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Global stability of a Leslie–Gower predator–prey model with feedback controls

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ABSTRACT

In this work the global stability of a unique interior equilibrium for a Leslie–Gower predator–prey model with feedback controls is investigated. The main result together with its numerical simulations shows that feedback control variables have no influence on the global stability of the Leslie–Gower model, which means that feedback control variables only change the position of the unique interior equilibrium and retain its global stability.

1. Introduction

Recently, much scholars studied the predator–prey system with the Leslie–Gower scheme [1–4]. Numerical studies for a Leslie–Gower type tritrophic model were done in [1,2]. In [3], the authors obtained a set of sufficient conditions which ensure the global stability of the positive equilibrium for a predator–prey model with modified Leslie–Gower and Holling type II schemes. Korobeinikov [4] considered the following Leslie–Gower predator–prey model:

$$\begin{aligned} \dot{N}_1(t) &= (r_1 - a_1N_2 - b_1N_1)N_1, \\ \dot{N}_2(t) &= \left(r_2 - a_2\frac{N_2}{N_1}\right)N_2. \end{aligned} \tag{1}$$

By introducing the Lyapunov function

$$V(N_1, N_2) = \ln\frac{N_1}{N_1^0} + \frac{N_1^0}{N_1} + \frac{a_1N_1^0}{a_2}\left(\ln\frac{N_2}{N_2^0} + \frac{N_2^0}{N_2}\right),$$

where $N_1^0 = \frac{r_1a_2}{a_1r_2+a_2b_1}$, $N_2^0 = \frac{r_1r_2}{a_1r_2+a_2b_1}$, he was able to show that the unique coexisting point $E^0(N_1^0, N_2^0)$ of system (1) is globally stable.

On the other hand, in some situations, one may wish to alter the positions of positive equilibrium, but to retain its stability. This is of significance in the control procedure of ecology balance. One of the techniques used to achieve the aim is to alter system (1) structurally by introducing "indirect control" variables. Though there are many works on the single-species or multispecies competition systems with feedback controls [5–7], to the best of the authors' knowledge, to this day,

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still no scholars are investigating the stability property of the Leslie–Gower predator–prey model with feedback controls; this motivates us to propose and study such a model. In this work, we introduce and study the following Leslie–Gower predator–prey model with feedback controls:

$$\begin{aligned}
\dot{N}_1(t) &= (r_1 - a_1N_2 - b_1N_1 - c_1u_1)N_1,\\
\dot{N}_2(t) &= \left(r_2 - a_2\frac{N_2}{N_1} - c_2u_2\right)N_2,\\
\dot{u}_1(t) &= -f_1u_1 + g_1N_1,\\
\dot{u}_2(t) &= -f_2u_2 + g_2N_2,
\end{aligned} \tag{2}$$

where $N_1(t)$ and $N_2(t)$ denote the density of prey and predator populations at time t, respectively; $u_1(t)$ and $u_2(t)$ are feedback control variables; the term $a_2\frac{N_2}{N_1}$ describes the "carrying capacity" of the predator's environment which is proportional to the number of prey and a_2 is the maximum value which a per capita reduction rate of $N_1(t)$ can attain; r_1 and r_2 describe the intrinsic growth rates of $N_1(t)$ and $N_2(t)$, respectively; a_1 denotes the capturing rate of predator species $N_2(t)$; b_1 measures the strength of competition among individuals of $N_1(t)$. $r_i, a_i, c_i, f_i, g_i, i = 1, 2$, and b_1 are all positive constants. For more detailed adjustment of the Leslie–Gower system and the meanings of coefficients of the system, one could refer to [8–10] and the references cited therein.

With the restriction of their analysis technique, traditional works on the feedback control ecosystem (see [5,6] and the references cited therein) showed that feedback control variables play important roles as regards the persistency and stability properties of the system. Recently, by giving a detailed analysis of the right-hand side functional of the system, Hu, Teng and Jiang [7] were able to show that feedback control variables have no influence on the persistence property of the competition system that they considered. Their success motivated us to propose the following **conjecture**: Maybe the feedback control variables have no influence on the stability property of the system (2), that is, we can show that feedback controls only change the position of the unique interior equilibrium and retain its stability property.

The work is organized as follows. In the next section, we state and prove the global stability property of model (2). In Section 3, numerical simulations are presented to illustrate the feasibility of our results. We end this work with a brief discussion.

2. Global stability

For practical biological meaning, we simply study system (2) in $R^4_+ = \{(N_1, N_2, u_1, u_2) \in R^4 | N_i > 0, u_i > 0, i = 1, 2\}$ or in $\overline{R^4_+}$. From the first equation of system (2), it is easy to derive that $\limsup_{t\to\infty} N_1(t) \le \frac{r_1}{b_1}$.

Lemma 1. *The solutions $(N_1(t), N_2(t), u_1(t), u_2(t))^T$ of system (2) with initial values $N_i(0) > 0, u_i(0) > 0, i = 1, 2$, are positive and bounded for all $t \ge 0$.*

Proof. Obviously, the solutions $(N_1(t), N_2(t), u_1(t), u_2(t))^T$ of system (2) with initial values $N_i(0) > 0, u_i(0) > 0, i = 1, 2$, are positive for all $t \ge 0$. Given any $\epsilon > 0$, $N_1(t) \le \frac{r_1}{b_1} + \epsilon$ for t sufficiently large, from the second equation of system (2), it follows that $\dot{N}_2(t) \le \left(r_2 - \frac{a_2N_2}{\frac{r_1}{b_1}+\epsilon}\right)N_2$, which implies that $\limsup_{t\to\infty} N_2(t) \le \frac{r_1r_2}{a_2b_1}$. For above $\epsilon > 0$, $N_1(t) \le \frac{r_1}{b_1} + \epsilon$ and $N_2(t) \le \frac{r_1r_2}{a_2b_1} + \epsilon$ for t sufficiently large, from the third and fourth equations of system (2), it follows that

$$\begin{aligned}
\dot{u}_1(t) &\le -f_1u_1 + g_1\left(\frac{r_1}{b_1} + \epsilon\right),\\
\dot{u}_2(t) &\le -f_2u_2 + g_2\left(\frac{r_1r_2}{a_2b_1} + \epsilon\right).
\end{aligned}$$

By a standard comparison argument and basic ODE theory, it follows that $\limsup_{t\to\infty} u_1(t) \le \frac{g_1r_1}{f_1b_1}$ and $\limsup_{t\to\infty} u_2(t) \le \frac{g_2r_1r_2}{a_2b_1f_2}$, which completes the proof. □

Define $\delta = f_1(a_1r_2f_2 - r_1c_2g_2) + a_2f_2(b_1f_1 + c_1g_1)$, $\Delta = \delta^2 + 4r_1f_1a_2f_2c_2g_2(b_1f_1 + c_1g_1)$.

Lemma 2. *Model (2) admits a unique interior equilibrium $E^*(N_1^*, N_2^*, u_1^*, u_2^*)$, where $N_1^* = \frac{-\delta+\sqrt{\Delta}}{2c_2g_2(b_1f_1+c_1g_1)}$, $N_2^* = \frac{(\delta+2f_1r_1c_2g_2)-\sqrt{\Delta}}{2a_1f_1c_2g_2}$ and $u_i^* = \frac{g_i}{f_i}N_i^*, i = 1, 2$.*

Proof. We now consider positive equilibria of model (2). By the third and the fourth equation of model (2), we have $u_i = g_iN_i/f_i,\ i = 1, 2$. Substituting them into the right-hand side of the first and the second equations of model (2), respectively, we obtain

$$N_1 = \frac{f_1(r_1 - a_1N_2)}{b_1f_1 + c_1g_1}, \tag{3}$$

and

$$r_2f_2N_1 - a_2f_2N_2 - c_2g_2N_1N_2 = 0. \tag{4}$$

Substituting (3) into (4), we have

$$a_1f_1c_2g_2N_2^2 - (\delta + 2r_1f_1c_2g_2)N_2 + r_1r_2f_1f_2 = 0. \tag{5}$$

Since the discriminant of (5) satisfies

$$\Delta = (\delta + 2r_1f_1c_2g_2)^2 - 4a_1c_2g_2r_1r_2f_2f_1^2 = \delta^2 + 4r_1f_1a_2f_2c_2g_2(b_1f_1 + c_1g_1) > 0,$$

it is easy to see that (5) has two positive roots

$$N_2^{\pm} = \frac{(\delta + 2f_1r_1c_2g_2) \pm \sqrt{\Delta}}{2a_1f_1c_2g_2} > 0.$$

Note that $r_1 - a_1N_2^+ = \frac{-\delta-\sqrt{\Delta}}{2f_1c_2g_2} < 0$ and hence $N_1 = \frac{f_1(r_1-a_1N_2^+)}{b_1f_1+c_1g_1} < 0$; then model (2) has a unique positive equilibrium $E^*(N_1^*, N_2^*, u_1^*, u_2^*)$, where $N_2^* = N_2^- = \frac{(\delta+2f_1r_1c_2g_2)-\sqrt{\Delta}}{2a_1f_1c_2g_2}$, $N_1^* = \frac{f_1(r_1-a_1N_2^*)}{b_1f_1+c_1g_1} = \frac{-\delta+\sqrt{\Delta}}{2c_2g_2(b_1f_1+c_1g_1)}$ and $u_i^* = g_iN_i^*/f_i,\ i = 1, 2$, which completes the proof. □

Before we state and prove the global stability of this work, we need to state a definition and a useful lemma.

Definition 3 (*Chen, Song and Lu [11]*). A matrix $A = (a_{ij})_{n\times n}$ is said to be an M matrix if $a_{ij} \le 0, i \ne j, i, j = 1, 2, \ldots, n$, and any one of the following conditions holds:

(I) all of the eigenvalues of matrix A have positive real parts;
(II) the order principal minor of matrix A is positive;
(III) matrix A is nonsingular and $A^{-1} \ge 0$;
(IV) there exists a vector $x > 0$ such that $Ax > 0$;
(V) there exists a vector $y > 0$ such that $A^{\mathrm{T}}y > 0$.

Lemma 4 (*Araki and Kondo [12]*). *If A is an M matrix, then there exists a positive diagonal matrix $D = diag(d_1, d_2, \ldots, d_n),\ d_i > 0,\ i = 1, \ldots, n$, such that matrix $B = \frac{1}{2}(DA + A^{\mathrm{T}}D)$ is positive definite.*

Theorem 5. *Assume that $a_2b_1 \ge a_1r_2$ holds; then the unique interior equilibrium $E^*(N_1^*, N_2^*, u_1^*, u_2^*)$ of model (2) is globally stable.*

Proof. Note that $r_1 = a_1N_2^* + b_1N_1^* + c_1u_1^*, r_2 = a_2N_2^*/N_1^* + c_2u_2^*, f_iu_i^* = g_iN_i^*,\ i = 1, 2$; then model (2) can be rewritten as

$$\begin{aligned}
\dot{N}_1(t) &= [-b_1(N_1 - N_1^*) - a_1(N_2 - N_2^*) - c_1(u_1 - u_1^*)]N_1,\\
\dot{N}_2(t) &= \left[\left(-N_1^*(N_2 - N_2^*) + N_2^*(N_1 - N_1^*)\right)\frac{a_2}{N_1N_1^*} - c_2(u_2 - u_2^*)\right]N_2,\\
\dot{u}_1(t) &= [-N_1(u_1 - u_1^*) + u_1(N_1 - N_1^*)]\frac{g_1}{u_1^*},\\
\dot{u}_2(t) &= [-N_2(u_2 - u_2^*) + u_2(N_2 - N_2^*)]\frac{g_2}{u_2^*}.
\end{aligned} \tag{6}$$

Now let's construct a Lyapunov function

$$V(t) = d_1V_1(t) + d_2V_2(t) + e_1\phi_1(t) + e_2\phi_2(t),$$

where $V_i(t) = N_i - N_i^* - N_i^* \ln \frac{N_i}{N_i^*}$, $\phi_i(t) = u_i - u_i^* - u_i^* \ln \frac{u_i}{u_i^*}$, $e_i = \frac{d_ic_iu_i^*}{g_i}$, $i = 1, 2$, and d_i, $i = 1, 2$, are positively undetermined coefficients. Obviously, $V(t)$ is well defined and continuous for all $N_i, u_i > 0, i = 1, 2$. The time derivative of the function $V(t)$ along the solutions of model (6) is

$$\begin{aligned}
\dot{V}(t) &= d_1(N_1 - N_1^*)[-b_1(N_1 - N_1^*) - a_1(N_2 - N_2^*) - c_1(u_1 - u_1^*)]\\
&\quad + d_2(N_2 - N_2^*)\left[\left(-N_1^*(N_2 - N_2^*) + N_2^*(N_1 - N_1^*)\right)\frac{a_2}{N_1N_1^*} - c_2(u_2 - u_2^*)\right]\\
&\quad + \frac{g_1e_1(u_1 - u_1^*)}{u_1u_1^*}[-N_1(u_1 - u_1^*) + u_1(N_1 - N_1^*)] + \frac{g_2e_2(u_2 - u_2^*)}{u_2u_2^*}[-N_2(u_2 - u_2^*) + u_2(N_2 - N_2^*)]\\
&= -d_1b_1(N_1 - N_1^*)^2 + \left(\frac{a_2d_2N_2^*}{N_1^*N_1} - a_1d_1\right)(N_1 - N_1^*)(N_2 - N_2^*)\\
&\quad - \frac{a_2d_2}{N_1}(N_2 - N_2^*)^2 - \frac{c_1d_1N_1}{u_1}(u_1 - u_1^*)^2 - \frac{c_2d_2N_2}{u_2}(u_2 - u_2^*)^2,
\end{aligned}$$

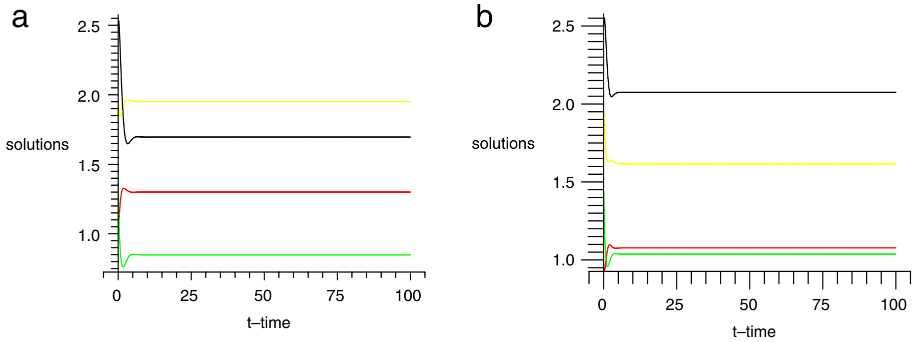

Fig. 1. Dynamic behaviors diagram of model (2) for $a_2b_1 \geq a_1r_2$. (a) $E^*(1.3, 0.848, 1.95, 1.696)$ is globally stable for $a_1 = 0.5, a_2 = 1, b_1 = 2$ and $r_2 = 1.5$. (b) $E^*(1.077, 1.037, 1.6155, 2.074)$ is globally stable for $a_1 = a_2 = 1, b_1 = r_2 = 2$.

and

$$\begin{aligned}
&-d_1b_1(N_1 - N_1^*)^2 + \left(\frac{a_2d_2N_2^*}{N_1^*N_1} - a_1d_1\right)(N_1 - N_1^*)(N_2 - N_2^*) - \frac{a_2d_2}{N_1}(N_2 - N_2^*)^2 \\
&\quad \leq -d_1b_1(N_1 - N_1^*)^2 + \left(\frac{a_2d_2N_2^*}{N_1^*N_1} + a_1d_1\right) \mid N_1 - N_1^* \mid \cdot \mid N_2 - N_2^* \mid - \frac{a_2d_2}{N_1}(N_2 - N_2^*)^2 \\
&\quad \stackrel{\Delta}{=} \frac{1}{2}Y^{\mathrm{T}}(DG + G^{\mathrm{T}}D)Y,
\end{aligned}$$

where $Y = \left(\mid N_1 - N_1^* \mid, \mid N_2 - N_2^* \mid\right)^{\mathrm{T}}$, $D = \mathrm{diag}(d_1, d_2)$, $d_i > 0$, $i = 1, 2$, and

$$G = \begin{pmatrix} -b_1 & a_1 \\ \dfrac{a_2N_2^*}{N_1^*N_1} & -\dfrac{a_2}{N_1} \end{pmatrix}.$$

To show that conclusion of Theorem 5 holds, for all $N_i, u_i > 0, i = 1, 2$, it is enough to prove that there exists a positive diagonal matrix $D = \mathrm{diag}(d_1, d_2), d_i > 0, i = 1, 2$, such that matrix $(DG + G^{\mathrm{T}}D)$ is negative definite. Note first that both of the off-diagonal elements of matrix G are positive and

$$b_1 - \frac{a_1N_2^*}{N_1^*} = \frac{2a_2f_2(2b_1f_1 + c_1g_1) - (\sqrt{\Delta} + \delta)}{2a_2f_1f_2}.$$

Simple algebraic computations show that, under the assumption $a_2b_1 \geq a_1r_2$, $b_1 - \frac{a_1N_2^*}{N_1^*} > 0$, the two order principal minors of matrix $-G$ are b_1 and $\frac{a_2}{N_1}\left(b_1 - \frac{a_1N_2^*}{N_1^*}\right)$, which are positive. From Definition 3, it follows that $-G$ is an M matrix; according to Lemma 4, there exists a positive diagonal matrix $D = \mathrm{diag}(d_1, d_2), d_i > 0, i = 1, 2$, such that matrix $(DG + G^{\mathrm{T}}D)$ is negative definite. So, under the assumption $a_2b_1 \geq a_1r_2$, $\dot{V}(t) < 0$ strictly for all $N_i, u_i > 0, i = 1, 2$, except the positive equilibrium $E^*(N_1^*, N_2^*, u_1^*, u_2^*)$, where $\dot{V}(t) = 0$. The above analysis shows that $V(t)$ satisfies Lyapunov's asymptotic stability theorem and the unique interior equilibrium $E^*(N_1^*, N_2^*, u_1^*, u_2^*)$ of model (2) is globally stable, which completes the proof. □

3. Numerical examples

Let $f_1 = f_2 = 1, c_1 = c_2 = 0.5, r_1 = 4, g_1 = 1.5$ and $g_2 = 2$. For these values of parameters, we verify the existence and stability properties of the positive equilibrium for model (2). In Fig. 1, we clearly observe that the unique interior equilibrium of model (2) is globally stable under the assumption that $a_2b_1 \geq a_1r_2$ holds.

Since it is difficult to make further analytical studies, we invoke numerical calculations to find asymptotical behaviors of model (2) for $a_2b_1 < a_1r_2$. Fig. 2 shows that the unique interior equilibrium of model (2) is globally stable also under the condition that $a_2b_1 < a_1r_2$ holds.

4. Discussion

Figs. 1 and 2 and Theorem 5 show that the unique interior equilibrium is globally stable for the Leslie–Gower predator–prey model with feedback controls. Compared to the global stability of the unique interior equilibrium E^0 for model (1) in [4], it is found that feedback controls have no influence on the existence and stability properties of the unique positive equilibrium for the Leslie–Gower predator–prey model. Moreover, in the cases of Fig. 1(a), (b) and Fig. 2(a), (b), the unique

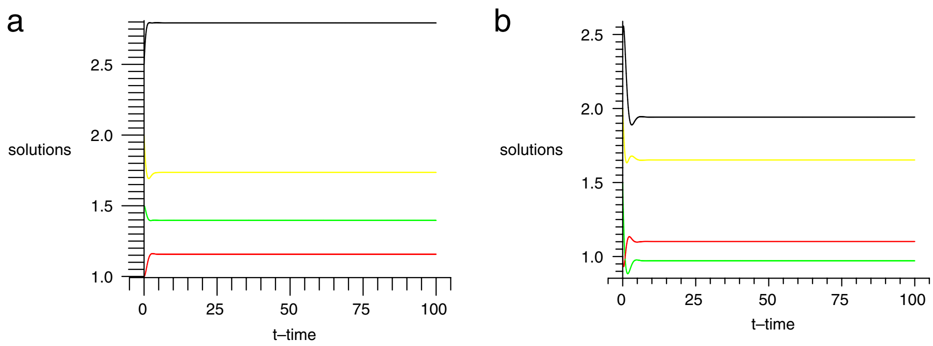

Fig. 2. Dynamic behaviors diagram of model (2) for $a_2b_1 < a_1r_2$. (a) $E^*(1.157, 1.397, 1.736, 2.794)$ is globally stable for $a_1 = 1$, $a_2 = 0.5$, $b_1 = 1.5$ and $r_2 = 2$. (b) $E^*(1.101, 0.971, 1.6515, 1.942)$ is globally stable for $a_1 = 1$, $a_2 = 0.6$, $b_1 = 2$ and $r_2 = 1.5$.

interior equilibrium E^0 of model (1) is $E^0(1.455, 2.182)$, $E^0(1, 2)$, $E^0(0.727, 2.909)$ and $E^0(0.889, 2.222)$, respectively. Note that $N_2^* < N_2^0$ in the above four cases. Hence, for the Leslie–Gower predator–prey model, feedback controls only change the position of the unique interior equilibrium and help the predator species to stability, while retaining global stability of the unique interior equilibrium. This indicates that, in the realistic environment, predator species could extend their survival space by introducing the feedback control variables and both predator species and prey species would finally reach a 'good' state, which is suitable for them to survive and develop in. This aids permanence of ecosystems.

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