Sufficient conditions for the incompressibility of the boundary of an \( n \)-relator 3-manifold

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Abstract
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In this paper we give sufficient conditions for the incompressibility of the boundary of an \( n \)-relator 3-manifold. The conditions are those conjectured to be sufficient by Przytycki, with one additional indispensable condition.

Keywords: Incompressibility, bind, coplanar with.


1. Introduction

In 1983, Przytycki [5] first gave sufficient conditions for the boundary of a 1-relator 3-manifold to be incompressible, and in 1984, Jaco [2] extended Przytycki's result by using a geometric approach. But there exist examples (see [5]) which indicate that a direct generalization of the above results to the case of \( n \)-relator 3-manifolds is not possible. In 1987, Przytycki [6] proposed a set of conditions which might be sufficient.
Przytycki's conjecture. Let $C = \{J_1, \ldots, J_n\}$ be a family of 2-sided, pairwise disjoint, simple closed curves in the boundary of a handlebody $H$ (with genus $k > 0$). Assume that the following conditions are satisfied (see Section 2 for necessary definitions):

1. $\partial H - C$ is incompressible in $H$,
2. for each $j$, $\partial H - (C - J_j)$ is compressible in $H$ (or equivalently, the family of elements of $\pi_1(H) = F_k$ represented by $C - J_j$ does not bind the free group $F_k$),
3. for each pair $j, s$ ($j \neq s$), $C - \{J_j, J_s\}$ does not bind any free factor $F_{k-1}$ of $F_k = F_{k-1} \times F_1$,
4. no $(n - p)$-element subfamily of $C$ binds a free factor $F_{k-p+1}$ of $F_k = F_{k-p+1} \times F_{p-1}$,
5. $(n-1)$ no curve $J_i$ of $C$ binds a free factor $F_{k-n+2}$ of $F_k = F_{k-n+2} \times F_{n-2}$.

Then the $n$-relator 3-manifold $H_C$ has incompressible boundary, or it is equal to $D^3$.

In [6] Przytycki proved his conjecture for $n = 1, 2,$ and $3$. When $n > 3$, he had examples to show that all the assumptions in his conjecture are necessary.

In this paper we shall prove Przytycki’s conjecture assuming the following additional condition:

1. for each $j, J_j$ is not contained in the normal subgroup of $\pi_1(H) = F_k$ generated by any $(n-2)$-subfamily of $C - J_j$.

We shall also show that some such additional condition is required.

2. Preliminaries

We work in the PL-category and use scs as an abbreviation of simple closed curve (or curves).

**Definition 2.1.** Let $M$ be a 3-manifold and $S$ a surface which is either properly embedded in $M$ or contained in $\partial M$. We say that $S$ is compressible (in $M$) if one of the following conditions is satisfied:

1. $S$ is a 2-sphere which bounds a 3-cell in $M$, or
2. $S$ is a 2-cell and either $S \subset \partial M$ or there is a 3-cell $X \subset M$ with $\partial X \subset S \cup \partial M$, or
3. there is a 2-cell $D \subset M$ with $D \cap S = \partial D$ and with $\partial D$ not contractible in $S$.

In case (3), $D$ is also called a compressing disk for $S$ (in $M$). We say that $S$ is incompressible if it is not compressible.

**Definition 2.2.** Let $M$ be a 3-manifold and $J$ a 2-sided scs on $\partial M$. Let $A_J$ be a regular neighbourhood of $J$ in $\partial M$, $(D^3, A)$ a 3-cell with an annulus $A \subset \partial D^3$, and $h$ a homeomorphism $A_J \rightarrow A$. Then the 3-manifold $(M, A_J) \cup_h (D^3, A)$ is denoted $M_J$. If $C = \{J_1, \ldots, J_n\}$ is a collection of pairwise disjoint, 2-sided scs on $\partial M$, then
we denote $M_C = (\ldots ((M_{J_i})_{J_i}) \ldots )_{J_i}$. In particular, when $M$ is a handlebody $H$ with genus $k > 0$, $H_C$ is called an $n$-relator 3-manifold.

Clearly, the definition of $M_C$ does not depend on the order of the $J_i$.

**Definition 2.3.** Let $C = \{J_1, \ldots, J_n\}$ be a family of pairwise disjoint 2-sided scs on a surface $S$. We say that an sc $J \subset S - C$ is coplanar with $C$ if $J$ cuts a disk with holes from $S$ cut open along $C$ (i.e., $S - C$).

**Definition 2.4.** Let $W \subset F_k$ be a set of cyclic words in the free group $F_k$ with a basis $X$. The incidence graph $J(W)$ is the graph whose vertices are in 1-1 correspondence with the nontrivial words in $W$, with an edge joining vertices $w_1$ and $w_2$ if there exists $x \in X$ such that $x$ or $x^{-1}$ lies in $w_1$, and $x$ or $x^{-1}$ lies in $w_2$. $W$ is connected with respect to the basis $X$ if $J(W)$ is connected, and is connected if it is connected with respect to each basis of $F_k$. If the set $W$ of cyclic elements is not contained in any proper free factor of $F_k$ and if $W$ is connected, we say that $W$ binds $F_k$.

For convenience, we shall refer to disks with holes as “planar surfaces”. We shall also abuse notation slightly by using the symbol $C$, which represents a family of closed curves in the 3-manifold $M$, also to represent the corresponding elements of $\pi_1(M)$ when this causes no confusion.

The following two lemmas will be used in our proof:

**Lemma 2.5** (Due to Przytycki [6]). Let $C = \{J_1, \ldots, J_n\}$ be a family of pairwise disjoint, 2-sided scs on $\partial M$ and let the following conditions be satisfied:

(i) $\partial M - C$ is incompressible in $M$,
(ii) for each $J_i$, $\partial M - (C - J_i)$ is compressible in $M$,
(iii) for each $J_i$, a compressing disk from (ii), say $D$, can be chosen in such a way that $\partial D$ is not coplanar with $C - J_i$.

Then $M_C$ has incompressible boundary or it is equal to $D^3$.

**Lemma 2.6** (Due to Lyon [4]). Let $C$ be a family of pairwise disjoint scs on the boundary of a handlebody $H$. Then $S = \text{cl}(\partial H - N(C))$ is incompressible if and only if $C$ binds $\pi_1(H)$ and no curve in $C$ is contractible in $\partial H$.

3. The proof

**Theorem 3.1.** Let $C = \{J_1, \ldots, J_n\}$ be a family of pairwise disjoint, 2-sided scs on the boundary of a handlebody $H$ with genus $k > 0$. Suppose the following conditions are satisfied:

(I) the conditions (0)–(n−1) in Przytycki’s conjecture, and
(II) for each $J_i$, $J_i$ is not contained in the normal subgroup of $\pi_1(H) = F_k$ generated by any $(n−2)$-subfamily of $C - J_i$.

Then the n-relator 3-manifold $H_C$ has incompressible boundary, or it is equal to $D^3$. 

Proof. Without condition (II), the theorem was proved for $n = 1, 2$ and $3$ by Przytycki (see [5, 6]). Here we only need to consider the case of $n > 3$.

From assumption (I)(0) we easily have

**Assertion 1.** No curve in $C$ is contractible in $H$.

By (I)(1), for each $j$, $\partial H - (C - J_j)$ is compressible in $H$. If for each $J_j$, a compressing disk of $\partial H - (C - J_j)$, say $D_j$, can be chosen in such a way that $\partial D_j$ is not coplanar with $C - J_j$, then the given conditions (I)(0) and (I)(1) and Lemma 2.5 imply that $H_C$ has incompressible boundary or it is equal to $D^3$, the proof is already complete. Henceforth, it is sufficient to consider the other case.

In the following, without loss of generality, we suppose that

$$\text{each compressing disk of } \partial H - (C - J_n) \text{ in } H \text{ has boundary coplanar with } C - J_n.$$  

Let $S$ denote the surface $\partial H$ cut open along $C - J_n$, and let $J'_1$ and $J''_n$ denote the two boundary components of $S$ corresponding to the curve $J_n$, $1 \leq i \leq n - 1$. Let $\Delta$ be a compressing disk of $\partial H - (C - J_n)$. By (*), $\partial \Delta$ is coplanar with $C - J_n$, that is, $\partial \Delta$ and a subset (with at least two elements, by Assertion 1) of $\partial S = \{J'_i: 1 \leq i \leq n - 1\} \cup \{J''_n: 1 \leq i \leq n - 1\}$ bound a planar surface $S^*$. First we consider the case that $\partial \Delta$ does not separate $\partial H$ (therefore $\Delta$ does not separate $H$). In this situation, the curves of $\partial S^*$ cannot all be paired, thus there exists some $J'_i$ (or $J''_n$) $\in \partial S^*$ but $J''_n$ (or $J'_i$) $\notin \partial S^*$. $S^* \cup \Delta$ is an embedded planar surface in $H$, hence $J_n$ is contained in the normal subgroup of $\pi_1(H)$ generated by the subfamily $\{J'_i: 1 \leq i \leq n - 1, i \neq i_0\}$ of $C$. This contradicts assumption (II). So we have

**Assertion 2.** Every compressing disk in $H$ of $\partial H - (C - J_n)$ separates $H$.

Thus we know that $\Delta$ does separate $H$, therefore the curves contained in $\partial S^*$ are all paired, that is, if $J'_i$ (or $J''_n$) $\in \partial S^*$, then $J''_n$ (or $J'_i$) $\in \partial S^*$. So, without loss of generality we assume that

$$\partial S^* = \partial \Delta \cup \{J''_n, J''_{n-1}, \ldots, J'^{1}_{n-1}, J'^{1}_n\},$$

where $i_0 > 1$ (otherwise $\partial \Delta$ is contractible on $S$), and $\Delta$ divides $H$ into two handlebodies $H_1$ and $H_2$, with genus $k_1$ and $k_2$, respectively, and $k_1 = k + i_0 + 1$, $k_2 = n - 1 - i_0$, $k_1, k_2 > 0$, and $\{J'_1, \ldots, J'_n\} \subset \partial H_1 - \Delta$, and $\{J''_{n-1}, \ldots, J'_n\} \subset \partial H_2$. By assumption (I)(n - 1), $C_1 = \{J'_1, \ldots, J'_n\}$ does not bind $F_{i_0} = \pi_1(H_1)$ of $F_k$, therefore from Lemma 2.6 we know that $\partial H_1 - C_1$ is compressible in $H_1$ (hence in $H$), and by Assertion 2, each compressing disk of $\partial H_1 - C_1$ (after a small isotopy, if necessary) is a compressing disk of $\partial H - (C - J_n)$ in $H$, and by Assertion 2, each compressing disk of $\partial H_1 - C_1$ separates $H$ (therefore $H_1$). After applying this argument finitely often, we can reduce to the following situation (say): $J_i \subset$ a handlebody $H'$, cut out from $H$, with genus of $H' = k' = k - n + 2 > 0$. By assumption (I)(n - 1), $J_i$ does not bind the free factor $F_{k-n+2} = \pi_1(H')$ of $\pi_1(H)$, and again by Lemma 2.6 and Assertion 1, we obtain a compressing disk of $\partial H' - J_i$ which is also a compressing disk of $\partial H - (C - J_n)$. By Assertion 2, a compressing disk of $\partial H' - J_i$, say $\Delta'$, divides $H'$.
into two handlebodies $H'_1$ and $H'_2$ with positive genus. Suppose $J_i \subset \partial H'_i$, then $\partial H'_2$ has a nonseparating compressing disk in $H'_2$, which is also a nonseparating compressing disk of $H$ (after a small isotopy, if necessary). This contradicts Assertion 2, which followed from the assumption condition (*), so we have a contradiction to (*).

Thus we have finished the proof. \(\square\)

Remark. From the proof of Theorem 3.1 we know that without assumption (II), we can always choose a nonseparating compressing disk $\Delta$ of $\partial H - (C - J_n)$ and obtain $S^*$ as before with $\partial S^* = \{J'_i, \ldots, J'_{i_0}, J_{i_0+1}', \ldots, J_{i_1}', J_{i_1+1}', \ldots, J_{i_n}', J_{i_n+1}'\} \cup \partial \Delta$, say, where $1 \leq i_0 < j_0 \leq n - 1$. It is not hard to show that $J_i$ bounds a disk on $\partial H_{C - J_i}$, therefore $H_c = H_c - J_i \# D^3$, where $\#$ denotes the connected sum, in other words, $H_c$ can be obtained from $H_{C - J_i}$ by removing an open 3-cell in the interior of $H_{C - J_i}$, hence $H_c$ has incompressible boundary if and only if $H_{C - J_i}$ does, unless $H_{C - J_i} = D^3$. But for $H_{C - J_i}$, assumption (I)(1) $\ldots$ (n-1) cannot guarantee that $H_{C - J_i}$ has incompressible boundary, since there exist examples (see [6]) which show that none of the conditions in Przytycki’s conjecture can be deleted. Thus assumption (II) in Theorem 3.1 is needed.

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References