\alpha)}(x))'(1-x^2)^{\alpha-\frac{1}{2}} dx, \quad k = 0, 1, \dots, n-1, \\ \gamma_{2n+1}^{(\alpha)} &= \int_{-1}^1 x^{2k+2} (G_{2n+1}^{(\alpha)}(x))'(1-x^2)^{\alpha-\frac{1}{2}} dx, \quad k = 0, 1, \dots, n-1. \end{aligned}$$

Proof. We prove the first equality in a recursive way. The result holds for $k = 0$ since the symmetry of Gegenbauer polynomials yields

$$\gamma_{2n}^{(\alpha)} = \int_{-1}^1 (1+x)(G_{2n}^{(\alpha)}(x))'(1-x^2)^{\alpha-\frac{1}{2}} dx = \int_{-1}^1 x(G_{2n}^{(\alpha)}(x))'(1-x^2)^{\alpha-\frac{1}{2}} dx.$$

Now, we suppose the result is true for $k = j - 1$ with $1 \leq j < n$, that is

$$\gamma_{2n}^{(\alpha)} = \int_{-1}^1 x^{2j-1} (G_{2n}^{(\alpha)}(x))'(1-x^2)^{\alpha-\frac{1}{2}} dx,$$

and we show it for $k = j$. We observe that $x^{2j-1} + x^{2j-2} - (x^{2j+1} + x^{2j}) = x^{2j-2}(1-x^2)(1+x)$. Then,

$$\int_{-1}^1 (x^{2j-1} + x^{2j-2} - (x^{2j+1} + x^{2j}))(G_{2n}^{(\alpha)}(x))'(1-x^2)^{\alpha-\frac{1}{2}} dx = \int_{-1}^1 x^{2j-2}(1-x^2)(1+x)(G_{2n}^{(\alpha)}(x))'(1-x^2)^{\alpha-\frac{1}{2}} dx.$$

Therefore, we have

$$\gamma_{2n}^{(\alpha)} - \int_{-1}^1 x^{2j+1} (G_{2n}^{(\alpha)}(x))'(1-x^2)^{\alpha-\frac{1}{2}} dx = 2n \int_{-1}^1 x^{2j-2}(1+x)G_{2n-1}^{(\alpha+1)}(x)(1-x^2)^{\alpha+\frac{1}{2}} dx = 0,$$

where the last equality holds because $j < n$. We have proved the first identity. The second one can be established in an analogous way. \square

Proposition 1. For $\alpha > -1/2$, we have

$$\begin{aligned} G_n^{(\alpha,-1)}(x) &= \delta_n^{(\alpha)} x(G_n^{(\alpha)}(x))' + (1 - n\delta_n^{(\alpha)})G_n^{(\alpha)}(x), \\ &= G_n^{(\alpha)}(x) + n\delta_n^{(\alpha)} \left(xG_{n-1}^{(\alpha+1)}(x) - G_n^{(\alpha)}(x) \right), \quad n \geq 2, \end{aligned}$$

where

$$\delta_n^{(\alpha)} = -\frac{M^{(-1)}}{\frac{2^{2\alpha-1}\Gamma(n+1)\Gamma^2(\alpha+1/2)}{\Gamma(n+2\alpha)} + \frac{M^{(-1)}n(n-1)}{2\alpha+1}}. \tag{24}$$

Proof. Consider the polynomials

$$W_n(x) = \delta_n^{(\alpha)} x(G_n^{(\alpha)}(x))' + \sigma_n^{(\alpha)} G_n^{(\alpha)}(x), \quad n \geq 2,$$

where $\sigma_n^{(\alpha)} := 1 - n\delta_n^{(\alpha)}$. Clearly, W_n are monic polynomials. We are going to prove that W_n are orthogonal with respect to the inner product associated with the measure $\psi^{(\alpha,-1)}$.

For $k = 0, 1, \dots, n - 1$ using Lemma 3 we get

$$\begin{aligned} \langle W_{2n}, x^{2k} \rangle_{\psi^{(\alpha,-1)}} &= \langle \delta_{2n}^{(\alpha)} x(G_{2n}^{(\alpha)}(x))' + \sigma_{2n}^{(\alpha)} G_{2n}^{(\alpha)}(x), x^{2k} \rangle_{\psi^{(\alpha,-1)}} \\ &= \delta_{2n}^{(\alpha)} \int_{-1}^1 x^{2k+1} (G_{2n}^{(\alpha)}(x))' (1-x^2)^{\alpha-\frac{1}{2}} dx + \sigma_{2n}^{(\alpha)} \int_{-1}^1 x^{2k} G_{2n}^{(\alpha)}(x) (1-x^2)^{\alpha-\frac{1}{2}} dx \\ &\quad + M^{(-1)} \left(\delta_{2n}^{(\alpha)} (G_{2n}^{(\alpha)}(1))' + \sigma_{2n}^{(\alpha)} G_{2n}^{(\alpha)}(1) - \delta_{2n}^{(\alpha)} (G_{2n}^{(\alpha)}(-1))' + \sigma_{2n}^{(\alpha)} G_{2n}^{(\alpha)}(-1) \right) \\ &= \delta_{2n}^{(\alpha)} \gamma_{2n}^{(\alpha)} + 2M^{(-1)} \left(\delta_{2n}^{(\alpha)} (G_{2n}^{(\alpha)}(1))' + \sigma_{2n}^{(\alpha)} G_{2n}^{(\alpha)}(1) \right). \end{aligned} \tag{25}$$

In the same way, we get for $k = 1, 2, \dots, n$,

$$\langle W_{2n+1}, x^{2k-1} \rangle_{\psi^{(\alpha,-1)}} = \delta_{2n+1}^{(\alpha)} \gamma_{2n+1}^{(\alpha)} + 2M^{(-1)} (\delta_{2n+1}^{(\alpha)} (G_{2n+1}^{(\alpha)}(1))' + \sigma_{2n+1}^{(\alpha)} G_{2n+1}^{(\alpha)}(1)). \tag{26}$$

Thus, if we assume

$$\delta_n^{(\alpha)} \gamma_n^{(\alpha)} + 2M^{(-1)} [\delta_n^{(\alpha)} (G_n^{(\alpha)}(1))' + \sigma_n^{(\alpha)} G_n^{(\alpha)}(1)] = 0, \tag{27}$$

and take into account (25) and (26), then we obtain that W_n are the monic polynomials orthogonal with respect to the inner product $\langle \cdot, \cdot \rangle_{\psi^{(\alpha,-1)}}$. Therefore, $W_n(x) = G_n^{(\alpha,-1)}(x)$, $n \geq 2$.

To conclude we just need to solve Eq. (27). We obtain

$$\delta_n^{(\alpha)} = -\frac{2M^{(-1)} G_n^{(\alpha)}(1)}{\gamma_n^{(\alpha)} + 2M^{(-1)} \left((G_n^{(\alpha)}(1))' - nG_n^{(\alpha)}(1) \right)}.$$

According to (3) and since $P_n^{(\alpha-\frac{1}{2}, \alpha-\frac{1}{2})}(1) = \frac{\Gamma(n+\alpha+1/2)}{\Gamma(n+1)\Gamma(\alpha+1/2)}$, we have

$$G_n^{(\alpha)}(1) = \frac{\sqrt{\pi} \Gamma(n+2\alpha)}{2^{n+2\alpha-1} \Gamma(\alpha+1/2) \Gamma(n+\alpha)}. \tag{28}$$

The result (24) follows using (23) and (28). □

Using Proposition 1 we can derive a Mehler–Heine type formula for these polynomials.

Proposition 2. It holds, for $M^{(-1)} > 0$ and $\alpha > -1/2$ and a fixed $j \in \mathbb{Z}$,

$$\lim_{n \rightarrow \infty} \frac{2^n G_n^{(\alpha,-1)}(\cos(x/(n+j)))}{n^\alpha} = -\sqrt{\pi} (2x)^2 (2x)^{-(\alpha+\frac{3}{2})} J_{\alpha+\frac{3}{2}}(x),$$

uniformly on compact subsets of the complex plane. Furthermore, if we denote the $m = [n/2]$ positive zeros of $G_n^{(\alpha,-1)}$ by $x_{n,i}^{(\alpha,-1)}$ in decreasing order, that is, $x_{n,m}^{(\alpha,-1)} < x_{n,m-1}^{(\alpha,-1)} < \dots < x_{n,2}^{(\alpha,-1)} < x_{n,1}^{(\alpha,-1)}$, then

$$\lim_{n \rightarrow \infty} x_{n,1}^{(\alpha,-1)} = 1, \quad \lim_{n \rightarrow \infty} n \arccos(x_{n,i}^{(\alpha,-1)}) = j_{i-1}^{(\alpha+\frac{3}{2})}, \quad i = 2, 3, \dots, m,$$

where $0 < j_1^{(\alpha+\frac{3}{2})} < j_2^{(\alpha+\frac{3}{2})} < \dots < j_m^{(\alpha+\frac{3}{2})}$ denote the first m positive zeros of the Bessel function $J_{\alpha+\frac{3}{2}}$.

Proof. From (24) and taking (8) into account we obtain

$$\lim_{n \rightarrow \infty} n^2 \delta_n^{(\alpha)} = -(2\alpha + 1). \tag{29}$$

Applying the above limit and Proposition 1 for $\alpha > -1/2$, we have

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{2^n G_n^{(\alpha, -1)}(\cos(x/(n+j)))}{n^\alpha} &= \lim_{n \rightarrow \infty} \frac{2^n G_n^{(\alpha)}(\cos(x/(n+j)))}{n^\alpha} \\ &+ \lim_{n \rightarrow \infty} 2n^2 \delta_n^{(\alpha)} \cos(x/(n+j)) \frac{2^{n-1} G_{n-1}^{(\alpha+1)}(\cos(x/(n+j)))}{n^{\alpha+1}} \\ &- \lim_{n \rightarrow \infty} n \delta_n^{(\alpha)} \frac{2^n G_n^{(\alpha)}(\cos(x/(n+j)))}{n^\alpha}. \end{aligned}$$

The Mehler–Heine type formula is obtained using the one for Gegenbauer polynomials (9) and the well-known relation for the Bessel functions of the first kind $(2\alpha + 1)J_{\alpha+\frac{1}{2}}(x) = x \left(J_{\alpha-\frac{1}{2}}(x) + J_{\alpha+\frac{3}{2}}(x) \right)$ (e.g. see [10, p. 15]).

The asymptotic behaviour of the zeros follows from Hurwitz’s Theorem and the fact that all the zeros of $G_n^{(\alpha, -1)}$ are real and different as proved in [16]. \square

Remark 3. Observe that the case $M^{(-1)} = 0$ is trivial because $G_n^{(\alpha, -1)}(x) = G_n^{(\alpha)}(x)$, and therefore asymptotic results follow from (9).

We need some additional results to establish ours. It is necessary to know the order of convergence to zero of the sequence $\left\{ b_{n-2}^{(\alpha, -1)} - d_{n-2}^{(\alpha-1, -1)} \right\}_n$.

Lemma 4. Let $M^{(-1)} > 0$ be. Then, we have

$$\lim_{n \rightarrow \infty} n \left(b_{n-2}^{(\alpha, -1)} - d_{n-2}^{(\alpha-1, -1)} \right) = -1.$$

Proof. Using (5) and (6) in Proposition 1, we get

$$G_{n-1}^{(\alpha, -1)} = G_{n-1}^{(\alpha+1)} + \omega_{n-3}^{(\alpha)} G_{n-3}^{(\alpha+1)}, \quad n \geq 3,$$

where

$$\omega_{n-3}^{(\alpha)} = \frac{(n-2)(n-1)}{4(n+\alpha-1)(n+\alpha-2)} \left(2(n+\alpha-1)\delta_{n-1}^{(\alpha)} - 1 \right).$$

Now, we pay attention to (11) and we deduce $d_{n-3}^{(\alpha, -1)} = \omega_{n-3}^{(\alpha)}$. Therefore, taking into account (14) we obtain

$$b_{n-2}^{(\alpha, -1)} - d_{n-2}^{(\alpha-1, -1)} = \frac{n(n-1)}{2(n+\alpha-2)} \left(\delta_{n-1}^{(\alpha)} - \delta_n^{(\alpha-1)} \right).$$

It just remains to use (29) to prove the result. \square

Finally, we can give an useful Mehler–Heine type formula for the polynomials S_n when $q = -1$ and $\alpha > 1/2$.

Theorem 2. Let us define $g_\alpha(x) := (2x)^{-\alpha} J_\alpha(x)$. Then, for $q = -1$ and $\alpha > 1/2$, it holds

$$\lim_{n \rightarrow \infty} \frac{2^n S_n(\cos(x/n))}{n^{\alpha-1}} = \begin{cases} -\frac{\sqrt{\pi}}{1 + 4a^{(-1, \kappa_1, \kappa_2)}} \left((2x)^2 g_{\alpha+\frac{1}{2}}(x) + 4g_{\alpha-\frac{1}{2}}(x) \right), & M^{(-1)} > 0, \\ \frac{\sqrt{\pi}}{1 + 4a^{(-1, \kappa_1, \kappa_2)}} g_{\alpha-\frac{3}{2}}(x), & M^{(-1)} = 0, \end{cases}$$

uniformly on compact subsets of the complex plane.

Proof. For $M^{(-1)} > 0$, $\alpha > 1/2$ and $q = -1$, using (11) and (19), we have

$$R_n^{(-1)}(x) = G_n^{(\alpha-1, -1)}(x) + \left(b_{n-2}^{(\alpha, -1)} - d_{n-2}^{(\alpha-1, -1)} \right) G_{n-2}^{(\alpha)}(x), \quad n \geq 2. \tag{30}$$

Then, using (9), Proposition 2, and Lemma 4 we get for a fixed $j \in \mathbb{Z}$

Table 6

$n \arccos(s_{n,i})$ for $i = 1, 2, 3, \alpha = 3, q = -1, M^{(-1)} = 1, \kappa_1 = 2, \kappa_2 = 20$. h_i are roots of the limit function $h(x)$. Notice that S_n has 2 zeros outside $(-1, 1)$.

$n \arccos(s_{n,i})$	$i = 1$	$i = 2$	$i = 3$	Zero > 1
$n = 25$	6.264901398	9.52712014	12.61864187	1.012123412
$n = 100$	6.602701863	10.04082257	13.30001753	1.000834435
$n = 200$	6.665245950	10.13590229	13.42596359	1.000212480
$n = 350$	6.692671240	10.17759909	13.48119229	1.000069946
h_i	6.729838906	10.23411504	13.55605099	$\xi_{-1} = 1$

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{2^n R_n^{(-1)}(\cos(x/(n+j)))}{n^{\alpha-1}} &= \lim_{n \rightarrow \infty} \frac{2^n G_n^{(\alpha-1,-1)}(\cos(x/(n+j)))}{n^{\alpha-1}} \\ &+ \lim_{n \rightarrow \infty} 4n \left(b_{n-2}^{(\alpha,-1)} - d_{n-2}^{(\alpha-1,-1)} \right) \frac{2^{n-2} G_{n-2}^{(\alpha)}(\cos(x/(n+j)))}{n^\alpha} \\ &= -\sqrt{\pi} \left((2x)^2 g_{\alpha+\frac{1}{2}}(x) + 4g_{\alpha-\frac{1}{2}}(x) \right), \end{aligned}$$

uniformly on compact subsets of \mathbb{C} .

Thus, we only need to apply the same technique used in Theorem 1 and the above limit to prove the result when $M^{(-1)} > 0$.

For the case $M^{(-1)} = 0$ we have $G_n^{(\alpha,-1)}(x) = G_n^{(\alpha)}(x)$. Then, from (6) and (11) we can deduce

$$d_{n-2}^{(\alpha,-1)} = -\frac{n(n-1)}{4(n+\alpha)(n+\alpha-1)}.$$

Now, from the above expression and (14) we get

$$b_{n-2}^{(\alpha,-1)} = \frac{n}{n-2} d_{n-3}^{(\alpha,-1)} = d_{n-2}^{(\alpha-1,-1)}, \quad n \geq 3.$$

Thus, from (30) (or gathering (6) and (19)) we obtain $R_n^{(-1)}(x) = G_n^{(\alpha-1,-1)}(x) = G_n^{(\alpha-1)}(x)$ and so Mehler–Heine type formula for the polynomials $R_n^{(-1)}(x)$ is the same as the one for Gegenbauer orthogonal polynomials with parameter $\alpha - 1$. Again, we go on in the same way as Theorem 1 to prove this case. \square

Remark 4. Observe that when $M^{(-1)} > 0$ the limit function in the previous theorem

$$h(x) := -\frac{\sqrt{\pi}}{1 + 4a^{(-1,\kappa_1,\kappa_2)}} \left((2x)^2 g_{\alpha+\frac{1}{2}}(x) + 4g_{\alpha-\frac{1}{2}}(x) \right)$$

is symmetric, that is, $h(-x) = h(x)$, and furthermore we can express it as

$$h(x) = -\frac{\sqrt{\pi}}{1 + 4a^{(-1,\kappa_1,\kappa_2)}} \left(\frac{1}{2^{2\alpha-3} \Gamma(\alpha + 1/2)} + \sum_{i=1}^{\infty} \frac{(-1)^{i-1} (i-1)x^{2i}}{2^{2\alpha+2i-3} i! \Gamma(i + \alpha + 1/2)} \right).$$

Remember that, under the restrictions (18), when $q = -1$ all the zeros of S_n are real and simple.

Corollary 2. Let $\alpha > 1/2$ and $q = -1$ and let κ_1 and κ_2 which satisfy the restrictions given in (18). We denote by

$$m = \begin{cases} [n/2] - 1, & \text{if } S_n \text{ has 2 zeros outside } (-1, 1), \\ [n/2], & \text{otherwise} \end{cases}$$

and by $s_{n,i}, i = 1, 2, \dots, m$, the m positive zeros of S_n within $(0, 1)$ in decreasing order, i.e., $s_{n,m} < s_{n,m-1} < \dots < s_{n,2} < s_{n,1}$. If $M^{(-1)} > 0$, then

$$\lim_{n \rightarrow \infty} n \arccos(s_{n,i}) = h_i, \quad i = 1, 2, \dots, m,$$

where $0 < h_1 < h_2 < \dots < h_m$ denote the first m positive real zeros of the function $h(x)$.

If $M^{(-1)} = 0$, then

$$\lim_{n \rightarrow \infty} n \arccos(s_{n,i}) = j_i^{(\alpha-\frac{3}{2})}, \quad i = 1, 2, \dots, m,$$

where $0 < j_1^{(\alpha-\frac{3}{2})} < j_2^{(\alpha-\frac{3}{2})} < \dots < j_m^{(\alpha-\frac{3}{2})}$ denote the first m positive zeros of the Bessel function of the first kind $J_{\alpha-\frac{3}{2}}$.

In Tables 6–8 numerical examples are given to illustrate Corollary 2.

Table 7Largest zeros inside $(0, 1)$ with $\alpha = 3, q = -1, M^{(-1)} = 1, \kappa_1 = 2, \kappa_2 = 20$. h_i are roots of the limit function $h(x)$ and $e_{n,i} = |s_{n,i} - \cos(h_i/n)|$.

n	$s_{n,1}$	$\cos(h_1/n)$	$e_{n,1}$	$s_{n,2}$	$\cos(h_2/n)$	$e_{n,2}$
25	0.9687647830	0.9639856868	0.477910e-2	0.9282617127	0.9173739134	0.108878e-1
100	0.9978210082	0.9977363180	0.846902e-4	0.9949633278	0.9947677136	0.195614e-3
200	0.9994447326	0.9994339193	0.108133e-4	0.9987160684	0.9986910718	0.249966e-4
350	0.9998171817	0.9998151456	0.203614e-5	0.9995772399	0.9995725320	0.470788e-5

Table 8Largest zeros inside $(0, 1)$ with $\alpha = 3, q = -1, M^{(-1)} = 0, \kappa_1 = 2, \kappa_2 = 20$. $e_{n,i} = |s_{n,i} - \cos(j_i^{(\alpha-3/2)}/n)|$.

n	$s_{n,1}$	$\cos(j_1^{(\alpha-3/2)}/n)$	$e_{n,1}$	$s_{n,2}$	$\cos(j_2^{(\alpha-3/2)}/n)$	$e_{n,2}$
25	0.9861182623	0.9838908547	0.222741e-2	0.9591602611	0.9526350888	0.652517e-2
100	0.9990288134	0.9989906334	0.381800e-4	0.9971302945	0.9970175080	0.112787e-3
200	0.9997524742	0.9997476265	0.484775e-5	0.9992684256	0.9992540988	0.143268e-4
350	0.9999185006	0.9999175800	0.910637e-6	0.9997591115	0.9997564200	0.269152e-5

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