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Periodic Competitive Differential Equations and the Discrete Dynamics of Competitive Maps

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0. Introduction

Mathematical models of competition between species of organisms frequently involve systems of ordinary differential equations having the form

$$\begin{aligned} x_i' &= x_i f_i(t, x), \qquad 1 \leqslant i \leqslant n, \\ x &= (x_1, x_2, \ldots, x_n) \end{aligned} \tag{0.1}$$

and for which

$$\frac{\partial}{\partial x_j}(x_i f_i(t, x)) = x_i \frac{\partial f_i}{\partial x_j} \leqslant 0 \tag{0.2}$$

for $i \neq j$, $(t, x) \in R \times R^n_+$, $R^n_+ = \{x\colon x_i \geqslant 0\}$. The biological interpretation of (1.2) is that an increase in a competitor's population size or density can only have a negative effect on a species per capita growth rate due, perhaps, to the competitor consuming resources which then become more scarce. The class of systems (0.1) satisfying (0.2) is a subset of a more general class of systems referred to as competitive systems by Hirsch [6]. Much of the early literature consists of the study of particular examples of systems having the form (0.1) and satisfying (0.2), such as the so-called Gauss–Lotka–Volterra-type systems in which the f_i are linear time-independent functions of x. For these systems, there is an extensive literature [2, 5, 11]. Important among these is a paper of May and Leonard [11] which pointed out that complex asymptotic behavior is possible in the three species Gauss–Lotka–Volterra model. For particular values of the parameters, solutions approach a triangular cycle consisting of three one-species equilibria together with three connecting heteroclinic orbits.

The full range of asymptotic behavior possible for general vector fields can be realized with competitive systems. This, essentially, is the content of

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a note by Smale [13] who showed that any vector field on the standard $n-1$ simplex in R^n can be imbedded in a smooth competitive vector field on R^n for which the simplex is an attractor. On the positive side, Hirsch, in a series of papers [6–9] establishes that the limit sets of competitive systems can be no more complicated than those of general systems in one fewer dimension. The papers of Hirsch are especially important for introducing some useful techniques for treating monotone flows.

The above-mentioned work focuses on autonomous systems. Our focus in this paper is on nonautonomous, periodic, competitive systems:

$$f_i(t+2\pi, x) = f_i(t, x) \tag{0.3}$$

of normalized period 2π. In the applications, periodicities in the parameters are introduced to model day–night or seasonal forcing. Of course, one cannot expect the asymptotic behavior of general periodic competitive systems to be any more tame than autonomous ones and indeed one might expect far worse. Important results for periodic systems were obtained by de Mottoni and Schiaffino [12] for the two-dimensional Gauss–Lotka–Volterra system. They showed that all solutions are asymptotic to 2π-periodic systems, described where these periodic solutions must lie, and obtained results on the basis of attraction of the one species periodic solutions. This paper is especially beautiful for its geometric approach; the authors study the discrete dynamical system generated by the Poincare map: $x(0) \to {}^{T}x(2\pi)$. The methods employed in [12], although for a particular set of equations, turn out to have general applicability as was pointed out by Hale and Somolinos [4].

In [15], the present author introduced invariant manifold techniques which together with ideas of Hirsch, proved to be moderately successful for both competitive and cooperative periodic systems (reverse the inequalities (0.2)), at least for the discussion of periodic solutions of (0.1). However, the focus in [15] was primarily on cooperative systems. The purpose of the present paper is to employ many of the same ideas used in [15] to the study of competitive periodic systems. Competitive systems are slightly more difficult to work with than cooperative ones. The solution map of the latter preserves the natural partial ordering on R^n (the Kamke theorem [1]), whereas the solution map of the time-reversed competitive system preserves the partial ordering. In addition, the property of competitiveness, (0.2), places no restriction on the one-dimensional subsystems on each coordinate axis. To obtain general results for competitive systems, we are forced to introduce four important additional restrictions on (0.1). First, a reasonable restriction on the one-dimensional systems, from the point of view of applications, is that either the origin is a global attractor or there is a unique nontrivial globally stable 2π-periodic solution. Also motivated by the applications, we assume that no periodic solutions of (0.1) are repelling (all multipliers outside the unit circle). For this to be true, it suffices, for example, that $\partial f_i/\partial x_i \leqslant 0$ along periodic solutions. In [7], Hirsch indicated the importance of the assumption that $((\partial/\partial x_j)(x_i f_i))$ be an irreducible matrix. We require this to hold for each subsystem of (0.1) obtained by setting some of the $x_i = 0$. Finally, for our consideration of two-dimensional competitive systems we assume that all 2π-periodic solutions are nondegenerate.

For general n-dimensional periodic competitive systems satisfying the above-mentioned restrictions, we come very close to concluding that the limit set of every nontrivial orbit $O^+(x) = \{T^n x\colon n \geqslant 0\}$, where T is the Poincare map for (0.1), lies on a certain lower dimensional manifold which we describe in some detail. This lower dimensional manifold does contain all the p-periodic points of T, for every p. Under the additional assumption that the fixed points of T are hyperbolic we are able to significantly extend the results of de Mottoni and Schiaffino for two-dimensional systems. In fact, we describe completely the possible "phase portraits" for the discrete dynamical system generated by the Poincare map T.

We proceed as follows. In Section 1, various notations and conventions are introduced. In Section 2, the Poincare map (period 2π map) for (0.1) is introduced and the various hypotheses mentioned above concerning (0.1) are translated into properties of the Poincare map. In Section 3, the class of competitive maps on R^n_+ is introduced and the discrete dynamical system generated by such a map is studied from a geometrical viewpoint. In the final section attention is restricted to the two-dimensional case where we catalogue the realizable phase portraits assuming all fixed points are nondegenerate.

As we mentioned earlier, many of the important ideas used in this work are due to Hirsch [6–9] and to de Mottoni and Schiaffino [12] (see also Hale and Somolinos [4]). In addition to these, an invariant curve theorem for mappings, proved in [4] and first used in our earlier work on competitive and cooperative systems is crucial to our approach. This theory together with the Perron–Frobenius theory of positive matrices allows us to obtain monotone invariant curves associated with periodic points of competitive maps.

The example of Leonard and May [11] and the theorem of Smale [13] indicate that it is not likely that all possible "phase portraits" of competitive maps on R^3 can be described even assuming the hyperbolicity of all periodic points. On the other hand, we have essentially complete information on the dynamics of the competitive map on each of the three two-dimensional faces of R^3_+ and Section 3 provides considerable information on the location of limit sets. We are led to believe that, at least in several biologically interesting cases, a fairly complete description of the dynamics of a competitive map in R^3_+ is possible. It is our intention to pursue this goal in a future paper.

1. Notation and Definitions

Throughout this paper, the letter n is reserved for the dimension of the euclidean space R^n. Given two vectors (matrices) x and y we write $x < y$ or $x \leqslant y$ in case the order relation holds component-wise. Two vectors x and y are related (weakly related) if $x < y$ or $y < x$ ($x \leqslant y$ or $y \leqslant x$). Let $R^n_+ = \{x \in R^n\colon x \geqslant 0\}$ be the usual nonnegative cone and $\dot{R}^n_+$ denote its interior. The usual basis for R^n will be denoted by $\{e_1, e_2, ..., e_n\}$. We reserve the letter N for the set $N = \{1, 2, ..., n\}$. If $I \subseteq N$, let $\# I$ denote the cardinality of I and $C(I)$ the complement of I in N. Given $I \subseteq N$, let H_I be the subspace of R^n generated by $\{e_i\}_{i \in I}$. If x and y are two n-vectors in H_I we write $x <_I y$ provided $x_i < y_i$ for every $i \in I$ (the subscript on $<$ is dropped if $I = N$). Let $H_I^+ = \{x \in H_I\colon x \geqslant 0\}$ and $\dot{H}_I^+ = \{x \in H_I\colon x >_I 0\}$. If $x \leqslant y$, let $[x, y]$ be the order interval $\{z\colon x \leqslant z \leqslant y\}$.

An $n \times n$ matrix A is reducible if it leaves invariant one of the subspaces H_I, where I is a nonempty proper subset of N. Otherwise A is irreducible. If I is a nonempty subset of N and A is an $n \times n$ matrix, let A_I be the $\# I \times \# I$ submatrix of A obtained from A by deleting from A the rows and columns indexed by elements of $C(I)$. The spectrum of A will be denoted $\mathrm{sp}(A)$.

If A is a subset of a topological space X, $\bar{A}$ denotes the closure of A in X and $\partial_X A$ denotes the boundary of A relative to X.

2. The Poincare Map for Competitive Systems

In this section we will be concerned with the nonautonomous system of differential equations

$$x_i' = x_i f_i(t, x) \equiv F_i(t, x), \qquad 1 \leqslant i \leqslant n, \\ x = (x_1, x_2, ..., x_n), \tag{2.1}$$

where the function $F(t, x) = (F_1, ..., F_n)$ is defined and continuous, together with its first and second derivatives with respect to x, for $(t, x) \in R \times U$, where U is an open subset of R^n containing R^n_+. We write $\phi(t, s, x_0)$ for the solution map of (2.1), that is $\phi(\cdot, s, x_0)$ is the unique solution of (2.1) satisfying $x(s) = x_0$. We will assume without further mention that the domain of $\phi(\cdot, s, x_0)$ includes $[s, \infty]$ in case $x_0 \in R^n_+$. In addition to the special form of (2.1) we make two additional assumptions.

(I) F is periodic in t of normalized period 2π

$$F(t + 2\pi, x) = F(t, x).$$

(II) The system (2.1) is competitive in the sense that the following two conditions hold:

(A) $(\partial F_i/\partial x_j)(t, x) \leqslant 0$ for $i \neq j$, $(t, x) \in R \times R^n_+$.

(B) For each $I \subset N$, $I \neq \phi$, the matrix

$$\left(\frac{\partial F_i}{\partial x_j}(t, x)\right)(i, j) \in I \times I$$

is irreducible for each $(t, x) \in R \times \dot{H}_I^+$.

Hirsch introduced the notion of a competitive system (2.1) (although he only required (A) hold) and stressed the importance of the assumption that $(\partial F_i/\partial x_j)_{(i,j) \in N \times N}$ be irreducible in two important papers [6, 7] (see also Krasnoselskii [10]). Our assumption (B) is motivated by considering the subsystem of (2.1) on H_I, $\phi \neq I \subseteq N$:

$$x_i' = F_i(t, x), \qquad i \in I, x \in H_I^+. \tag{2.1$_I$}$$

Our assumption on the form of the F_i guarantees that H_I is an invariant set for (2.1). Clearly the inequalities in (II)(A) are inherited by the subsystem (2.1$_I$). However, the assumption that $(\partial F_i/\partial x_j)_{(i,j) \in N \times N}$ is irreducible for $(t, x) \in R \times \dot{R}^n_+$ is not inherited by the submatrix $(\partial F_i/\partial x_j)_{(i,j) \in I \times I}$. We want the strong properties that irreducibility implies (see Proposition 2.3) to hold for each subsystem (2.1$_I$). This is the motivation for (II)(B).

Hereafter we will assume (I) and (II) hold and refer to (2.1) as a periodic competitive system. We will want to make additional assumptions on (2.1) motivated by the applications. To motivate these additional assumptions, consider the n one-dimensional subsystems

$$x_i' = x_i f_i(t, (0,..., 0, x_i, 0,..., 0)). \tag{2.1$_i$}$$

Our assumptions so far say essentially nothing about these one-dimensional subsystems, they could have any number of nontrivial 2π-periodic solutions (recall that for a scalar equation each bounded solution is either 2π-periodic or asymptotic to a 2π-periodic solution). One should not expect to go far towards a qualitative understanding of (2.1) with this much freedom in each one-dimensional subsystem. Taking our cue from the applications we will make the following assumption:

(III) For each $i \in N$ the trivial solution of (2.1_i) is hyperbolic, i.e., the Floquet multiplier, λ_i, of the trivial solution, a positive number, is not unity. Either $\lambda_i < 1$ in which case we assume the trivial solution is a global attractor for (2.1_i) or $\lambda_i > 1$ in which case we assume there is a *unique*, nontrivial, 2π-periodic solution which is hyperbolic and every nontrivial solution of (2.1_i) is attracted to this 2π-periodic solution.

If (2.1) represents a community of competing species whose densities are given by x, then each species, in isolation from its competitors, will either become extinct or asymptotically approach a nontrivial oscillating population density which is independent of initial conditions. This is the content of our assumption (III). It is satisfied by most of mathematical models of competition which arise in the applications [2, 4, 5, 11]. By confining our considerations to systems satisfying (III), we will be ignoring some interesting cases, for example, when one or more of the species has several possible stable regimes available to it depending on initial conditions. Most of the techniques we use will apply to these other cases as well.

We will make an additional assumption on (2.1) but before doing so it is convenient to define the fundamental object of study in this paper, namely, the Poincare map for (2.1). Let

$$T(x) = \phi(2\pi, 0, x)$$

for those x for which the right-hand side is defined, namely an open set containing R^n_+, which we again label U. Let us state the consequences of our assumptions concerning (2.1) for the Poincare map T. Proofs for these assertions can be found in [15]. First, the form of (2.1) implies that T fixes the origin and maps each of the sets H_I, H_I^+, $\dot{H}_I^+$ into itself where $\phi \neq I \subset N$. It is well known that T is a C^2-diffeomorphism of U onto the open set TU which is orientation preserving. Our assumption (II)(A) implies that the time-reversed system (2.1) satisfies the Kamke condition [1] and results in

PROPOSITION 2.1. *If $x, y \in R^n_+$ and $Tx < Ty$ then $x < y$.*

PROPOSITION 2.2. *If $x \in R^n_+$ and $Tx = y$ then $[0, y] \subseteq T([0, x])$.*

The proofs of the two propositions can be found in [15, Corollaries F and G]. Proposition 2.1 says that T^{-1} preserves the $<$ relation and Proposition 2.2 describes a property of the domain of T^{-1}, the open set $T(U)$ containing $T(R^n_+)$, which will prove to be important. As we mentioned earlier, Hirsch [7] pointed out the stronger monotonicity properties that obtain if one assumes that $(\partial F_i/\partial x_j)_{(i,j) \in N \times N}$ is irreducible. In (II)(B) we have assumed more in order that the restriction of T to H_I^+ enjoys the same properties as T. In the following proposition we state the result for T on $\dot{R}^n_+$ with the relations $<$ and $\leqslant$. An identical statement holds for $T|_{H_I}$ on H_I^+ with the relations $<_I$ and $\leqslant_I$.

PROPOSITION 2.3. *If* $x \in \dot{R}^n_+$ *then* $DT(x)^{-1} > 0$. *If* $x, y \in R^n_+$, $Tx \leqslant Ty$, $x \neq y$ *and* $y > 0$ *then* $x < y$.

Proposition 2.3 is contained in [15, Theorem D and Corollary E]. To clarify the remark preceding the statement of Proposition 2.3, we describe the first assertion of Proposition 2.3 as it relates to $T|_{H_I}$. If $x \in \dot{H}_I^+$ then $D(T|_{H_I})(x)$ is the $\#I \times \#I$ matrix obtained from $DT(x)$ by deleting rows and columns indexed by $j \in C(I)$. Proposition 2.3 together with the remark preceding it implies that $D(T|_{H_I})(x)^{-1} > 0$. Note that if $x \in \dot{H}_I^+$, $DT(x)$ is a reducibile matrix ($\partial T_j/\partial x_i = 0$ for $i \in I$, $j \in C(I)$) with $D(T|_{H_I})(x)$ as a submatrix.

The significance of the fact that $DT(x)^{-1} > 0$ if $x \in \dot{R}^n_+$ ($D(T|_{H_I})(x)^{-1} > 0$ if $x \in \dot{H}_I^+$) is that the Perron–Frobenius theory of positive matrices applies. Recall that if $A > 0$ is an $n \times n$ matrix then the spectral radius of A is a positive simple eigenvalue of A exceeding in modulus all other eigenvalues of A. Moreover, corresponding to the spectral radius is an eigenvector in $\dot{R}^n_+$ which is the only eigenvector of A, up to scalar multiple, which belongs to R^n_+. If $A \geqslant 0$ and nonsingular then the spectral radius is still a positive eigenvalue though not necessarily simple nor exceeding in modulus other eigenvalues. There is an eigenvector in R^n_+ corresponding to the spectral radius though it need not be positive nor unique. We refer the reader to [16] for the theory of nonnegative matrices. If $DT(x)^{-1} > 0$, the Perron–Frobenius theory implies that $DT(x)$ has a positive simple eigenvalue which is strictly smaller in modulus than all other eigenvalues of $DT(x)$ and a corresponding positive eigenvector. We will write $\mu(x)(\mu_I(x))$ for this uniquely determined eigenvalue of $DT(x)(D(T|_{H_I})(x))$. Notice that $\mu(x)$ is well defined even if $x \in \dot{H}_I^+$ ($DT(x)^{-1} \geqslant 0$) and $\mu(x) \leqslant \mu_I(x)$.

The eigenvalue $\mu(x)(\mu_I(x))$ has a special significance if x is a fixed point of T. In that case $\phi(t, 0, x)$ is a 2π-periodic solution of (2.1) and $\mu(x)$ is the smallest Floquet multiplier corresponding to the variational equation along $\phi(t, 0, x)$. Of course, $\mu(x)$ does not determine the stability of $\phi(t, 0, x)$ as a solution of (2.1) (or of x as a fixed point of T) but in case $\mu(x) > 1$ we say that $x(\phi(t, 0, x))$ is a repelling fixed point of T (repelling periodic solution of (2.1)). If x is a periodic point of T, that is, $T^p x = x$ for some positive integer p and $T^j x \neq x$ for $j = 1, 2, \ldots, p-1$, then $\phi(t, 0, x)$ is a $2\pi p$-periodic solution of (2.1). The Floquet multipliers of the variational equation along (2.1) are just the eigenvalues of $D(T^p)(x) = DT(T^{p-1}x)\, DT(T^{p-2}x) \cdots DT(x)$ which has a positive (nonnegative) inverse if $x > 0$ ($x \geqslant 0$). Consequently, $D(T^p)(x)(D(T|_{H_I})^p(x))$ has a smallest positive eigenvalue

(Floquet multiplier) which we label $\mu_p(x)(\mu_{I,p}(x))$. Note that as before $\mu_p(x) \leqslant \mu_{I,p}(x)$ if $x \in H_I^+$.

Our final hypothesis concerning (2.1) is most easily motivated from the point of view of the applications. Before doing this we state the hypothesis:

(IV) For each I, $\phi \neq I \subseteq N$, and for each $2\pi p$-periodic solution, $x(t)$, of $(2.1)_I$ in $\dot{H}_I^+$, p a positive integer, we have

$$\int_0^{2\pi p} \left[\sum_{i \in I} x_i(t) \frac{\partial f_i}{\partial x_i}(t, x(t)) \right] dt \leqslant 0.$$

Hypothesis (IV) is obviously satisfied if $\partial f_i/\partial x_i \leqslant 0$, $1 \leqslant i \leqslant n$ and this is in fact the case for most models of competition in ecology. The inequality $\partial f_i/\partial x_i \leqslant 0$ expresses the usual assumption in ecology that increasing a species' density does not increase its per capita growth rate, indeed it usually decreases due to crowding or other effects. Hypothesis (IV) has the following important implication, which is really our sole reason for assuming it. In the next section we will merely assume the following result holds.

PROPOSITION 2.4. *If $x \in \dot{H}_I^+$ is a p-periodic point of T then $\mu_{p,I}(x) < 1$, $\phi \neq I \subset N$.*

Proof. We write $x(t) = \phi(t, 0, x)$ for the $2\pi p$-periodic solution of (2.1_I) which lies in $\dot{H}_I^+$. In case $I = \{i\}$, the result follows from our hypothesis (III). Assume I contains at least two elements. Then

$$\begin{aligned} \operatorname{Det} D((T|_{H_I})^p)(x) &= \operatorname{Det} \frac{\partial \phi}{\partial x_I}(2\pi p, 0, x) \\ &= \exp \int_0^{2\pi p} (\operatorname{Div} F_I)(t, x(t))\, dt \\ &= \exp \int_0^{2\pi p} \left(\sum_{i \in I} x_i \frac{\partial f_i}{\partial x_i}(t, x(t)) \right) dt \\ &\leqslant 1. \end{aligned}$$

In the above calculation, $\partial\phi/\partial x_I$ represents the $\# I \times \# I$ submatrix of $\partial\phi/\partial x$ obtained by deleting the jth row and column of $\partial\phi/\partial x$ if $j \in C(I)$. In other words, $\partial\phi/\partial x_I$ is the fundamental matrix solution of the variational equation of (2.1_I) along $x(t)$. Similarly, Div F_I is the divergence of the vector field $F|_{H_I}$. Now, since the product of the moduli of the eigenvalues of $D((T|_{H_I})^p)(x)$ cannot exceed unity and the smallest eigenvalue, $\mu_{I,p}(x)$, is positive and strictly smaller in modulus than the remaining eigenvalues, the result follows.

Observe that Proposition 2.4 is stronger than the assertion that $T(T|_{H_I})$ has no nontrivial repelling periodic points. One could probably give a plausible biological argument for assuming the latter.

Proposition 2.4 may be known to other researchers who have worked with competitive systems. The author believes the genesis of his idea of assuming (2.2) occurred after listening to a talk by G. Butler at the University of Alberta.

If is worth noting that if, for each nonempty set $I \subseteq N$, $(\partial f_i/\partial x_j)(t, x) < 0$ for $(t, x) \in R \times \dot{H}_I^+$ and $(i, j) \in I \times I$ then (II)(A) and (B) hold, (IV) holds, and the uniqueness and *local* asymptotic stability hold in the final statement in (III). It turns out that, as a consequence of assuming (III), all forward orbits of (2.1) are bounded so our assumption that solutions are extendable into the future is redundant (see Proposition 3.4 in the next section).

3. The Dynamics of Competitive Maps

In this section we study the discrete dynamical system generated by what we term competitive maps. The class of competitive maps include the Poincare maps for periodic competitive systems (2.1) satisfying (I) and (II)(A) and (B) of Section 2. More precisely, let U be an open neighborhood of R^n_+ and T: $U \to R^n$. We say that T is a competitive map provided the following hold:

(H1) T is an injective, C^2 diffeomorphism onto $T(U)$.

(H2) For each nonempty $I \subseteq N$, the sets $A = H_I$, H_I^+ and $\dot{H}_I^+$ have the property that $T(A) \subseteq A$ and $T^{-1}(A) \subseteq A$.

(H3) For each nonempty subset $I \subset N$ and $x \in \dot{H}_I^+$, $D(T|_{H_I})(x)^{-1} = DT(x)_I^{-1} > 0$.

(H4) If $x \in R^n_+$ and $y = Tx$ then $[0, y] \subset T[0, x]$.

(H1) implies that $DT(x)$ is nonsingular for every $x \in U$. It follows that for $x \in \dot{H}_I^+$, $DT(x)_I$ is nonsingular so that the content of (H3) is that $DT(x)_I$ has a positive inverse.

Our first three results below are rather technical in nature and might well be skipped over in a first reading. They are the tools which are useful in the results to follow. The study of the discrete dynamical system generated by T begins following the proof of Proposition 3.3 where we make some assumptions about the action of T on each coordinate axis. In our first important result, Proposition 3.4, we show that orbits of points under the action of T approach a bounded set. Thereafter we may restrict our attention to this bounded set.

Most of the following properties of competitive maps follow immediately from (H2)–(H4).

PROPOSITION 3.1. *Let T be a competitive map. Then*

(i) $T0 = 0$.

(ii) *T is orientation preserving.*

(iii) *For each nonempty subset $I \subset N$, $T|_{H_I}$ is competitive; T^p is competitive for each positive integer p.*

(iv) *For each nonempty subset $I \subset N$ and distinct points x and y in H_I^+ with $y >_I 0$ satisfying $Tx \leqslant Ty$ we have $x <_I y$.*

Proof. (H2) implies that $T0 = 0$ and that $DT(0) = \operatorname{diag}\{\lambda_1, ..., \lambda_n\}$, where $\lambda_i = (d/dx_i)|_{x_i=0} T|_{H_{\{i\}}} \geqslant 0$. By (H1), $\lambda_i > 0$, $1 \leqslant i \leqslant n$, and (ii) follows. Part (iii) is easily checked. To prove (iv) we follow Hirsch [7]. For each s, $0 \leqslant s \leqslant 1$, $sTy + (1-s)\,Tx \in T(H_I^+)$ by (H4) and $x_s \equiv T^{-1}(sTy + (1-s)\,Tx) \in \dot{H}_I^+$ for $0 < s \leqslant 1$ by (H2). Consequently we have

$$y - x = \left[\int_0^1 D(T^{-1})(Tx_s)\,ds\right](Ty - Tx)$$

$$= \left[\int_0^1 DT(x_s)^{-1}\,ds\right](Ty - Tx).$$

(H3) and the above calculation imply that (iv) holds.

The following very simple result turns out to be of fundamental importance.

LEMMA 3.2. *Let a be a positive number, $y \in R_+^n$, $i \in N$ and suppose y and ae_i are distinct points with $y_i \leqslant a$. Then $T_i(y) < T_i(ae_i)$.*

Proof. If $y \in H_{\{i\}}$ then by assumption $y_i < a$ and the result follows by (H3) which implies that $(d/dx_i)(T|_{H_{\{i\}}}) > 0$. If $y_i = 0$ then the result follows from (H2). We suppose $y_i > 0$ and $y_j > 0$ for some $j \neq i$. Let $I \subset N$ be such that $y \in \dot{H}_I^+$. Clearly $ae_i \in H_I^+$ and if $T_i(y) \geqslant T_i(ae_i)$ then $Ty \geqslant T(ae_i)$ so by Proposition 3.1(iv) we have $y >_I ae_i$, hence $y_i > a$ contradicting a hypothesis.

Let a be a positive number and $i \in N$ and let $P_i(a) = ae_i + H_{C\{i\}} = \{x\colon x_i = a\}$ and $P_i^+(a) = P_i(a) \cap R_+^n$. Then $TP_i^+(a)$ is a C^2 $(n-1)$-manifold with boundary in R_+^n. The next result describes some of the geometry of $TP_i^+(a)$, in particular it can be parametrized by $x_i = X_i(x_1, ..., x_{i-1}, x_{i+1}, ..., x_n)$, X_i a smooth function.

PROPOSITION 3.3. *$TP_i^+(a)$ is the graph of a C^2 function*

$$x_i = X_i(x_1, x_2, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n),$$

where X_i is defined on a domain D_i, which is relatively open in $H_{C\{i\}}^+$, and satisfies the following:

(1) *If $\hat{x}_j = (x_1^j, \ldots, x_{i-1}^j, x_{i+1}^j, \ldots, x_n^j)$, $j = 1, 2$, are distinct points of D_i with $\hat{x}_1 \leqslant \hat{x}_2$, then*

$$X_i(\hat{x}_1) > X_i(\hat{x}_2).$$

In particular, if $\hat{x}_1 = 0$ and $\hat{x}_2 \neq 0$, $T_i(ae_i) = X_i(0) > X_i(\hat{x}_1) > 0$.

(2) *If $I \subset N$ with $i \notin I$, $x \in \dot{H}_I^+ \cap D_i$ then*

$$\frac{\partial X_i}{\partial x_j}(x) < 0 \qquad \text{if} \quad j \in I.$$

Figure 3.1 depicts $TP_i(a)$ and $P_i(a)$.

If we define the orthogonal projection Q_i of R^n onto $H_{C\{i\}}$ along e_i then one of the assertions of Proposition 3.3 is essentially that $Q_i \mid_{TP_i^+(a)}$: $TP_i^+(a) \to D_i \subset H_{C\{i\}}$ is a smooth diffeomorphism whose inverse is given by $(x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_n) = x \to x + X_i(\hat{x})e_i$.

Proof of Proposition 3.3. First, note that there cannot be two points of $TP_i(a)$ having the same projection onto $H_{C\{i\}}$ along e_i. For if $Q_i Tx = Q_i Ty$ where $x \neq y$, x, $y \in P_i(a)$ then clearly, either $Tx \leqslant Ty$ or $Ty \leqslant Tx$. Assume $Tx \leqslant Ty$ and let $I \subset N$ be such that $y \in \dot{H}_I^+$. It follows that $x \in H_I^+$ and by

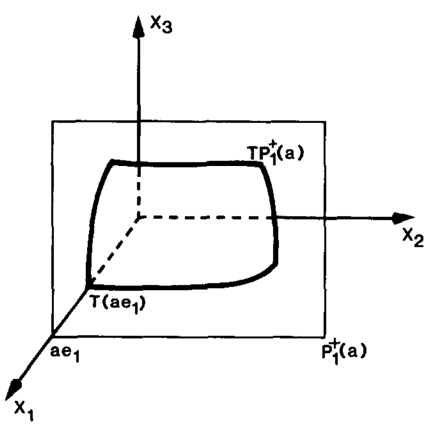

FIG. 3.1. Portions of $P_1^+(a)$ and $TP_1^+(a)$ in the case that $T(ae_1) <_{\{1\}} ae_1$.

Proposition 3.1(iv), $x <_I y$. But $i \in I$ so we have the contradiction $a < a$. It follows that the functions X_i: $Q_i(TP_i(a)) = D_i \to R$ is well defined. If $\hat{x}_j \in D_i$, $j = 1, 2$, are distinct, $\hat{x}_1 \leqslant \hat{x}_2$ and if $X_1(\hat{x}_1) \leqslant X_1(\hat{x}_2)$ then we have two points x and y in $P_i(a)$ with $Tx \leqslant Ty$. But now we can again argue as above to the contradiction $a < a$. Thus we have proved (1).

For the remainder of the proof, take $i = 1$ (there are no special coordinates) and consider what the implicit function theorem says about the map $((y_1, x_2,..., x_n), (y_2, y_3,..., y_n)) \to^F T(a, x_2,..., x_n) - (y_1, y_2,..., y_n)$ of $R^n \times R^{n-1}$ into R^n at a particular zero of the map $((\bar{y}_1, \bar{x}_2,..., \bar{x}_n), (\bar{y}_2,..., \bar{y}_n))$. We may assume $\bar{x} = (a, \bar{x}_2,..., \bar{x}_n) \in \dot{H}_I^+$ for some I, $\{1\} \in I \subset N$ and $(\bar{y}_1,..., \bar{y}_n) \in \dot{H}_I^+$ follows. The jacobian, $(\partial F/\partial(y_1, x_2,..., x_n))(\bar{x}) = (-e_1, DT(\bar{x})e_2,..., DT(\bar{x})e_n)$ is singular if there exists $h \neq 0$, $h \in H_{C\{1\}}$, such that $DT(\bar{x})h = e_1$. Now by (H2) and (H3), there is a permutation σ of N fixing one such that in the basis $\{e_1, e_{\sigma(2)},..., e_{\sigma(n)}\}$, $DT(\bar{x})$ has the form

$$\begin{array}{cc} & \begin{array}{cc} \# I & \# C(I) \end{array} \\ \begin{array}{c} \# I \\ \# C(I) \end{array} & \begin{bmatrix} A & B \\ 0 & C \end{bmatrix}, \end{array}$$

where $A = DT(\bar{x})_I$ and A and C are invertible since $DT(\bar{x})$ is invertible. We have $h = \operatorname{col}(h^1, h^2)$ with $h^1 = (0, h_2^1,..., h_{\# I}^1)$ and $e_1 = \operatorname{col}((1, 0,..., 0), (0,..., 0))$ so $DT(\bar{x})h = e_1$ implies

$$Ah^1 + Bh^2 = \operatorname{col}(1, 0,..., 0),$$

$$Ch^2 = 0,$$

or, since C is invertible,

$$h^1 = A^{-1}(\operatorname{col}(1, 0,..., 0)).$$

But $A^{-1} > 0$ implies $h^1 > 0$ contradicting that the first component of h^1 is zero. Hence our jacobian is nonzero and the implicit function theorem implies that the zero set of F in a neighborhood of $((\bar{y}_1, \bar{x}_2,..., \bar{x}_n), (\bar{y}_2,..., \bar{y}_n))$ is the graph of a C^2-function $(y_1, x_2,..., x_n) = X(y_2,..., y_n) = (X_1,..., X_n)$ and $y_1 = X_1(y_2,..., y_n)$. For the proof of (2) one must implicitly differentiate and we leave this to the reader. In case $I = N$ in (2), the reader should find that

$$\frac{\partial X_1}{\partial x_j}(x) = -\frac{M_{1j-1}}{M_{11}},$$

where $DT(x)^{-1} = (M_{rs})_{(r,s) \in N \times N} > 0$ if $x > 0$.

Before beginning our study of the dynamics generated by the map T, we impose some restrictions on the behavior of $T|_{H_{\{i\}}}$ for each $i \in N$ motivated by our discussion in the previous section. We have already noted that

$$DT(0) = \operatorname{diag}[\lambda_1, \lambda_2, ..., \lambda_n], \qquad \lambda_i > 0.$$

We will assume from here on that the following hold:

(H5) $\lambda_i \neq 1$ for all i (hyperbolicity).

(H6) If $\lambda_i < 1$ then 0 is a globally attracting fixed point for the dynamics generated by $T|_{H_{\{i\}}}$.

If $\lambda_1 > 1$ then $T|_{H_{\{i\}}}$ has a unique fixed point $u_i > 0$ and $0 < d/dx_i(T|_{H_{\{i\}}})(u_i) < 1$. Hence u_i attracts all orbits with nontrivial initial condition in $H_{\{i\}}$.

For simplicity of notation we let u_i denote both a scalar and the vector $u_i e_i \in R^n_+$, the context in which it is used will determine the appropriate meaning.

If $x \in R^n_+$, let $O(x) = \{T^p x\colon p \text{ an integer}\}$, $O^+(x) = \{T^p x\colon p \text{ a non-negative integer}\}$, $O^-(x) = \{T^p x\colon p \text{ a nonpositive integer}\}$ denote the orbit, the positive orbit and the negative orbit of x. Only $O^+(x)$ is guaranteed to exist, of course. Let $\Lambda(x)$ denote the limit points of $O^+(x)$, $\Lambda(x) = \{y\colon y = \lim_{i\to\infty} T^{p_i}x$, where $\{T^{p_i}x\}_{i \geqslant 0}$ is a convergent subsequence of $\{T^p x\}_{p \geqslant 0}\}$, and $\alpha(x)$ denote the limit points of $O^-(x)$ if $O^-(x)$ is defined. If $O^+(x)$ is bounded then $\Lambda(x)$ is a nonempty compact set invariant under T: $T\Lambda(x) = \Lambda(x)$. A similar statement holds for $\alpha(x)$ if $O^-(x)$ is defined and bounded.

We now begin our study of the discrete dynamical system generated by T. Our first result says that all orbits are bounded and approach a certain compact set.

PROPOSITION 3.4. *Let $I^* = \{i \in N\colon \lambda_i > 1\}$ and, if $I^* \neq \phi$, let $u_{I^*} = \sum_{i \in I^*} u_i \in \dot{H}^+_{I^*}$. If $x \in R^n_+$ then $O^+(x)$ is bounded. If $I^* = \phi$ then $\Lambda(x) = 0$ for all $x \in R^n_+$. If I^* is a nonempty proper subset of N then $\Lambda(x) = 0$ for $x \in H^+_{C(I^*)}$ and $\Lambda(x) \subset [0, u_{I^*}] - \{0\}$ for $x \notin H^+_{C(I^*)}$. If $I^* = N$ then $\Lambda(x) \subset [0, u_N] - \{0\}$ for $x \neq 0$.*

Proof. Let $x = (x_1, x_2, ..., x_n)$ and suppose $x_j > 0$. If $j \in I^*$, $T^p(x_j e_j) \to u_j$ as $p \to \infty$ while if $j \in C(I^*)$, $T^p(x_j e_j) \to 0$ as $p \to \infty$ by (H6). Lemma 3.2 implies $(T^p)_j(x) < (T^p)_j(x_j e_j)$ for $p = 1, 2, ...$. Consequently, if $j \in C(I^*)$, $(T^p)_j(x) \to 0$ while if $j \in I^*$ then $\limsup_{p\to\infty} (T^p)_j(x) \leqslant u_j$. This, together with the hyperbolicity of the trivial fixed point, implies the result.

Proposition 3.4 can be improved in the following way in case $I^* \neq \phi$. If $D \subseteq [0, u_{I^*}]$ has the property that every T invariant subset of $[0, u_{I^*}]$ is contained in D, then necessarily $\Lambda(x) \subset D$ for every $x > 0$ since $T\Lambda(x) = \Lambda(x)$. We will now construct such a set.

Hereafter, we will assume $I^* \neq \phi$ and restrict our attention to $T \mid H^+_{I^*}$: $H^+_{I^*} \to H^+_{I^*}$. Equivalently, we may and do take $I^* = N$. We make some remarks on the general case where I^* is a nonempty proper subset of N at the end of this section. Let $u = u_N$

PROPOSITION 3.5. *If* $y \in [0, u]$, $y \neq u_i$, $1 \leqslant i \leqslant n$, *then* $T_i(y) < u_i$, $1 \leqslant i \leqslant n$. $T[0, u]$ *is a compact set contained in* $[0, u]$ *and bounded by the* $2n$ $(n-1)$*-manifolds* $H^+_{C\{i\}}$, $T(P_i^+(u))$, $1 \leqslant i \leqslant n$. *The set*

$$D = \bigcap_{p \geqslant 0} T^p[0, u]$$

is a nonempty compact invariant set with the property that every invariant subset of $[0, u]$ *is contained in* D. *If* $v \in D$ *then* $[0, v] \subset D$.

Proof. The first assertion is immediate from Lemma 3.2. Hence $T[0, u] \subset [0, u]$ and $T^{p+1}[0, u] \subset T^p[0, u]$, $p = 0, 1, 2, \ldots$. It follows that D is a nonempty compact subset of $[0, u]$. The remaining assertions are easily checked (the last assertion follows since T^p satisfies (H4)).

Since we are assuming $\lambda_i > 1$, $1 \leqslant i \leqslant n$, the zero fixed point of T is repulsive and we expect that its domain of repulsion is contained in D. Let $B(0)$ denote the domain of repulsion, $B(0) = \{ y \in \bigcap_{p \geqslant 0} T^p(R^n_+)\colon T^{-j}y \to 0$ as $j \to \infty \}$. The following result gives some properties of $B(0)$.

PROPOSITION 3.6. $B(0)$ has the following properties.

(i) *If* $y \in B(0)$ *then* $[0, y] \subset B(0)$,

(ii) $\overline{B(0)} \subset D$,

(iii) $B(0)$, $\overline{B(0)}$ *are* T *invariant*,

(iv) $B(0)$ *is relatively open in* R^n_+,

(v) $B(0) \cap H_{\{i\}} = [0, u_i)$.

Proof. The reader may check that (H4) holds for T^p, $p = 1, 2, \ldots$, by induction on p. If $y \in B(0)$ then for each $p \geqslant 0$ there exists $x_p \in R^n_+$ such that $y = T^p x_p$. It follows that $[0, y] \subset T^p[0, x_p]$ and thus $[0, y] \subset \bigcap_{p \geqslant 0} T^p(R^n_+)$. If $z \in [0, y]$, then $T^{-j}z \leqslant T^{-j}y$ for $j = 1, 2, \ldots$, since T^{-1} is order preserving. Since $T^{-j}y \to 0$ as $j \to \infty$, it follows that the same holds for z and $z \in B(0)$.

If $y \in B(0)$ then $T^{-j}y \in [0, u]$ for all large $j (u > 0)$. Equivalently,

$y \in T^j[0, u]$ for all large j and hence $y \in D$. Thus $B(0) \subset D$ and $\overline{B(0)} \subset D$ since D is closed.

Clearly $T(B(0)) = B(0)$ and $T(\overline{B(0)}) = \overline{B(0)}$ follows since T is a homeomorphism.

Using the fact that $DT(0) = \text{diag}\{\lambda_1, \ldots, \lambda_n\}$ with $\lambda_i > 1$, $1 \leqslant i \leqslant n$, it is easily established that there exists $x > 0$, $x \in [0, u]$, satisfying $Tx > x$ and with $[0, x]$ containing no nontrivial fixed point of T. Since $[0, x] \subset [0, Tx] \subset T[0, x]$, we have $T^p[0, x] \subset T^{p+1}[0, x]$, $p = 0, 1, 2, \ldots$. Clearly $W = \bigcup_{p \geqslant 0} T^p[0, x]$ is a T invariant subset of D with nonempty interior. Now $Tx > x$ implies $T^{-(p-1)}x > T^{-p}x$ for $p = 1, 2, \ldots$, by Proposition 3.1(iv). It follows that $\lim_{p \to \infty} T^{-p}x$ exists and is a fixed point of T in $[0, x]$. Hence this fixed point must be the trivial fixed point. We may conclude that $O(x) \subset B(0)$ and hence $W \subset B(0)$. In fact $W = B(0)$ as the reader may show but at least we may conclude that $B(0)$ contains a neighborhood of zero in R^n_+. If $y \in B(0)$ then $T^{-p}y$ lies in this neighborhood of zero so by continuity of T^{-p}, all points sufficiently near y have images under T^{-p} in this neighborhood. It follows that $B(0)$ is open in R^n_+.

Assertion (v) is obvious from (H6).

The fixed points u_i of T lie on the boundary of $B(0)$ relative to R^n_+ according to Proposition 3.6(v). We will show that all nontrivial fixed points and periodic points of T lie on the boundary of $B(0)$ relative to R^n_+ with an additional assumption on T. Before doing this we describe the geometry of this set which we label S.

Proposition 3.7. *$S = \partial B(0)$ is a T invariant set containing the fixed points u_i, $1 \leqslant i \leqslant n$, and satisfying*:

(i) *S is homeomorphic to the $n-1$ simplex*

$$\left\{ \sum_{i=1}^{n} t_i u_i : t_i \geqslant 0, \sum_{i=1}^{n} t_i = 1 \right\}.$$

(ii) *If $x, y \in S$ are distinct points then x and y cannot be weakly related.*

(iii) *For each $j \in N$, the orthogonal projection Q_j: $R^n \to H_{C\{j\}}$ along e_j is a homeomorphism when restricted to S: $Q_j|_S$: $S \to Q_j(S)$. If $u, v \in Q_j(S)$ with $u \leqslant v$ then $[u, v] \subset Q_j(S)$. The map Q_j^{-1}; $Q_j(S) \to S$ has the form $Q_j^{-1}(\hat{x}) = \hat{x} + h_j(\hat{x})e_j$, $\hat{x} = (x_1, \ldots, x_{j-1}, 0, x_{j+1}, \ldots, x_n)$ where h_j is strictly decreasing: $h_j(\hat{x}) < h_j(\hat{y})$ if $\hat{x} \geqslant \hat{y}$, $\hat{x} \neq \hat{y}$.*

Proof. S is T invariant since both $B(0)$ and $\overline{B(0)}$ are T invariant. We verify (ii) first. Suppose $x, y \in S$ are distinct with $x \leqslant y$. Let $I \neq \phi$, $I \subset N$ be such that $y \in \dot{H}_I^+$. Then $x \in H_I^+$ and $x \leqslant y$. Since $TS = S$, $x = Tu$, $y = Tv$, where $u, v \in S$ are distinct and Proposition 3.1(iv) implies that $u <_I v$. Now $v \in S \cap \dot{H}_I^+$ implies there exists $w \in B(0) \cap \dot{H}_I^+$ so close to v that $u <_I w$. But then $u \in B(0)$ since $[0, w] \subset B(0)$. This contradiction proves (ii).

To verify (i), note that if $h \geqslant 0$, $|h| = 1$, then there exists atmost one point of intersection of the ray $\{th\colon t \geqslant 0\}$ with S by (ii) and there exists at least one point of intersection of the ray with S since $B(0)$ is relatively open in R^n_+ and bounded. Thus the map $h \to t_h h$ from $\{x \in R^n_+ : |x| = 1\}$ to S is well defined. It is injective by (ii) and obviously onto S. The continuity of the map is easily verified so S is homeomorphic to the intersection of the unit ball with R^n_+ which, in turn, is homeomorphic to the simplex.

The verification of (iii) is essentially contained in [15, Proposition 2.4].

It should be clear that (i) and (iii) of Proposition 3.7, suitably modified, hold for $S \cap H_I$, $\phi \neq I \subset N$.

At this point the reader must wonder whether the sets $\overline{B(0)}$ and D are not the very same. We have already shown that $\overline{B(0)} \subset D$. We now indicate that the boundary of D relative to R^n_+ possesses some of the same properties as S.

PROPOSITION 3.8. *$M \equiv \partial_{R^n_+} D$ is a T invariant set containing the fixed points u_i, $1 \leqslant i \leqslant n$. M satisfies* (i)–(iii) *of Proposition* 3.7. *In addition, if $x \notin D$ then $\Lambda(x) \subset M$.*

The proof of Proposition 3.8 is entirely similar to that of Proposition 3.7.

The set S is clearly an important invariant set for T: it contains the fixed points u_i and $\Lambda(x) \subset S$ for every $x \in B(0)$, $x \neq 0$. Under an additional assumption on T, mentioned in the previous section, we will show that all fixed points and periodic points of T belong to S.

Recall that if $x \neq 0$ is a periodic point of T of period p, then $D(T^p)(x) = DT(T^{p-1}x) \cdots DT(x)$ has a nonnegative inverse: $D(T^p)(x)^{-1} \geqslant 0$. Consequently, by the Perron–Frobenius theorem, $D(T^p)(x)$ has a positive eigenvalue with the property that no other eigenvalue of $D(T^p)(x)$ has smaller modulus. We label this eigenvalue $\mu_p(x)$ (or simply $\mu(x)$ if $p = 1$). Corresponding to $\mu_p(x)$ there is an eigenvector for $D(T^p)(x)$ in R^n_+. If $x \notin \dot{R}^n_+$ then there is some I, $\phi \neq I \subset N$ such that $x \in \dot{H}_I^+$. It then follows from (H3) that $D((T|_{H_I})^p)(x) = D(T|_{H_I})(T^{p-1}x) \cdots D(T|_{H_I})(x) = [D(T^p)(x)]_I$ has a positive inverse. The Perron–Frobenius theory implies that this submatrix of $D(T^p)(x)$ has a positive, simple eigenvalue, which we label $\mu_{I,p}(x)$, which is strictly smaller in modulus than all other eigenvalues of $[D(T^p)(x)]_I$ and corresponding to which there is an eigenvector in $\dot{H}_I^+$. Since $\mu_{I,p}(x)$ is also an eigenvalue of $D(T^p)(x)$, we have $\mu_p(x) \leqslant \mu_{I,p}(x)$.

The following theorem is of fundamental importance to our discussion of fixed points and periodic points of T. It is the sole reason that we assume $T \in C^2$ (see [14]).

THEOREM 3.9. *Let x_1 be a nontrivial, p-periodic point of T and $I \subset N$ such that $x \in \dot{H}_I^+$. Assume $\mu_{I,p}(x_1) \equiv \mu_1 < 1$ and denote by v the unique (up to scalar multiple) eigenvector of $D(T^p)(x_1)$ which belongs to $\dot{H}_I^+$. Then there exists t_0, $0 < t_0 \leqslant \infty$, and a unique C^1 function $y_+ : [0, t_0) \to \dot{H}_I^+$ satisfying*:

(A$_+$) $y_+(t) = x_1 + tv + O(t^2)$ *as* $t \to 0$.

(B$_+$) $0 < t_1 < t_2 < t_0$ *implies* $y_+(t_1) <_I y_+(t_2)$.

(C$_+$) $T^p(y_+(t)) = y_+(\mu_1 t)$, $0 \leqslant t < t_0$.

(D$_+$) *Either* $\lim_{t \to t_0^-} |y_+(t)| = \infty$ *or* $\lim_{t \to t_0^-} y_+(t) = x_2 \in \dot{H}_I^+$. *In the latter case,* $t_0 = \infty$, $T^p x_2 = x_2$, $\mu_{I,p}(x_2) \geqslant 1$, *and* $\lim_{t \to \infty} y'_+(t)/|y'_+(t)| = w \in \dot{H}_I^+$, *where* $D(T^p)(x_2)w = \mu_{I,p}(x_2)w$.

(E$_+$) *If* $\lim_{t \to t_0^-} |y_+(t)| = \infty$ *then for all* $x \in \dot{H}_I^+$, $x \neq x_1$, $x \geqslant x_1$ *either there exists* N *such that* $T^{-mp}x \in T^p(H_I^+)$, $0 \leqslant m \leqslant N$ and $T^{-(N+1)p}x \notin T^p(H_I^+)$ *or* $T^{-mp}x \in T^p(H_I^+)$ *for all* $m \geqslant 0$ *and* $|T^{-mp}x| \to \infty$ *as* $m \to \infty$. *If* $\lim_{t \to \infty} y_+(t) = x_2$ *then* $T^{-mp}x \to x_2$ *as* $m \to \infty$ *for all* $x \neq x_1$, $x \in [x_1, x_2]$.

There exists a C^1 function $y_- : [0, \infty) \to \dot{H}_I^+$ satisfying

(A$_-$) $y_-(t) = x_1 - tv + O(t^2)$ *as* $t \to 0$

(B$_-$) $0 \leqslant t_1 \leqslant t_2$ *implies* $y_-(t_2) <_I y_-(t_1)$

(C$_-$) $T^p(y_-(t)) = y_-(\mu_1 t)$, $t \geqslant 0$

(D$_-$) $\lim_{t \to \infty} y_-(t) = x_0 \in H_I^+$, $T^p x_0 = x_0$, *and* $\mu_{I,p}(x_0) \geqslant 1$. *If* $x_0 >_I 0$, *then*

$$\lim_{t \to \infty} \frac{-y'_-(t)}{|y'_-(t)|} = w \in \dot{H}_I^+, \qquad \text{where} \quad D(T^p)(x_0)w = \mu_{I,p}(x_0)w.$$

(E$_-$) *If* $x \neq x_1$, $x \in [x_0, x_1]$ *then* $T^{-mp}x \to x_0$ *as* $m \to \infty$.

Theorem 3.9, a rather long-winded result, essentially says that to every periodic point $x_1 \in \dot{H}_I^+$ with $\mu_{I,p}(x_1) < 1$, there are two monotone invariant curves $C^+(x_1) = \{y_+(t)\colon 0 \leqslant t < t_0\} \subseteq \dot{H}_I^+$ and $C^-(x_1) = \{y_-(t)\colon t \geqslant 0\} \subseteq \dot{H}_I^+$ for T, which, according to (C_+) and (C_-) might be called the "most stable" manifolds of x_1. $C^+(x_1)$ either connects x_1 to ∞ or to a periodic point $x_2 >_I x_1$ in $\dot{H}_I^+$ while $C^-(x_1)$ connects x_1 to a fixed point $x_0 \in H_I^+$, $x_0 <_I x_1$. These periodic points x_0 and x_2 (if it exists) are quite unstable for $T|_{H_I^+}$ since $\mu_{I,p} \geqslant 1$ for each. The periods of these periodic points divide p, they need not be p-periodic points.

The proof of Theorem 3.9 is contained in [15, Theorem 3.3] for the case $I = N$ and $p = 1$. This proof contains all the essential ideas since $(T|_{H_I^+})^p$ has the same properties as T.

Motivated by our discussion in the previous section we now make our final assumption.

(H7) If x is a nontrivial periodic point of T and $I \subset N$ is such that $x \in \dot{H}_I^+$ then $\mu_{I,p}(x) < 1$.

A brief glance at Theorem 3.9 shows that (H7) has important implications. We list some of these implications in

THEOREM 3.10. *Let* (H7) *hold. Then all nontrivial periodic points of T belong to S. If x_1 is a nontrivial periodic point of T and $I \subset N$ is such that $x_1 \in \dot{H}_I^+$, then $C^-(x_1) \subset B(0) \cap \dot{H}_I^+$ connects x_1 to the trivial fixed point: $\lim_{t \to \infty} y_-(t) = 0$. $C^+(x_1) \subset \dot{H}_I^+$ connects x_1 to ∞: $\lim_{t \to t_0^-} y_+(t) = \infty$. Moreover, x_1 belongs to the boundary of $D \cap H_I^+$ relative to $\dot{H}_I^+$.*

Proof. All nontrivial periodic points of T belong to $D - B(0)$. Let x_1 be a nontrivial periodic point of T, $x_1 \in \dot{H}_I^+$. Theorem 3.9 implies that $C^-(x_1)$ connects x_1 to a periodic point $x_0 \in H_I^+$, $x_0 <_I x_1$, with $\mu_{I,p}(x_0) \geqslant 1$. By (H7), the trivial fixed point is the only such point so $\lim_{t \to \infty} y_-(t) = 0$. But then Theorem 3.9($C_-$) implies $y_-(t) \in B(0)$ for $t > 0$. In fact, for $m = 1, 2, \ldots$, $T^{-mp} y_-(t) = y_-(\mu_{I,p}^{-m} t)$ so $T^{-mp} y_-(t) \to 0$ as $m \to \infty$. It follows that $x_1 \in S \cap \dot{H}_I^+$.

According to Theorem 3.9, to prove $\lim_{t \to t_0^-} y_+(t) = \infty$ we need only rule out the possibility that $t_0 = \infty$ and $\lim_{t \to \infty} y_+(t) = x_2$, where $x_2 \in \dot{H}_I^+$, $x_2 >_I x_1$, $T^p x_2 = x_2$, and $\mu_{I,p}(x_2) \geqslant 1$. But the latter cannot occur since by our previous argument, $x_2 \in S$ and x_1 and x_2 are weakly related, a contradiction to Proposition 3.7(ii).

Finally x_1 must belong to the boundary of $D \cap H_I^+$ relative to H_I^+. If not, there is a neighborhood V of x_1, relative to H_I^+, contained in D. If $v \in \dot{H}_I^+$ is the positive eigenvector of $D(T^p)(x_1)$, described in Theorem 3.9, corresponding to $\mu_{I,p}(x_1)$ then $x_1 + tv \in V \subset D$ for all small values of t. It follows from [14, Remark 3, Theorem 1.1] that $C^+(x_1) \subset D \cap H_I^+$ contradicting that $\lim_{t \to t_0^-} y_+(t) = +\infty$.

Are the sets $\overline{B(0)}$ and D the same? Equivalently, are $M = \partial_{R_+^n} D$ and $S = \partial_{R_+^n} B(0)$ the same? We saw in Propositions 3.7 and 3.8 that they share several important properties together with the fixed points u_i, $1 \leqslant i \leqslant n$. Theorem 3.10 implies that every periodic point of T lies on M and S. We are led to *conjecture* that, generically, these sets are the same although we give an example in the next section where they are not. If the conjecture were true then, generically, $\Lambda(x) \subset S$ for every nontrivial x, i.e., all the action occurs on S, topologically, an $(n-1)$ simplex in R_+^n. As it stands, we may conclude that $\Lambda(x) \subset S$ for $x \in \overline{B(0)}$, $\Lambda(x) \subset M$ for $x \notin D$, and $\Lambda(x) \subset D - B(0)$ for $x \in D - \overline{B(0)}$.

In the following result we deduce some order properties of orbits and limit sets for points $x \in D - B(0)$.

THEOREM 3.11. *Let* (H7) *hold and let* $x \in D - B(0)$. *Then each of the sets* $O(x)$, $\Lambda(x)$ *and* $\alpha(x)$ *contain no pair of weakly related points.*

Proof. Suppose $O(x)$ contains two weakly related points $T^p x$ and $T^l x$, $l > p$. By appropriate choice of $I \subset N$ we may assume $O(x) \subset \dot{H}_I^+$.

Suppose $T^l x \geqslant T^p x$ so $T^p T^{l-p} x \geqslant T^p x$. By Proposition 3.1, $T^{l-p} x >_I x$. Let $r = l - p > 0$. Then $x <_I T^r x$ and the monotonicity of T^{-r} implies $\cdots <_I T^{-mr} x <_I T^{-(m-1)r} x <_I \cdots <_I x <_I T^r x$. It follows that $\lim_{m \to \infty} T^{-mr} x = y \in H_I^+$ exists and $T^r y = y$. Now $y \in S \cap H_I^+$, $y <_I x$, so, since $[0, x] \subset D$, this contradicts Theorem 3.10.

Suppose $T^l x \leqslant T^p x$ so $T^p T^{l-p} x \leqslant T^p x$. Again, by Proposition 3.1, $T^r x <_I x$, where $r = l - p$. Since T^{-r} is order preserving,

$$T^r x <_I x <_I T^{-r} x <_I T^{-2r} x <_I \cdots <_I T^{-mr} x <_I \cdots.$$

We may conclude that $\lim_{m \to \infty} T^{-mr} x = y$ exists where $y \in \dot{H}_I^+$, $y >_I x$, and $T^r y = y$. But then $y \in S \cap \dot{H}_I^+$ and so $x \in B(0)$ contradicting our assumption that $x \in D - B(0)$.

We have shown that $O(x)$ can contain no pair of weakly related points. Suppose $\Lambda(x)$ contains two weakly related points, $\bar{y}$ and $\bar{z}$, $\bar{y} \leqslant \bar{z}$. Let $J \subset I$ be such that $z \in \dot{H}_J^+$ so $\bar{y} \in H_J^+$ and $\bar{y} \leqslant \bar{z}$. Since both H_J^+ and $\Lambda(x)$ are mapped into themselves by T, $\bar{z} = Tz$, and $\bar{y} = Ty$, where $z \in \dot{H}_I^+ \cap \Lambda(x)$, $y \in H_I^+ \cap \Lambda(x)$, and $y <_J z$ by Proposition 3.1. We can choose a positive integer p such that $T^p x$ is so close to z that $(T^p x)_j > y_j$ for $j \in J$. Fixing p we can now choose a positive integer l such that $T^l x$ is so close to y that $T^l x \leqslant T^p x$. This contradicts that $O(x)$ contains no pair of weakly related points. Thus $\Lambda(x)$ contains no pair of weakly related points. A similar argument can be used to show that $\alpha(x)$ contains no pair of weakly related points.

An immediate corollary of Theorem 3.11 and our discussion leading up to its statement is

COROLLARY 3.12. *If* $I^* = N$, *then for every* $x \in R_+^n$, $\Lambda(x)$ *contains no pair of weakly related points.*

Let us briefly review what we have been able to prove in this section. We consider the general situation where $I^* = \{i \in N\colon \lambda_i > 1\}$. If $I^* = \phi$ then $\Lambda(x) = 0$ for every $x \in R_+^n$ by Proposition 3.4. If I^* is a nonempty, proper subset of N, then $\Lambda(x) = 0$ if and only if $x \in H_{C(I^*)}^+$ by Proposition 3.4. If $x \notin H_{C(I^*)}^+$ then $\Lambda(x) \subset D - \overline{B(0)}$ where $\overline{B(0)}$ and D are subsets of $H_{I^*}^+$. This follows partly from Proposition 3.4, which asserts that $\Lambda(x) \subset [0, u_{I^*}] - \{0\}$, and partly from the fact that $\Lambda(x) \cap B(0) = \phi$ while every invariant subset of $[0, u_{I^*}]$ is a subset of D. In both the case where I^* is a nonempty proper subset of N and when $I^* = N$, all the nontrivial fixed points and periodic points of T lie on $S \cap M$, where S and M are the boundaries relative to $H_{I^*}^+$ of the sets $B(0)$ and D. The sets S and M have very similar properties, both being homeomorphic to the standard $(\# I - 1)$-dimensional simplex $\{\sum_{i \in I^*} t_i u_i\colon\ t_i \geqslant 0,\ \sum_{i \in I^*} t_i = 1\}$ in $H_{I^*}^+$. Associated with each periodic point x of T on S are two invariant curves $C^-(x)$ and $C^+(x)$ connecting x to 0, respectively, ∞. $C^-(x) - \{x\}$ lies in $\dot{H}_I^+ \cap B(0)$ and $C^+(x) - \{x\} \subset \dot{H}_I^+ \cap D$ where $I \subset N$ is such that $x \in \dot{H}_I^+$. $C^-(x) \cup C^+(x)$ belongs to the stable manifold of the periodic point x.

Briefly, we have been able to say something about where periodic points lie, and more generally, where limit sets lie. There are many open problems remaining. For example, it would be very nice to have sufficient conditions for $\Lambda(x) \subset S$ for all nontrivial x. One would then want to know how smooth is the manifold S. A question related to this is whether the unstable manifold of an unstable periodic point must lie on S. The answer should be yes for a hyperbolic periodic point (see the next section for a counterexample in the nonhyperbolic case). Clearly the unstable manifold cannot impinge on $B(0)$ and in [15] we showed that no point of the unstable manifold of a periodic point x distinct from x can be weakly related to x.

Finally, in the next section we will give a complete qualitative description of the dynamics of two-dimensional competitive maps in the case that all fixed points are nondegenerate. It follows that we have a fairly complete description of the dynamics for $n > 2$ if $\# I^* = 2$.

4. Competitive Maps on R_+^2

In two dimensions, to say x and y are not related is to give the same amount of information as to say x and y are related. This fact, not true in higher dimensions, allows us to obtain significantly stronger results for two-dimensional competitive maps than the general case considered in the previous section. In fact, for every $x \in R_+^2$, either x is a fixed point of T or $\lim_{p \to \infty} T^p x$ is a fixed point of T and the convergence of $T^p x$ to the fixed point is eventually monotone. This very important result is due to de Mottoni and Schiaffino [12] who proved the result for a special class of competitive maps T, the Poincare map of a periodic Lotka–Volterra system, but the proof of which remains valid in the general case. We reproduce that proof here for completeness (see also[4]). We assume in this section that T satisfies (H1)–(H7), although for the result of de Mottoni and Schiaffino we use only (H1)–(H4). Some additional notation will prove to be useful. For $i = 1, 2, 3, 4$, let Q_i denote the usual open quadrants of R^2, in counterclockwise order, e.g., $Q_1 = \{x \in R^2\colon\ x > 0\}$. Given $x \geqslant 0$, let $Q_i(x) =$

$(x+Q_j)\cap R^2_+$, $j=1,2,3,4$. The following lemma, due to de Mottoni and Schiaffino, is instrumental in proving the above-mentioned result.

LEMMA 4.1. *If* $x>0$ *then for* $i=2,4$, $T(\overline{Q_i(x)}-x)\subset Q_i(Tx)$; $T(\overline{Q_1(x)}-x)\cap\overline{Q_3(Tx)}=\phi$ *and* $T(\overline{Q_3(x)}-x)\cap\overline{Q_1(Tx)}=\phi$.

Proof. If $y\in\overline{Q_2(x)}-x$ then Tx and Ty cannot be weakly related in view of Proposition 3.1(iv). Thus $T(\overline{Q_2(x)}-x)\subset Q_2(Tx)\cup Q_4(Tx)$. Since $T(\overline{Q_2(x)}-x)$ is connected, it must be contained entirely in one of the two sets $Q_2(Tx)$ or $Q_4(Tx)$. Now note that Proposition 3.3 implies that the two rays emanating from x and forming part of the boundary of $Q_2(x)$ lie in $Q_2(Tx)$. A similar argument shows $T(\overline{Q_4(x)}-x)\subset Q_4(Tx)$.

If $z\in T(\overline{Q_1(x)}-x)\cap\overline{Q_3(Tx)}$ then $z=Ty$ for $y\neq x$, $y\geqslant x$ and $Ty\leqslant Tx$. But Proposition 3.1(iv) produces the contradiction $y<x$.

We can now state the theorem of de Mottoni and Schiaffino [12]. Note that we are assuming only that (H1)–(H4) hold for T.

THEOREM 4.2. *If* T *is a competitive map on* R^2_+, $x\geqslant 0$; *and* $x^{(m)}=(x_1^{(m)},x_2^{(m)})=T^mx$, $m=1,2,\ldots$, *then there exists a positive integer* M *such that if* $m\geqslant M$, $\{x_1^{(m)}\}$ *and* $\{x_2^{(m)}\}$ *are monotone sequences. Either* $\lim_{m\to\infty}T^mx=\infty$ *or* $\lim_{m\to\infty}T^mx=x_0$, *where* $Tx_0=x_0$.

Proof. We need only consider the case that $x>0$ and x is not a fixed point of T. Exactly one of the following must hold for $O^+(x)$: (1) for every $m\geqslant 0$, $T^{m+1}x\in\overline{Q_1(T^mx)}\cup\overline{Q_3(T^mx)}$, or (2) for some integer $M\geqslant 0$, $T^{M+1}x\in Q_2(T^Mx)\cup Q_4(T^Mx)$. If (1) holds and $Tx\in\overline{Q_1(x)}$ then $T^2x\in\overline{Q_1(Tx)}$ by Lemma 4.1 since $T(\overline{Q_1(x)}-x)\cap\overline{Q_3(Tx)}=\phi$. By induction on m, $T^{m+1}x\in\overline{Q_1(T^mx)}$ and the lemma is proved in this case; similarly if $Tx\in\overline{Q_3(x)}$. If (2) holds and $T^{M+1}x\in Q_2(T^Mx)$ then by Lemma 4.1, $T^{M+2}x\in T(Q_2(T^Mx))\subset Q_2(T^{M+1}x)$ and by induction $T^{m+1}x\in Q_2(T^mx)$ for $m\geqslant M$. Again the lemma follows.

If we assume (H1)–(H7) holds for T then all orbits are bounded. It follows from Theorem 4.2 that if $x\geqslant 0$ then either $Tx=x$ or $\Lambda(x)$ is a fixed point of T. Recall that $DT(0)=\operatorname{diag}[\lambda_1,\lambda_2]$ where $\lambda_i>0$, $\lambda_i\neq 1$ for $i=1,2$.

THEOREM 4.3. *If* $0<\lambda_i<1$ *for* $i=1,2$ *then* $\Lambda(x)=0$ *for every* $x\geqslant 0$. *If* $\{i,j\}=\{1,2\}$ *and* $0<\lambda_i<1$, $\lambda_j>1$ *then* $\Lambda(x)=0$ *for* $x\in H^+_{\{i\}}$ *and* $\Lambda(x)=u_j$ *for* $x\geqslant 0$, $x\notin H^+_{\{i\}}$. *If* $\lambda_i>1$, $i=1,2$, *then* $\Lambda(x)\subset S$ *is a fixed point for* $x\neq 0$, $x\geqslant 0$.

Proof. The first assertion is immediate (and holds in R^n_+ if $0<\lambda_i<1$, $1\leqslant i\leqslant n$) from Proposition 3.4. The remaining assertions follow from

Proposition 3.4, the hyperbolicity of the trivial fixed point and Theorem 4.2.

In the remainder of this section we concentrate on the case that $\lambda_i > 1$ for $i = 1, 2$. To be able to catalogue the possible "phase portraits" of the map T it will be assumed that all fixed points of T are nondegenerate. In Fig. 4.1 we describe a scenario in which not all fixed points are non-degenerate and consequently, $\overline{B(0)}$ and D are distinct sets. While confident that some competitive map satisfying (H1)–(H7) generates Fig. 4.1, we are unable to explicitly write down such a map.

The difference between $B(0)$ and D in Fig. 4.1 is due to the two degenerate positive fixed points x_1 and x_2 ($\mu(x_i) \in (0, 1)$ and 1 are the eigenvalues of $DT(x_i)$) and the "fan" of connecting invariant curves for T. Hereafter we assume:

(H8) If $Tx = x$ then $1 \notin \mathrm{sp}(DT(x))$, i.e., x is a nondegenerate fixed point.

One should be able to prove anything given eight hypotheses. Actually, with (H1)–(H8) we will be able to completely catalogue the "phase portraits" of T.

Consider the spectrum of $DT(x)$ for the nontrivial fixed points x of T in the light of (H8). If $x = u_i$ then $\mathrm{sp}(DT(u_i)) = \{\mu_i, \eta_i\}$ where $\mu_i = (d/dx_i)\,|_{x_i = \mu_i} T\,|_{H\{i\}}$ satisfies $0 < \mu_i < 1$ by (H6) and either $0 < \eta_i < 1$, i.e., u_i is stable or $\eta_i > 1$, so u_i is a saddle fixed point. If x is a positive fixed point of T then $\mathrm{sp}(DT(x)) = \{\mu(x), \rho(x)\}$ where $\mu(x)$ is the smallest positive eigenvalue of $DT(x)$, $0 < \mu(x) < 1$, by (H7) and $\rho(x)$ is the positive spectral radius of $DT(x)$ satisfying either $0 < \mu(x) < \rho(x) < 1$ or $0 < \mu(x) < 1 < \rho(x)$ by (H8) and the Perron–Frobenius theory. Corresponding to $\mu(x)$ there is an eigenvector for $DT(x)$ in $\dot{R}^2_+$. Since this eigenvector is the unique, up to

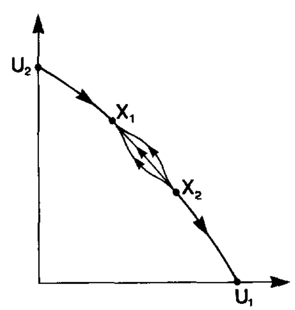

FIG. 4.1. A fan of connecting invariant curves for T join x_1 and x_2. $B(0)$ is bounded by the inside curve and D is bounded by the outer most curve.

scalar multiple, eigenvector of $DT(x)$ in R^2_+, it follows that corresponding to $\rho(x)$ there is an eigenvector in Q_2 (or Q_4). The following result is crucial for determining the phase portrait of two-dimensional competitive maps. Compare it with Theorem 3.9.

THEOREM 4.4. *Let $x_1 > 0$ be a fixed point of T with $\rho(x_1) = \rho_1 > 1$ and $DT(x_1)v = \rho_1 v$, $v \in Q_2$. Then there exists a unique C^1 function y_l: $[0, \infty] \to Q_2(x_1) \cap Q_1$ satisfying*

(A_l) $y_l(t) = x_1 + tv + O(t^2)$ *as* $t \to 0$.

(B_l) $0 < t < s$ *implies* $y_l(s) - y_l(t) \in Q_2$.

(C_l) $y_l(t) = T(y_l(\rho_1^{-1}t))$, $t \geqslant 0$.

(D_l) $\lim_{t\to\infty} y_l(t) = x_l \in Q_2(x_1)$, $Tx_l = x_l$, $\rho(x_l) \leqslant 1$ *and* $\lim_{t\to\infty} y_l'(t)/|y_l'(t)| = f$ *where* $DT(x_l)f = \rho(x_l)f$.

(E_l) *For every* $x \in \overline{Q_2(x_1)} \cap \overline{Q_4(x_l)}$ *different from* x_1, $T^m x \to x_l$ *as* $m \to \infty$.

There exists a unique C^1 function y_r: $[0, \infty) \to Q_4(x_1) \cap Q_1$ satisfying

(A_r) $y_r(t) = x_1 - tv + O(t^2)$ *as* $t \to 0$.

(B_r) $0 < t < s$ *implies* $y_r(s) - y_r(t) \in Q_4$.

(C_r) $y_r(t) = T(y_r(\rho_1^{-1}t))$, $t \geqslant 0$.

(D_r) $\lim_{t\to\infty} y_r(t) = x_r \in Q_4(x_1)$, $Tx_r = x_r$, $\rho(x_r) \leqslant 1$ and $\lim_{t\to\infty} y_r'(t)/|y_r'(t)| = f$ *where* $DT(x_r)f = \rho(x_r)f$.

(E_r) *For every* $x \in \overline{Q_4(x_1)} \cap \overline{Q_2(x_r)}$ *different from* x_1, $T^m x \to x_r$ *as* $m \to \infty$.

Theorems 3.9 and 4.4 give a very nice qualitative picture of the stable and unstable manifolds of a positive saddle fixed point of T. If we let $C^l(x_1) = \{y_l(t): t \geqslant 0\}$ and $C^r(x_1) = \{y_r(t): t \geqslant 0\}$, then C^l and C^r make up the unstable manifold of x_1 by (C_l) and (C_r) of Theorem 4.4. The four curves C^l, C^r, C^+, C^- are invariant, monotone curves connecting x_1 to four distinct unstable fixed points of T (counting ∞ as a fixed point). Figure 4.2 depicts the situation described by the two theorems.

Proof of Theorem 4.4. We merely outline the proof which is essentially similar to the proof of Theorem 4.5 in [15] and based on Theorem 1.1 in [14]. In fact, the latter theorem asserts the existence of y_l: $[0, t_l] \to \dot{R}^2_+$ satisfying (A_l) and (C_l) where $0 < t_l \leqslant \infty$. Since $y_l(t) = \lim_{p\to\infty} T^p(x_1 + \rho_1^{-p}tv)$ by [14, Theorem 1.1], $y_l(t) \in \overline{Q_2(x_1)}$ by Lemma 4.1. Since $y_l(t) \in Q_2(x_1)$ for small t it is easy to show that $y_l(t) \in Q_2(x_1) \cap Q_1$ for all t, $0 < t < t_l$. (B_l) follows essentially as in Theorem 4.5 in

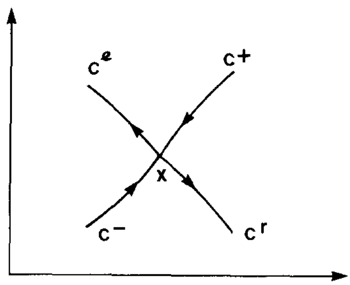

FIG. 4.2. The invariant manifolds C^l, C^+, C^r, and C^- at a saddle point x.

[15]. Since R^2_+ is a closed set mapped into itself by T, $t_l = \infty$ by Remark 3 following Theorem 1.1 in [14]. By (B_l), either $y_l(t) \to \infty$ as $t \to \infty$ or $y_l(t) \to x_l$ as $t \to \infty$ for some $x_l \in Q_2(x_1)$. But (C_l) and the fact that all orbits are bounded rules out the first alternative and x_l must be fixed point of T by (C_l) and continuity of T. The remainder of (D_l) is proved as in [15, Theorem 4.5]. Let $x \in \overline{Q_2(x_1)} \cap \overline{Q_4(x_l)}$ be distinct form x_1 and x_l. By Lemma 4.1 $T^p x \in Q_2(x_1) \cap Q_4(x_l)$ for $p = 1, 2, \ldots$. Now $Tx \in Q_2(x_1) \cap Q_4(x_l)$ implies that for some $t_0 > 0$, $Tx \in Q_2(y(t_0))$. Now, from Lemma 4.1 and (C_l), $T^2 x \in Q_2(y(\rho_1 t))$ and, by induction on m, $T^m x \in Q_2(y(\rho_1^{m-1} t))$ $m = 1, 2, \ldots$. It follows that $T^m x \to x_l$ as $m \to \infty$. This completes the proof of Theorem 4.4.

A slightly modified version of Theorem 4.4 also holds for the fixed points u_1 and u_2. We will consider briefly the fixed point u_2. We have $DT(u_2) = \left(\begin{smallmatrix} \eta_2 & 0 \\ b & \mu_2 \end{smallmatrix}\right)$, where $b \leqslant 0$ since $DT(u_2)^{-1} \geqslant 0$. Assume $\eta_2 > 1$ and recall that $0 < \mu_2 < 1$. The eigenvector corresponding to η_2 is $v = \mathrm{col}(\eta_2 - \mu_2, b) \in \bar{Q}_4$. There exists y_r: $[0, \infty) \to Q_4(u_2)$ satisfying $y_r(t) = u_2 + tv + O(t^2)$ as $t \to 0$ and (B_r), (C_r), and (D_r) of Theorem 4.4 where ρ_1 is replaced by η_2 in (C_r). In addition (E_r) holds except $x \in \overline{Q_4(u_2)} \cap \overline{Q_2(x_r)} - [0, u_2]$ must replace $x \in \overline{Q_4(x_1)} \cap \overline{Q_2(x_r)}$. There is no change in the proof unless $b = 0$. In this case it is a slightly more delicate task to insure that $y_r(t) \in Q_4(u_2)$. It is easy to show that $T(\overline{Q_4(u_2)} - [0, u_2]) \subset Q_4(u_2) - [0, u_2]$ and hence $T(\overline{Q_4(u_2)}) \subset \overline{Q_4(u_2)}$. Remark 3 following Theorem 1.1 in [14] implies $y_r(t) \in \overline{Q_4(u_2)}$ for $t \geqslant 0$. If $t > 0$ then by (C_r) and the above inclusion, $y_r(t) = T(y_r(\eta_2^{-1} t)) \in Q_4(u_2) - [0, u_2]$.

We need one more result to describe the phase portrait for T, namely, to show that if x is a saddle fixed point of T then $C^r(x)$ and $C^l(x)$ are contained in S. This is the content of

LEMMA 4.5. *If x is a saddle fixed point of T then $C^r(x)$ and $C^l(x)$ are contained in S.*

Proof. First, assume $x > 0$ and consider $C^l(x)$. Now x separates S into two pieces $S^l(x) = S \cap Q_2(x)$ and $S^r(x) = S \cap Q_4(x)$. We show $C^l(x) \subset S^l(x)$. Both curves lie in $Q_2(x)$, in fact, since $C^l(x) \cap B(0) = \phi$ one easily sees that $S^l(x)$ lies between the two curves $C^l(x)$ and $C^0 = \{(t, b)$: $0 \leqslant t \leqslant a\}$ where $x = (a, b)$. By a result of Hadamard [3], $T^m C^0 \to C^l(x)$ as $m \to \infty$ and one easily sees that the convergence is monotone in the sense that $T^{m+1}C^0$ lies between $T^m C^0$ and $C^l(x)$. On the other hand, since $S^l(x)$ is invariant under T, $S^l(x)$ must lie between $T^m C^0$ and $C^l(x)$ for every m. Moreover $T^m C^0 \to S$ since $C^0 \subset \overline{B(0)}$. It follows that $C^l(x) \subset S^l(x)$.

Now suppose $x = u_2$ and $\eta_2 > 1$. We want to show $C^r(u_2) \subset S$. Observe that S must lie between $C^r(u_2)$ and the segment $[0, u_2]$. Since these latter curves are, respectively, the stable and unstable manifold of u_2 under the action of T^{-1}, the nondegeneracy of u_2 implies that in some neighborhood of u_2 relative to R^2_+, S and $C^r(u_2)$ are identical. The invariance of each under T imply $C^r(u_2) \subset S$.

The discussion of the phase portrait of the discrete dynamical system generated by T naturally breaks up into four cases depending on the stability type of the two fixed points u_1 and u_2. We combine two of these cases in each of the next two theorems. The proofs of the theorems are to be found at the end of this section following some discussion. In each result we assume without mention that (H1)–(H8) hold for T; in particular, T has a finite number of fixed points. In our first result, Theorem 4.6, we assume that one of the fixed points u_i is a saddle and the other is stable. For definiteness, we take the former to be u_2 and the latter u_1.

THEOREM 4.6. *Assume u_2 is a saddle fixed point and u_1 is a stable fixed point of T. Then either* (a) *T has no positive fixed points or* (b) *T has an even number of positive fixed points $x_1, x_2, ..., x_{2m}$, listed in order of occurrence on S from $x_0 = u_2$ to $x_{2m+1} = u_1$. The odd indexed x_i's are stable while the even indexed x_i's are saddle fixed points. In case* (a), *$S = C^r(u_2) \cup \{u_1\}$ is a C^1 strictly decreasing curve. In case* (b), *S consists of $C^r(u_2)$ together with $C^l(x_{2i}) \cup C^r(x_{2i})$, $1 \leqslant i \leqslant m$, and the stable fixed points; S is a C^1 strictly decreasing curve. In case* (a), *$A(x) = \{u_1\}$ for all $x > 0$. In case* (b), *the set S together with the pairwise nonintersecting curves $C^-(x_i)$, $C^+(x_i)$, $0 \leqslant i \leqslant 2m+1$, partition R^2_+ into $4m+2$ connected open components each of which is invariant under T and contains exactly one stable fixed point on its boundary. This stable fixed point is $A(x)$ for all x in the open component.*

Figure 4.3 illustrates the two cases.

In our next result, u_1 and u_2 have the same stability type, either saddle type or stable.

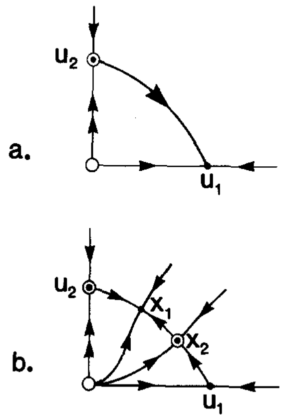

FIG. 4.3. Theorem 4.6, case (a) and (b) ($m=1$). ($\cdot$) stable, ($\circ$) repeller, and $\odot$ saddle point. See the remark following Fig. 4.4 for further explanation.

THEOREM 4.7. *Suppose that both u_1 and u_2 are stable (saddle) fixed points of T. Then S contains an odd number of positive fixed points, $x_1, x_2, \ldots, x_{2m+1}$, listed in order of occurrence on S from u_2 to u_1. The odd indexed x_i's are saddles (stable) while the even indexed x_i's are stable (saddles). S is a C^1 monotone decreasing curve consisting of fixed points together with $C^l(x_i)$ and $C^r(x_i)$ for odd (even) i ($x_0 = u_2$, $x_{2m+2} = u_1$). S, together with $C^-(x_i)$, $C^+(x_i)$, $0 \leqslant i \leqslant 2m+2$, partitions R^2_+ into $4(m+1)$ open connected components each of which is invariant under T and contains exactly one stable fixed point on its boundary. This stable fixed point is $A(x)$ for x in the open component.*

The two case contained in Theorem 4.7 are illustrated in Fig. 4.4.

In Figs. 4.3 and 4.4 the curves $C^-(x_i)$ are tangent to the horizontal coordinate axis. Of course, none of our hypotheses allow us to conclude this behavior for $C^-(x_i)$. However, we expect that typically $\lambda_1 \neq \lambda_2$ so that either $1 < \lambda_1 < \lambda_2$ or $1 < \lambda_2 < \lambda_1$ holds. In both figures we have assumed the former holds and have drawn the double arrows along the vertical coordinate axis to indicate this further assumption that the vertical axis is the most repelling direction for T. Consequently, the vertical direction is the most attractive direction for T^{-1} and we expect the curves $C^-(x)$ to be tangent to the horizontal axis. It should be possible to give a straightforward proof of this without additional assumptions but we have not been able to do so. Under so-called nonresonance conditions on the eigenvalues $\{\lambda_1, \lambda_2\}$ of $DT(0)$, which in our case become $\lambda_2 \neq \lambda_1^m$ for all positive integers $m \geqslant 2$, and assuming additional smoothness of T, one can linearize T^{-1} in a neighborhood of $x=0$ by a smooth map h: $h(0)=0$,

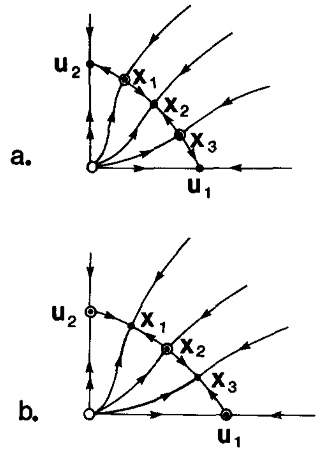

FIG. 4.4. An illustration of Theorem 4.7. In (a) u_1 and u_2 are stable while in (b) they are saddle points. For both cases $m=1$ and there are 8 invariant components.

$Dh(0)=I$ such that $T^{-1}(h(y))=h(DT(0)^{-1}y)$ (see Hartman [17, Theorem 12.1, p. 257]).

It is easy to see that h must send coordinate axes to coordinate axes since T^{-1} leaves them invariant. One can now see that $h^{-1}(C^-(x_i))$ are tangent to the horizontal axis in the new coordinates and hence $C^-(x_i)$ must be tangent to the horizontal axis.

Theorems 4.6 and 4.7 give a complete qualitative description of the dynamics of the discrete dynamical system generated by the competitive map T in the case that all fixed points are nondegenerate. Given that one knows the fixed point set of T, the stability type of u_1 and u_2, the curves S and $C^+(x_i)$, $C^-(x_i)$, then one can determine $\Lambda(x)$ for every $x \in R^2_+$.

We have made clear, in the paragraph following the proof of Theorem 4.4, the manner in which the curve S intersects the coordinate axis at a saddle point ($(u_1, 0)$ or $(0, u_2)$). However, we have not discussed the nature of this intersection in the case that $(0, u_2)$ (or $(u_1, 0)$) is a stable fixed point. We consider this question briefly in this paragraph. Refering back to the remarks following the proof of Theorem 4.4, we see that there are several different possibilities for $DT(u_2)$ in case u_2 is stable: (i) $0<\mu_2<\eta_2<1$, (ii) $0<\eta_2<\mu_2<1$, and (iii) $0<\mu_2=\eta_2<1$, the latter being nongeneric. In case (i), Theorem 4.4 implies that the unstable manifold of a saddle fixed point on S is asymptotic to u_2 and tangent to the eigenvector $(\eta_2-\mu_2, b) \in \bar{Q}_4$ at u_2. In other words, S is tangent at u_2 to this eigenvector. This is the case that we have assumed to hold in sketching our phase portraits in Figs. 4.3 and 4.4. In case (ii), however, Theorem 4.4 implies that an unstable manifold of a saddle point on S is asymptotic to u_2

and tangent to the vertical axis. Hence S meets the vertical axis at u_2 so as to be tangent to the vertical axis. One can show that this tangency of S with the vertical axis also prevails in case (iii) if $b<0$. However, if (iii) holds and $b=0$ there is no unique direction at u_2 which is singled out and we cannot assert that S is tangent at u_2 to any direction.

A simple example of a two dimensional competitive map to which the foregoing theory applies is given by

$$T(x_1, x_2) = \left(\frac{\lambda_1 x_1}{1+a_1 x_1}, \frac{\lambda_2 x_2}{1+a_2 x_2}\right),$$

$$a_1 = a_1(x_2) \equiv a_{10} + m_1 \tan^{-1}(x_2),$$

$$a_2 = a_2(x_1) \equiv a_{20} + m_2 \tan^{-1}(x_1),$$

$$a_{i0} > 0,\ m_i > 0,\ m_1 m_2 < 1,\ \lambda_i > 1,\ i = 1, 2.$$

It is a straightforward calculation to check that $DT(x)^{-1} > 0$ for $x > 0$ and $\det DT(x) > 0$. T has four fixed points: $(0, 0)$, $(0, u_2)$, $(u_1, 0)$, $(\bar{x}_1, \bar{x}_2)$, $u_i = (\lambda_i - 1)/a_{i0}$, $i = 1, 2$, where $(\bar{x}_1, \bar{x}_2)$ is the unique positive solution of $x_1 = \eta_1/a_1(x_2)$ and $x_2 = \eta_2/a_2(x_1)$.

Note that (H8) is equivalent to

$$Tx = x \Rightarrow \det DT(x) - \operatorname{tr} DT(x) \neq 1$$

which can be verified in our example. One easily checks that $(0, u_2)$ and $(u_1, 0)$ are saddle fixed points, hence $(\bar{x}_1, \bar{x}_2)$ must be stable. The phase portrait of the map is described in Fig. 4.5, where it is assumed that $\lambda_2 > \lambda_1$ and the above-mentioned nonresonance condition holds.

Proof of Theorem 4.6. We may view S as a line segment on which T acts, fixing the two endpoints. If there are interior fixed points, they must be finite in number with alternating stability type since they are hyperbolic. Since u_2 is repelling with respect to S and u_1 is stable, there must be an

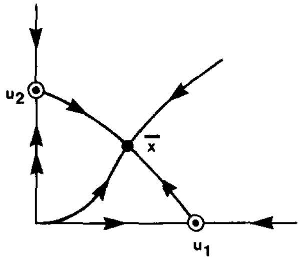

FIG. 4.5. An example illustrating Theorem 4.7; $\bar{x} = (\bar{x}_1, \bar{x}_2)$ is stable.

even number of interior fixed points. If there are no positive fixed points of T on S then S is a C^1 strictly decreasing curve since $S = C^r(u_2) \cup \{u_1\}$ (see Proposition 3.7). Even if S contains an even number of positive fixed points, S is a C^1 strictly decreasing curve. The reason for this is contained in Theorem 4.4(D_l) and (D_r). If x_{2i+1} is a stable positive fixed point, then $C^r(x_{2i})$ and $C^l(x_{2i+2})$ meet at x_{2i+1} and are tangent to the same eigendirection. By Lemma 4.5, S consists of the curves $C^r(u_2)$, $C^l(x_{2i})$, $C^r(x_{2i})$, $1 \leqslant i \leqslant m-1$, together with the stable fixed points. If there are no positive fixed points then $\Lambda(x) = \{u_1\}$ for $x > 0$. Indeed, if $x \in [0, u_1 + u_2]$, $x > 0$ then the result follows from Theorem 4.4(E_r) (see the remark following the proof of Theorem 4.4). Otherwise, if $x \notin [0, u_1 + u_2]$, $x > 0$, we know $\Lambda(x) \subset [0, u_1 + u_2]$, is a fixed point, and can not be the saddle point u_2 or the repeller 0.

In the case that S contains an even number of positive fixed points, one easily sees that the $C^+(x_i)$ and $C^-(x_i)$, $0 \leqslant i \leqslant 2m+1$, are pairwise disjoint and partition R^2_+ into $4m+2$ connected components each containing exactly one stable fixed point. The invariance follows since the boundary curves are invariant. The remaining assertions are obvious.

Proof of Theorem 4.7. As in the proof of Theorem 4.6, we may treat S as a line segment with attracting (repelling) fixed points as endpoints. The segment may contain a finite number of isolated fixed points according to our assumptions. By elementary topological degree theory (or simply note that $Tx - x$ changes sign from u_1 + to u_2 −), T must have a fixed point in the interior of the segment, i.e., a positive fixed point on S. It is easy to see that the number of positive fixed points of T on S must be odd in number and their stability type, relative to S, must alternate. The remaining assertions are proved exactly as in the proof of Theorem 4.6.

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