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On Principal Eigenvalues of p-Laplacian-like Operators

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1. INTRODUCTION

We study the existence of a principal eigenvalue to the problem

$$\begin{aligned} [r^{N-1}\phi(u')]' + \lambda r^{N-1}\psi(u) = 0, \qquad & r \in (0, R), \\ u'(0) = 0, \qquad & u(R) = 0, \end{aligned} \tag{1.1}$$

where ϕ is an increasing homeomorphism of $\mathbb{R}$ and ψ is nondecreasing with $\phi(0) = \psi(0) = 0$; i.e., we study the existence of values of λ such that problem (1.1) has a solution u with $u(r) > 0$, $r \in [0, R)$ and properties of the set of such eigenvalues. In particular, we are interested whether distinguished values λ_0 exist such that no nontrivial solutions of (1.1) exist for values of λ, $\lambda < \lambda_0$.

In the case that N is a positive integer and $\phi(u) = |u|^{p-2}\, u = \psi(u)$ the above problem is the problem of the existence of the principal eigenvalue of the p-Laplacian on a ball of radius R in $\mathbb{R}^N$, and subject to zero Dirchlet boundary data. As such it is well understood ([1, 2, 6, 7, 11]).

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The tools that have been used for establishing the existence of such (and higher) eigenvalues come from variational methods and are usually critical point theorems for smooth functionals defined in an appropriate Sobolev space; these methods consequently also yield theorems for the case the underlying ball domain is replaced by an arbitrary bounded domain.

Very detailed information is available in the case $N=1$, see e.g. [5, 9 and 10].

In this paper we treat the general case (1.1) and rely in our techniques and methods of proof on some earlier work in [3] and [8], i.e. fixed point and continuation techniques.

We will assume further that $N \geqslant 1$ (not necessarily an integer) and ϕ and ψ satisfy for all $x \neq 0$ and $\sigma > 0$

$$A(\sigma) \leqslant \frac{\phi(\sigma\,|x|)}{|\psi(x)|} \leqslant B(\sigma), \tag{1.2}$$

where $A(\sigma)$ and $B(\sigma)$ are positive constants depending on σ only

We call λ an eigenvalue of (1.1) provided (1.1) has a nontrivial solution and a principal eigenvalue if (1.1) has a positive (or negative) solution. We let E be the set of all principal eigenvalues. Our main result in this paper is as follows.

THEOREM 1.1. *Suppose the above. Then the set* $E \neq \varnothing$, *further there exists a smallest* $\lambda_0 > 0$ *such that for* $\lambda < \lambda_0$ *the eigenvalue problem* (1.1) *has no nontrivial solutions. For every* $d > 0$, *there exists* $\lambda \in E$ *and a positive solution* u *of* (1.1) *such that* $u(0) = d$. *If, for each* $d > 0$, *we denote by*

$$\Gamma(d, R) = \{\lambda > 0 \mid \textit{problem (1.1) has a positive solution with } u(0) = d\}$$

and set

$$\Gamma^-(d, R) = \inf \Gamma(d, R),$$

then $\Gamma^-(d, R) > 0$. *Also*

$$\Gamma_1^-(R) := \liminf_{d \to \infty} \Gamma^-(d, R),$$

$$\gamma_1^-(R) := \liminf_{d \to 0} \Gamma^-(d, R)$$

$$\lambda_1^-(R) := \inf_{d > 0} \Gamma^-(d, R),$$

are nonincreasing functions of R, and

$$\lim_{R\to 0+} \Gamma_1^-(R) = \lim_{R\to 0+} \gamma_1^-(R) = \lim_{R\to 0+} \lambda_1^-(R) = \infty,$$

$$\lim_{R\to \infty} \Gamma_1^-(R) = \lim_{R\to \infty} \gamma_1^-(R) = \lim_{R\to \infty} \lambda_1^-(R) = 0.$$

There is a result dual to Theorem 1.1 for principal eigenvalues with associated solutions negative on $[0, R)$, the set of such eigenvalues may, of course, be different from the set whose existence is asserted in Theorem 1.1.

We shall also consider the nonhomogeneous problem

$$\begin{aligned} [r^{N-1}\phi(u')]' + \lambda r^{N-1}\psi(u) &= r^{N-1}h(r), \qquad && r\in(0, R), \\ u'(0) &= 0, && u(R) = 0, \end{aligned} \tag{1.3}$$

where $h \in L^\infty(0, R)$ is a given function. We have the following result.

THEOREM 1.2. *There exists $\lambda_0 > 0$ such that for every $\lambda < \lambda_0$ and every function $h \in L^\infty(0, R)$ the nonhomogeneous problem* (1.3) *has a solution.*

As is to be expected, λ_0 in this theorem is to the left of the set of principal eigenvalues associated with both positive and negative eigenfunctions.

2. ABSTRACT FORMULATION OF THE PROBLEM

A quick calculation shows that finding positive solutions to problem (1.1) is equivalent to finding nontrivial solutions to the problem

$$\begin{aligned} [r^{N-1}\phi(u')]' + \lambda r^{N-1}\psi(|u|) &= 0, \qquad && r\in(0, R), \\ u'(0) &= 0, && u(R) = 0. \end{aligned} \tag{2.1}$$

Let $C_\#$ denote the closed subspace of $C[0, R]$ defined by

$$C_\# = \{u \in C[0, R] \mid u(R) = 0\}.$$

Then $C_\#$ is a Banach space for the norm $\|\ \| := \|\ \|_\infty$.

Let $u(r)$ be a solution of (2.1). By integrating the equation in (2.1) we see that that $u(r)$ satisfies the integral equation

$$u(r) = \int_r^R \phi^{-1}\left[\frac{1}{s^{N-1}}\int_0^s \xi^{N-1}\lambda\psi(|u(\xi)|)\, d\xi\right] ds. \tag{2.2}$$

Let us define $T(\cdot, \cdot)\colon C_{\#} \times [0, \infty) \to C_{\#}$ by

$$T(u, \lambda)(r) = \int_r^R \phi^{-1}\left[\frac{1}{s^{N-1}} \int_0^s \xi^{N-1} \lambda \psi(|u(\xi)|)\, d\xi\right] ds. \tag{2.3}$$

Clearly T is well defined and fixed points of $T(\cdot, \lambda)$ will provide solutions of (1.1). Define now the operator $T_\varepsilon\colon C_{\#} \times [0, \infty) \to C_{\#}$, by

$$T_\varepsilon(u, \lambda)(r) = \int_r^R \phi^{-1}\left[\frac{1}{s^{N-1}} \int_0^s \xi^{N-1} \lambda(\psi(|u(\xi)|) + \varepsilon)\, d\xi\right] ds, \tag{2.4}$$

where $\varepsilon > 0$ is a constant. We have that T_ε sends bounded sets of $C_{\#} \times [0, \infty)$ into bounded sets of $C_{\#}$. Moreover, T_ε is a completely continuous operator and $T_\varepsilon(\cdot, 0) = 0$.

Since

$$\deg_{LS}(I - T_\varepsilon(\cdot, 0), B(0, R_1), 0) = 1$$

there exists a solution continuum $\mathscr{C}_\varepsilon^+ \subset C_{\#} \times [0, \infty)$ of solutions of

$$u = T_\varepsilon(u, \lambda) \tag{2.5}$$

with $\mathscr{C}_\varepsilon^+$ unbounded in $C_{\#} \times [0, \infty)$ (see e.g. [4]).

3. A PRIORI BOUNDS FOR λ

We will now prove that solutions $(u, \lambda) \in \mathscr{C}_\varepsilon^+$ are a priori bounded in the λ-direction.

We have:

LEMMA 3.1. *There exists $\bar{\lambda}$ such that for any solution (u, λ) of* (2.1), *$\lambda \leqslant \bar{\lambda}$. This number is independent of ε for $0 < \varepsilon \leqslant \varepsilon_0$.*

Proof. We have that u satisfies

$$-r^{N-1}\phi(u'(r)) = \int_0^r \xi^{N-1} \lambda(\psi(|u(\xi)|) + \varepsilon)\, d\xi \geqslant 0$$

and

$$u(r) = \int_r^R \phi^{-1}\left[\frac{1}{s^{N-1}} \int_0^s \xi^{N-1} \lambda(\psi(|u(\xi)|) + \varepsilon)\, d\xi\right] ds \geqslant 0.$$

Hence, $u'(r) \leqslant 0$ and $u(r) \geqslant 0$ for all $r \in [0, R]$. Also

$$u(r) \geqslant \int_r^R \phi^{-1} \left[\frac{1}{s^{N-1}} \int_0^s \xi^{N-1} \lambda \psi(|u(\xi)|)\, d\xi \right] ds,$$

Thus, for all $r \in [R/4, 3R/4]$, we have that

$$u(r) \geqslant \frac{R}{4} \phi^{-1} \left[\frac{\lambda R}{N 4^N} \psi(u(r)) \right]$$

or equivalently,

$$\frac{\phi\left(\dfrac{4}{R} u(r)\right)}{\psi(u(r))} \geqslant \frac{\lambda R}{N 4^N}.$$

It follows from (3.1) and (1.2) that λ is bounded independent of ε. ∎

It therefore follows that for each $d > 0$ and each $0 < \varepsilon \leqslant \varepsilon_0$ there exists $(u_\varepsilon, \lambda_\varepsilon) \in \mathscr{C}_\varepsilon^+$, with $\|u_\varepsilon\| = d > 0$ and $0 < \lambda_\varepsilon \leqslant \bar{\lambda}$.

We now let $\varepsilon \to 0$ and obtain a nontrivial solution of (1.1) for some $\lambda^* \in (0, \bar{\lambda}]$.

Next for each $d > 0$, let us denote by

$$\Gamma(d, R) = \{\lambda > 0 \mid \text{problem (1.1) has a positive solution with } \|u\| = d\}$$

and set

$$\Gamma^-(d, R) = \inf \Gamma(d, R), \qquad \Gamma^+(d, R) = \sup \Gamma(d, R).$$

It follows from the above calculations that $\Gamma^+(d, R) < \infty$. We next prove the following:

Proposition 3.1. $\Gamma^-(d, R) > 0$.

Proof. We argue by contradiction and thus we assume there exist sequences $\{u_n\}$, $\{\lambda_n\}$, such that

$$u_n = T(u_n, \lambda_n), \qquad \text{with} \quad \|u_n\| = d,$$

and $\lambda_n \to 0$. Since T is completely continuous we can assume without loss of generality that $\{u_n\}$ is convergent, say, $u_n \to u$. Then letting $n \to \infty$ in both expressions of (3.2), we obtain a contradiction. ∎

Let us set

$$\Gamma_1^-(R) := \liminf_{d \to \infty} \Gamma^-(d, R) \quad \text{and} \quad \Gamma_1^+(R) := \limsup_{d \to \infty} \Gamma^+(d, R),$$

$$\gamma_1^-(R) := \liminf_{d \to 0} \Gamma^-(d, R) \quad \text{and} \quad \gamma_1^+(R) := \limsup_{d \to 0} \Gamma^+(d, R),$$

and

$$\lambda_1^-(R) := \inf_{d > 0} \Gamma^-(d, R) \quad \text{and} \quad \lambda_1^+(R) := \sup_{d > 0} \Gamma^+(d, R).$$

We have:

PROPOSITION 3.2. *Under the above hypotheses on ϕ and ψ,*

$$\begin{aligned} \Gamma_1^-(R) &> 0, & \Gamma_1^+(R) &< \infty, \\ \gamma_1^-(R) &> 0, & \gamma_1^+(R) &< \infty, \\ \lambda_1^-(R) &> 0, & \lambda_1^+(R) &< \infty. \end{aligned}$$

Proof. That the numbers in the second column are finite has already been established. To show that the numbers in the first column are positive we argue as follows. Assume (u, λ) is a solution of (2.1) with $u(0) = d$. Then

$$\|u\| = d,$$

and

$$u(r) = \int_r^R \phi^{-1}\left[\frac{1}{s^{N-1}} \int_0^s \xi^{N-1} \lambda \psi(|u(\xi)|)\, d\xi\right] ds. \tag{3.3}$$

Hence

$$d \leqslant \int_0^R \phi^{-1}\left[\lambda \psi(d) \frac{s}{N}\right] ds,$$

which implies

$$\frac{\phi(d/R)}{\psi(d)} \leqslant \lambda \frac{R}{N}.$$

Using condition (1.2) we obtain a lower bound for λ. ∎

The above proposition also has the following corollary:

COROLLARY 3.1. *Let* (u, λ) *be a solution of* (1.1) *with* $u(0)=d$ *and let* $\theta\in(0,1)$ *be fixed. Let* $r_0\in(0,R)$ *be such that* $u(r_0)=\theta d$. *Then*

$$r_0 \geqslant \frac{N}{\lambda} A\left(\frac{1-\theta}{R}\right). \tag{3.5}$$

Proof. Using Eq. (3.3) we obtain

$$\theta d = \int_{r_0}^{R} \phi^{-1}\left[\frac{1}{s^{N-1}}\int_0^s \xi^{N-1}\lambda\psi(|u(\xi)|)\,d\xi\right] ds, \tag{3.6}$$

and hence

$$(1-\theta)\,d = \int_0^{r_0} \phi^{-1}\left[\frac{1}{s^{N-1}}\int_0^s \xi^{N-1}\lambda\psi(|u(\xi)|)\,d\xi\right] ds,$$

from which follows

$$(1-\theta)\,d \leqslant \int_0^{r_0} \phi^{-1}\left[\frac{1}{s^{N-1}}\int_0^s \xi^{N-1}\lambda\psi(d)\,d\xi\right] ds,$$

and

$$(1-\theta)\,d \leqslant R\phi^{-1}\left[\frac{\lambda\psi(d)\,r_0}{N}\right],$$

from which the conclusion follows. ∎

The result just proved has the following consequence.

COROLLARY 3.2. *Let* $\{(u_n, \lambda_n)\}$ *be a sequence of solutions of* (1.1) *with* $u_n(0)=d_n$. *If* $d_n\to\infty$ *as* $n\to\infty$, *then* $u_n(r)\to\infty$ *uniformly with respect to* r *in compact subintervals of* $[0, R)$.

Proof. Since u_n is given by

$$u_n(r) = \int_r^R \phi^{-1}\left[\frac{1}{s^{N-1}}\int_0^s \xi^{N-1}\lambda_n\psi(|u_n(\xi)|)\,d\xi\right] ds,$$

we obtain for $r\geqslant r_0$ (viz. Corollary 3.1) that

$$u_n(r) \geqslant \int_r^R \phi^{-1}\left[\frac{1}{s^{N-1}}\int_0^{r_0} \xi^{N-1}\lambda_n\psi(\theta d_n)\,d\xi\right] ds.$$

The conclusion follows from this inequality. ∎

4. OSCILLATION OF SOLUTIONS

Let us consider the initial value problem

$$\begin{aligned} [r^{N-1}\phi(u')]' + \lambda r^{N-1}\psi(u) = 0, & \\ u'(0) = 0, \qquad u(0) = d > 0. & \end{aligned} \tag{4.1}$$

To obtain the (local) existence of solutions of (4.1) we obtain solutions of an equivalent integral equation whose solutions are fixed points of the operator S defined by

$$S(u)(r) = d - \int_0^r \phi^{-1}\left[\frac{1}{s^{N-1}}\int_0^s \xi^{N-1}\lambda\psi(u(\xi))\,d\xi\right] ds.$$

Since u is decreasing and ϕ and ψ are increasing we find that for given $\varepsilon > 0$, there exists R_0 such that

$$S\colon \{u \in C[0, R_0] \colon \|u - d\| \leqslant \varepsilon\} \to \{u \in C[0, R_0] \colon \|u - d\| \leqslant \varepsilon\}$$

and that S is completely continuous. The result thus follows from Schauder's fixed point theorem.

We next show that solutions of the initial value problem (4.1) exist globally and are oscillatory.

PROPOSITION 4.1. *For each $d > 0$ and each $\lambda > 0$ solutions of* (4.1) *exist globally on* $[0, \infty)$, *have only simple zeros and are oscillatory, i.e. the set of zeros is unbounded.*

Proof. It follows from the above existence argument that a solution u of (4.1), as long as it is positive, is decreasing. If $u(r) \geqslant 0$, $0 \leqslant r < \infty$ we find that for each $r > 0$

$$u(r) \geqslant r\phi^{-1}\left(\frac{\lambda r\psi(u(r))}{N2^{N-1}}\right),$$

and hence (since we may assume $r \geqslant 1$)

$$\frac{\phi(u(r))}{\psi(u(r))} \geqslant \frac{\lambda r}{N2^{N-1}},$$

which implies, using condition (1.2) that r cannot be unbounded. Therefore u must have a first zero. Easy arguments show that the zeros of u must be simple and that u is oscillatory.

5. MONOTONICITY OF EIGENVALUES

In this section we shall show that $\lambda_1^-(R)$, $\gamma_1^-(R)$ and $\Gamma_1^-(R)$ are nondecreasing functions of R. To this end we shall need the following elementary properties of the operator T defined by Eq. (2.3). We note that the space $C_\#$ is a partially ordered Banach space with respect to the partial order $\leqslant$, i.e. for $u, v \in C_\#$, $u \leqslant v$ whenever $u(r) \leqslant v(r)$, $r \in [0, R]$. Further if $[u, v] = \{w \in C_\# : u \leqslant w \leqslant v\}$ is an order interval in $C_\#$, then it is a bounded closed set in $C_\#$.

PROPOSITION 5.1. *The operator T defined by* (2.3) *is monotone with respect to the above partial order in $C[0, R]$ and hence in $C_\#$ and also monotone with respect to λ.*

From this proposition and the complete continuity of T immediately follows the following fixed point result.

PROPOSITION 5.2. *Assume there exists $[\alpha, \beta] \subset C[0, R]$ such that*

$$T(\lambda, \cdot): [\alpha, \beta] \to [\alpha, \beta].$$

Then $T(\lambda, \cdot)$ has a fixed point $u \in C_\# \cap [\alpha, \beta]$.

We note that the hypotheses of Proposition 5.2 will hold, whenever we can find a pair $\{\alpha, \beta\} \subset C[0, R]$ such that

$$\alpha \leqslant \beta$$

and

$$\alpha \leqslant T(\lambda, \alpha), \qquad T(\lambda, \beta) \leqslant \beta.$$

Using these facts we can now establish the following result.

THEOREM 5.1. *λ_1^-, γ_1^-, Γ_1^- are nonincreasing functions of R.*

Proof. Assume there exist R_1, R_2, $R_1 < R_2$ such that $\lambda_1^-(R_1) < \lambda_1^-(R_2)$. Then there exists μ, $\lambda_1^-(R_1) < \mu < \lambda_1^-(R_2)$, such that (2.1) has a nontrivial solution $\tilde{\alpha}$ for $R = R_1$ and $\lambda = \mu$. Furthermore, there exists $v \geqslant \lambda_1^-(R_2)$ and a nontrivial solution β of (2.1) for $R = R_2$ and $\lambda = v$, with $\beta(0) = d$ as large as desired. It follows from Corollary 3.2 that for d sufficiently large $\beta(r) > \tilde{\alpha}(r)$, $0 \leqslant r \leqslant R_1$. Define

$$\alpha = \begin{cases} \tilde{\alpha}, & 0 \leqslant r \leqslant R_1 \\ 0, & R_1 \leqslant r \leqslant R_2. \end{cases}$$

Then the operator $T(\mu, \cdot)$ for $R = R_2$ satisfies in the space $C[0, R_2]$

$$\alpha \leqslant T(\mu, \alpha), \qquad T(\mu, \beta) \leqslant \beta,$$

as may easily be verified. Thus by Proposition 5.2 this operator will have a fixed point in $[\alpha, \beta]$, contradicting that $\mu < \lambda_1^-(R_2)$. The monotonicity of the other functions is proved using virtually similar arguments. ∎

Remark 5.1. 1. Theorem 5.1 implies that problem (1.1) has no non-trivial solutions for $\lambda < \lambda_1^-(R)$.

2. Solutions of (1.1) are a priori bounded for λ in compact subintervals of $(-\infty, \Gamma_1^-(R))$.

3. We have proved all of Theorem 1.1 except the last conclusion which follows from inequalities (3.1) and (3.4).

4. Our calculations and results also imply that for $\delta > 0$, small, the degrees

$$\deg_{LS}(I - T(\cdot, a), B(0, \delta), 0)$$
$$\deg_{LS}(I - T(\cdot, b), B(0, \delta), 0)$$

are defined for $a < \gamma_1^-(R)$ and $b > \gamma_1^+(R)$ and equal 1 and 0, respectively. Hence it follows from global bifurcation theory (see [4]) that an unbounded continuum of positive solutions of (1.1) will bifurcate from $[\gamma_1^-(R), \gamma_1^+(R)]$.

6. THE NONHOMOGENEOUS PROBLEM

In this section we shall consider the nonhomogeneous problem (1.3) and prove Theorem 1.2.

The existence of solutions to the nonhomogeneous problem is equivalent to the existence of fixed points of the operator $T(\lambda, u, h)$ defined by

$$T(\lambda, u, h)(r) = \int_r^R \phi^{-1}\left[\frac{1}{s^{N-1}} \int_0^s \xi^{N-1}(\lambda\psi(u(\xi)) - h(r))\, d\xi\right] ds. \qquad (6.1)$$

We shall establish the existence of a fixed point of T in the space $C_{\#}$ using Proposition 5.2. To this end, we define

$$\lambda_0 = \inf\{\lambda\colon \lambda \text{ is a principal eigenvalue of (1.1)}\}$$

and consider problem (1.3) for values of $\lambda < \lambda_0$. Let λ be so chosen, and choose $\bar{\lambda}$ such that $\lambda < \bar{\lambda} < \lambda_0$.

Let us consider the boundary value problem

$$\begin{aligned} &[r^{N-1}\phi(u')]' + \mu r^{N-1}\psi(u) = 0, \\ &\qquad\qquad u'(0) = 0, \qquad u(R) = 0. \end{aligned} \tag{6.2}$$

It then follows from the results in Sections 4 and 5 that there exists a solution u and a value $\mu \geqslant \lambda_0$ with $u(0) = d$ as large as we like. Then u satisfies also

$$u(r) = \int_r^R \phi^{-1}\left[\frac{1}{s^{N-1}}\int_0^s \xi^{N-1}\mu\psi(u(\xi))\,d\xi\right] ds,$$

Further $u(r)$ may be made arbitrarily large uniformly on compact subintervals of $[0, R]$ by choosing d sufficiently large. Hence

$$\begin{aligned} u(r) &= \int_r^R \phi^{-1}\left[\frac{1}{s^{N-1}}\int_0^s \xi^{N-1}\mu\psi(u(\xi))\,d\xi\right] ds \\ &= \int_r^R \phi^{-1}\left[\frac{1}{s^{N-1}}\int_0^s \xi^{N-1}(\lambda\psi(u(\xi)) - h)\,d\xi\right. \\ &\qquad \left. + \frac{1}{s^{N-1}}\int_0^s \xi^{N-1}((\mu - \lambda)\,\psi(u(\xi)) + h)\,d\xi\right] ds \\ &\geqslant T(\lambda, u, h)(r) \end{aligned}$$

provided

$$\frac{1}{s^{N-1}}\int_0^s \xi^{N-1}((\mu - \lambda)\,\psi(u(\xi)) + h)\,d\xi \geqslant 0, \qquad 0 \leqslant s \leqslant R$$

which may be accomplished for given $h \in L^\infty(0, R)$, provided d is sufficiently large. In a similar manner we may construct a negative α such that $\alpha \leqslant T(\lambda, \alpha, h)$. We now apply Proposition 5.2 to complete the proof.

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