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Almost distance-hereditary graphs

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Abstract

Distance-hereditary graphs (graphs in which the distances are preserved by induced subgraphs) have been introduced and characterized by Howorka. Several characterizations involving metric properties have been obtained by Bandelt and Mulder. In this paper, we extend the notion of distance-hereditary graphs by introducing the class of almost distance-hereditary graphs (a very weak increase of the distance is allowed by induced subgraphs). We obtain a characterization of these graphs in terms of forbidden-induced subgraphs and derive other both combinatorial and metric properties. © 2002 Elsevier Science B.V. All rights reserved.

Keywords: Distance-hereditary; Forbidden configurations; Metric properties

1. Introduction

A distance-hereditary graph is a connected graph, which preserves the distance function for induced subgraphs. That is, the distance between any two non-adjacent vertices of any connected induced subgraph of such a graph is the same as the distance between these two vertices in the original graph. These graphs have been introduced by Howorka [4], who derived many basic properties and gave some characterizations in terms of forbidden subgraphs [5]. Other characterizations for distance-hereditary graphs involving either the metric properties or forbidden configurations were given independently by Bandelt and Mulder [1], by D'Atri and Moscarini [2] and by Hammer and Maffray [3]. For instance, a connected graph is distance-hereditary if and only if each cycle on five or more vertices has at least two crossing chords [1] (where a chord of a cycle C is an edge that joins two non-consecutive vertices of C, and two chords $\{u,v\}$ and $\{w,x\}$ cross if the vertices u,w,v,x are distinct and in this order on the cycle).

In this paper, we introduce the class of almost distance-hereditary graphs. In such graphs, we do not require, as in distance-hereditary graphs, that the distance between any pair of non-adjacent vertices of any connected induced subgraph to be the same

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as in the original graph, but we allow this distance increase by one unit. We give a characterization of such graphs by forbidden-induced subgraphs, and derive some other combinatorial and metric properties.

2. Preliminaries

We shall only consider finite, simple loopless, undirected connected graphs $G=(V,E)$, where V is the vertex set and E is the edge set. An induced path of G is a non-redundant connection of a pair of vertices and if the end vertices are the same one, we have an induced cycle. The distance $d_G(u,v)$ (or simply $d(u,v)$ if no ambiguity arises) between two vertices u and v of a connected graph G is the length of a shortest $\{u,v\}$-path of G, a path connecting the vertices u and v.

If G is connected and v is one of its vertices, the hanging of G at v is the function h_v that associates to every vertex u in V the value $d_G(u,v)$. Thus, the vertex set V of G is partitioned, with respect to a vertex v, into levels $N_k(v)$, where the kth level contains the set of vertices of G at distance k from v, that is,

$$N_k(v)=\{u\in V \mid d(u,v)=k\}.$$

The neighborhood of v, often denoted by $N(v)$, is simply $N_1(v)$, the set of vertices at distance one from v.

For convenience, the (x,y) section of a path P (respectively a cycle C) is denoted by $P(x,y)$ (respectively, $C(x,y)$). A chord of a cycle C is an edge that joins two non-consecutive vertices of C. Two chords are disjoint if they have no common end-point (e.g. these chords are not adjacent), and two disjoint chords $\{u,v\}$ and $\{w,x\}$ cross if one and only one among the vertices w and x is in $C(u,v)$.

A graph $G=(V,E)$ is almost distance-hereditary if for all connected induced subgraphs $H=(Y,F)$ of G, we have

$$\forall u,v\in Y,\quad d_H(u,v)\leqslant d_G(u,v)+1.$$

It is easy to see that equivalently, a graph $G=(V,E)$ is almost distance-hereditary if and only if for every pair of non-adjacent vertices u and v, the length of each induced $\{u,v\}$-path either is equal to $d_G(u,v)$ or to $d_G(u,v)+1$. For instance, it is easy to verify that the graphs in Fig. 1, the cycle on five vertices without chord or with either one or two adjacent chords, which we shall now refer to as the C_5-configurations are almost distance-hereditary but not distance-hereditary.

Observe that C_n, the cycle of length n, is not almost distance-hereditary if $n\geqslant 6$, because it can be seen as the union of two disjoint induced paths, the first one with length 2 and the other one with length $n-2$. This observation is also valid if the cycle contains chords, all of which are incident to a single vertex (see Fig. 2). Henceforth, we will refer to a cycle on n vertices without chords or with chords all of which are incident to a single vertex as a C_n-configuration. Dashed line represents an edge, which may be included in the graph.

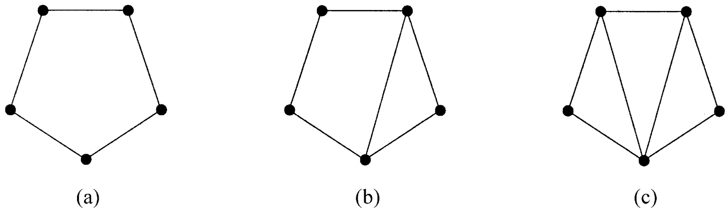


Fig. 1. C_5-configurations.

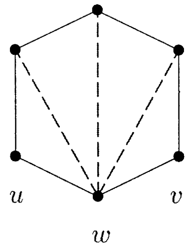


Fig. 2. C_6-configurations.

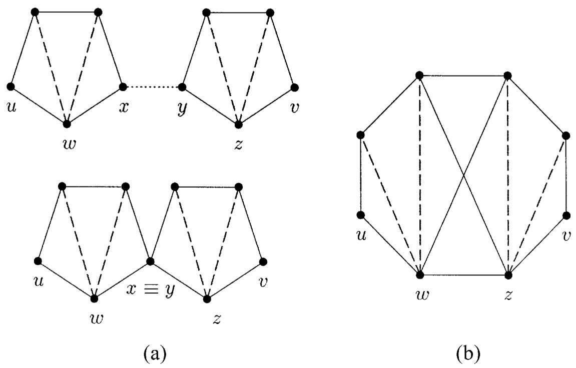


Fig. 3. $2C_5$-configurations.

Thus, a necessary condition for a graph G to be almost distance-hereditary is that every cycle on 6 or more vertices has chords which are not all incident to a single vertex.

Now, let us define a $2C_5$-configuration of type (a) to be a graph obtained by combining two C_5-configurations by connecting by an induced path two vertices which are not endpoints of chords of a cycle (Fig. 3(a)) (the induced path can be empty, and the related vertices are then identified) and a $2C_5$-configuration of type (b) to be the graph isomorphic to Fig. 3(b).

In Fig. 3 (and in all our figures), a dashed line represents an edge, which may be included in the graph, and a dotted line represents a path, eventually empty (vertices x and y coincide in this case).

One can easily see that each of these $2C_5$-configurations satisfies the necessary condition above (they do not contain a cycle on 6 or more vertices with all chords incident to a single vertex) but each of them contains two induced $\{u, v\}$-paths whose lengths differ by 2, and, consequently, cannot be contained in an almost distance-hereditary graph.

In fact, we have the following:

Lemma 1. *A necessary condition for a graph to be almost distance-hereditary is that it neither contains $2C_5$-configurations nor C_n-configurations, for $n \geqslant 6$, as induced subgraphs.*

Proof. The previous arguments can be used in the general case. □

3. Characterizations

Recall that a connected graph G is distance-hereditary if and only if every cycle in G of length at least 5 has a pair of crossing chords [1]. However, we do not have to require that these chords cross each other, since we have exactly the same result if we either impose to these chords to be disjoint or not to be incident to a single vertex.

More generally, let us define for every integer $k \geqslant 4$, the following classes of graphs.

$\mathscr{G}_c(k)$: The class of graphs such that every cycle on k or more vertices has a pair of crossing chords.

$\mathscr{G}_d(k)$: The class of graphs such that every cycle on k or more vertices has a pair of disjoint chords.

$\mathscr{G}_n(k)$: The class of graphs such that every cycle on k or more vertices has two or more chords that are not all incident to a single vertex.

$\mathscr{G}(k)$: The class of graphs such that every cycle on k or more vertices has a pair of chords.

It is well known that $\mathscr{G}(4)$ is simply the class of block graphs [1] (a block graph is a connected graph in which every 2-connected component is complete), and $\mathscr{G}_c(5)$ is the class of distance-hereditary graphs [1].

In fact, it is obvious that for every integer $k \geqslant 4$, $\mathscr{G}_c(k) \subseteq \mathscr{G}_d(k) \subseteq \mathscr{G}_n(k) \subseteq \mathscr{G}(k)$.

Moreover, we have the following.

Proposition 2. *Following the definitions and notations above, we have*

1. $\mathscr{G}_c(4) = \mathscr{G}_d(4) = \mathscr{G}_n(4) = \mathscr{G}(4)$,
2. $\mathscr{G}_c(5) = \mathscr{G}_d(5) = \mathscr{G}_n(5) \subset \mathscr{G}(5)$,
3. $\mathscr{G}_c(k) \subset \mathscr{G}_d(k) \subset \mathscr{G}_n(k) \subset \mathscr{G}(k)$, *for every* $k \geqslant 6$,

where the symbol '$\subset$' denotes the strict inclusion.

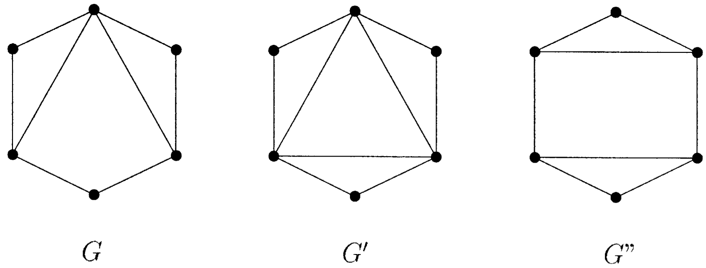


Fig. 4.

Proof. We just give a proof of the inclusion $\mathscr{G}_n(5) \subseteq \mathscr{G}_c(5)$. Let G be a connected graph in $\mathscr{G}_n(5)$. To prove that G belongs to $\mathscr{G}_c(5)$, we use induction over the length of cycles in G. Observe that if a cycle of length 5 has chords, which are not all incident to a single vertex, then necessarily it has two crossing chords.

In a cycle of length 6 with two chords, either these chords cross each other, or one of them is short (a chord $\{u, v\}$ in a cycle C is short if C contains a vertex w such that the edges $\{u, w\}$ and $\{w, v\}$ belong to C). In the last case, we obtain a cycle of length 5 and using the previous arguments (related to cycles of length 5), we deduce that the original cycle has two crossing chords. To finish the proof, observe that every cycle of length q, $q \geqslant 7$, with chords, necessarily contains a smaller cycle C_p (eventually with chords) as an induced subgraph, where $5 \leqslant p < q$, and consequently (by induction hypothesis) must have at least two crossing chords. Because the cycle on five vertices with two non-crossing chords (Fig. 1(c)) belongs to $\mathscr{G}(5)$ but not to $\mathscr{G}_c(5)$, we have the strict inclusion.

To prove assertion 3 of the proposition, we just have to observe that in Fig. 4, the graph G belongs to $\mathscr{G}(6)$ but not to $\mathscr{G}_n(6)$, G' belongs to $\mathscr{G}_n(6)$ but not to $\mathscr{G}_d(6)$ and G'' belongs to $\mathscr{G}_d(6)$ but not to $\mathscr{G}_c(6)$, and these constructions can easily be generalized for any $k \geqslant 7$. $\square$

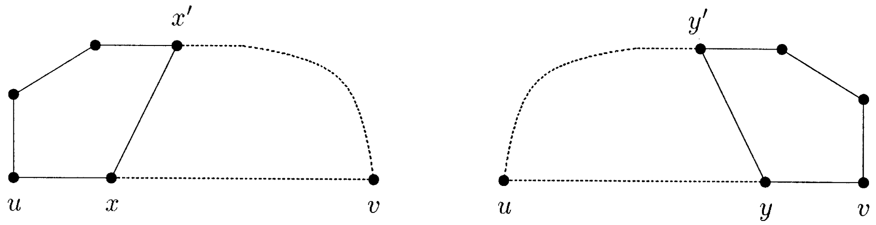
One can observe that the class of almost distance-hereditary graphs is contained in $\mathscr{G}_n(6)$.

For small graphs, it is easy to decide whether they are almost distance-hereditary or not. For instance, the graphs in Fig. 2 are not almost distance-hereditary (to see this, consider $d_G(u, v)$ and $d_{G-w}(u, v)$).

Theorem 3. *A graph G is almost distance-hereditary if and only if G neither contains a $2C_5$-configuration, nor a C_n-configuration, for $n \geqslant 6$, as induced subgraphs.*

Proof. '*Necessary condition*'. Just use Lemma 1.

'*Sufficient condition*'. Let G be a connected graph and assume that G is not almost distance hereditary. Therefore, G contains a pair $\{u, v\}$ of vertices linked by an induced path Q of length at least $d(u, v)+2$. Note that since the vertices u and v are not adjacent, then $d(u, v) \geqslant 2$.



(a) (b)

Fig. 5.

Choose such a pair of vertices u and v such that $d(u,v)$ is as small as possible. And then consider all shortest $\{u,v\}$-paths and $\{u,v\}$-paths of length at least $d(u,v)+2$. Let P be a shortest $\{u,v\}$-path and Q be an induced $\{u,v\}$-path of length at least $d(u,v)+2$ such that the vertices in these two paths are as small as possible. Henceforth, it will be referred to this choice of the vertices u and v and the paths P and Q as the minimality condition.

Thus, either u and v are contained in a cycle, induced by $P \cup Q$, say $C(u,v)$, of length at least $2d(u,v)+2$ or the paths P and Q contain two common vertices x and y (not necessarily distinct), such that

$$d_Q(u,x) \geqslant d_P(u,x)+1 \quad \text{and} \quad d_P(y,v) \geqslant d_Q(y,v)+1. \tag{1}$$

In fact, because of the minimality condition above, we have

$$d_Q(u,x) = d_P(u,x)+1 \quad \text{and} \quad d_P(y,v) = d_Q(y,v)+1. \tag{2}$$

Let us suppose that G contains no C_n-configuration, $n \geqslant 6$; we will show that G contains a 2-C_5-configuration.

Case 1: *u and v are contained in a cycle $C(u,v)$ of length at least $2d(u,v)+2$.* Since $d(u,v) \geqslant 2$, the length of the cycle $C(u,v)$ is at least six. Let x (respectively, y) be the vertex of $P \setminus \{u,v\}$ closest to u (respectively, v) having neighbors in Q, and let x' (respectively, y') be the vertex of Q closest to v (respectively, u) and adjacent to x (respectively, y) (Fig. 5). Observe that x and y cannot coincide (if such is the case, $C(u,v)$ induces a cycle on 6 or more vertices with chords coming out of the vertex $x \equiv y$, inducing a C_n-configuration), and hence the length $|P(u,v)|$ of $P(u,v)$ satisfies $|P(u,v)| \geqslant 3$.

If $d_P(u,x)+d_Q(u,x')+1 \geqslant 6$, the cycle induced by $P(u,x) \cup Q(u,x')$ is of length at least six, but all its chords are adjacent with x, a contradiction.

Therefore, $d_P(u,x)+d_Q(u,x') \leqslant 4$. If $d_Q(u,x') \leqslant d_P(u,x)+1$, then one can substitute u by x without violating the distance conditions, contradicting the minimality condition on $d(u,v)$.

Hence, $d_Q(u,x') > d_P(u,x)+1$, yielding $d_Q(u,x')=3$ and $d_P(u,x)=1$. By symmetry, we obtain $d_Q(v,y')=3$ and $d_P(v,y)=1$.

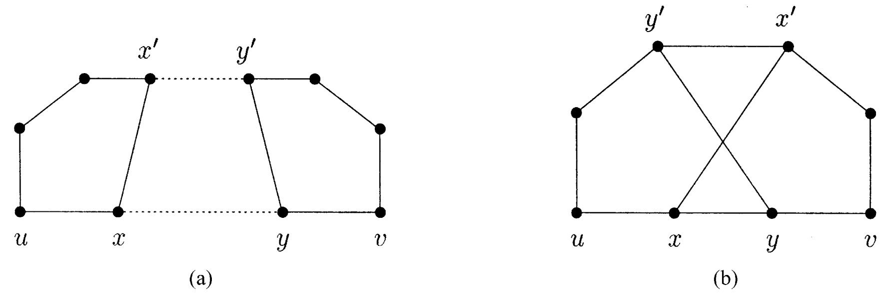


Fig. 6.

First assume $d_Q(u,x') \leqslant d_Q(u,y')$. By our assumption we have

$$|Q(u,v)| = 6 + d_Q(x',y') \geqslant |P(u,v)| + 2 = 4 + d_P(x,v) \tag{3}$$

implying $d_Q(x',y') \geqslant d_P(x,y) - 2$. Due to the minimality of $d(u,v)$ we must have $d_P(u,y) + 1 \geqslant d_Q(u,y') + 1$, yielding $d_Q(x',y') = d_P(x,y) - 2$.

If $d_Q(x',y') \geqslant 2$, then vertices x' and y' together with the $\{x',y'\}$-paths $P' = Q(x',y')$ and $Q' = \{x',x\} \cup P(x,y) \cup \{y,y'\}$ fulfill our assumption, contradicting the minimality of $d(u,v)$.

Therefore, the vertices x' and y' either coincide or are adjacent and we conclude $d_P(x,y) = 2$ or 3 (Fig. 6(a)). By removing the inner vertex (vertices) of $P(x,y)$ we obtain a $2C_5$-configuration of type (a), a contradiction.

Consequently, $d_Q(u,x') > d_Q(u,y')$. Hence, y' lies on $Q(u,x')$. Since $d_Q(u,x') = 3$, there is a vertex between u and y' or between y' and x'. Take the latter case. Since x' also lies on $Q(y',v)$, this makes $d_Q(u,v) = 4$ and $d_P(u,v) = 2$, a contradiction. If there is a vertex between u and y', then x' and y' are necessarily adjacent. It follows $|Q(u,v)| = |P(u,v)| + 2 = 5$, and the configuration pointed out in Fig. 6(b) implying a $2C_5$-configuration of type (b), which is impossible.

Case 2: *A shortest* u,v-*path* P *contains two vertices* x *and* y *such that*

$$d_Q(u,x) \geqslant d_P(u,x) + 1 \quad \text{and} \quad d_Q(y,v) \geqslant d_P(y,v) + 1. \tag{4}$$

Let x (respectively, y) be the first (respectively, last) vertex belonging both to P and to Q (and then satisfying the relations (4)). Hence, we can assume that the portions of P and Q between the vertices u and x and between y and v form elementary cycles Fig. 7. Let us use the notations $C(u,x)$ and $C(y,v)$ to designate these cycles. If the length of both $C(u,x)$ and $C(y,v)$ is equal to 5, then we are done, since we obtain a $2C_5$-configuration of type (a).

Let us consider the cycle $C(u,x)$ (the same arguments occur for $C(y,v)$), and assume that its length is greater than or equal to 7 (note that the length of such a cycle is odd and then cannot be equal to 6). $C(u,x)$ contains at least two chords, each one linking a vertex in P to a vertex in Q. Moreover, these chords do not have a common endpoint in P.

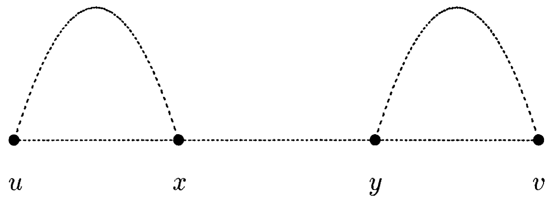


Fig. 7.

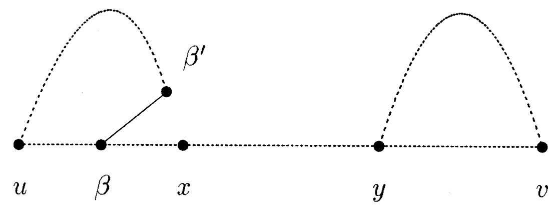


Fig. 8.

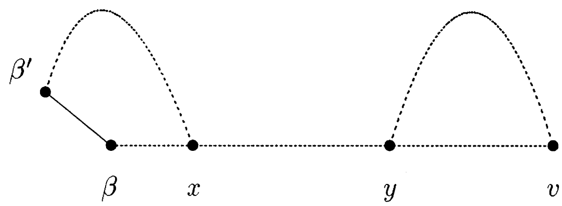


Fig. 9.

Let β be the vertex in $P(u,x)$ having neighbors in $Q(u,x)$ closest to x and let β' be the neighbour of β in $Q(u,x)$ closest to u. Observe that since the cycle $C(u,x)$ has at least two chords which are not adjacent to a single vertex, we have $d_P(u,\beta)\geqslant 2$.

If $d_Q(u,\beta')\leqslant d_P(u,\beta)$ then set $P'=P$ and $Q'=Q(u,\beta')\cup\{\beta,\beta'\}\cup P(\beta,x)\cup Q(x,v)$ (Fig. 8). Thus, P and Q' are two $\{u,v\}$-paths, satisfying the hypothesis, but $P'\cup Q'$ is strictly contained in $P\cup Q$, since $Q(\beta',x)$ contains other vertices than β' and x, and a contradiction arises.

If $d_Q(u,\beta')>d_P(u,\beta)$ then set $P'=\{\beta',\beta\}\cup Q(\beta,v)$ and $Q'=Q(\beta',v)$ (Fig. 9). Thus, P' and Q' are two $\{\beta',v\}$-paths, satisfying the hypothesis, but $P'\cup Q'$ is strictly contained in $P\cup Q$, and a contradiction arises. $\square$

Corollary 4. *If a graph G does not contain a path of length* 4, *then G is almost distance-hereditary.*

Proof. If G does not contain a path on 5 vertices, then G neither contains a cycle on 6 or more vertices all chords of which are incident to a single vertex, nor a

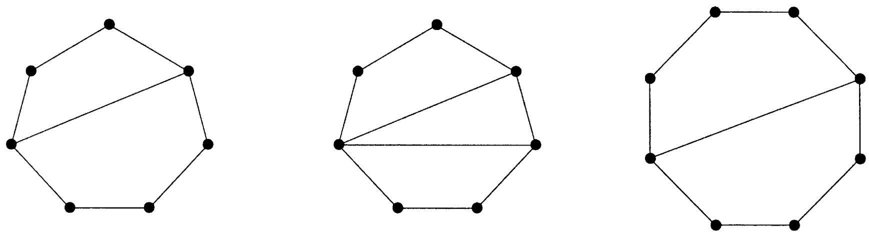


Fig. 10.

$2C_5$-configuration (observe that both $2C_5$-configurations of types (a) and (b) contain a path on 5 vertices) as induced subgraphs. □

Another characterization of almost distance-hereditary graphs can be obtained by solely induced subgraphs.

Corollary 5. *A graph G is almost distance-hereditary if and only if G contains none of the following configurations as induced subgraphs*:

(a) *chordless cycles of length greater than or equal to* 7,
(b) $2C_5$-*configurations*,
(c) *the graphs in Figs.* 2 *and* 10.

Proof. For the 'only if part', observe that cycles of length $n \geqslant 7$ containing chords necessarily contain smaller cycles, on 6 or more vertices, except the cycles in Fig. 10. Note that the 'if part' of the corollary is obvious, since if G contains a cycle of length greater than or equal to 7 or a $2C_5$-configuration or a graph in the Figs. 2 and 10, then it cannot be almost distance-hereditary. □

4. Other properties

Bandelt and Mulder [1] have shown that if G is a distance-hereditary graph, then there is no induced path of length 3 in any level $N_k(v)$ with respect to any vertex v. This property is obviously false in almost distance-hereditary graphs (it is not satisfied by the graph in Fig. 1(c)). Furthermore, levels of almost distance-hereditary graphs can contain induced paths of length 4 (Fig. 11).

However, the following proposition shows that no induced path of length 5 can be contained in levels of almost distance-hereditary graphs.

Proposition 6. *Let G be a graph with no C_n-configuration, for $n \geqslant 6$, as an induced subgraph, and let v be any vertex of G. Then for any $k \geqslant 1$, there is no induced path of length* 5 *in $N_k(v)$.*

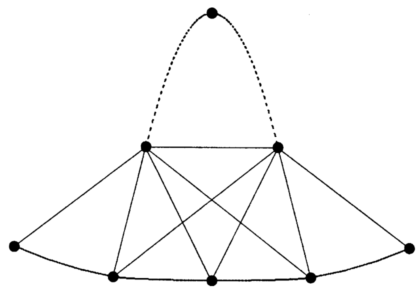


Fig. 11.

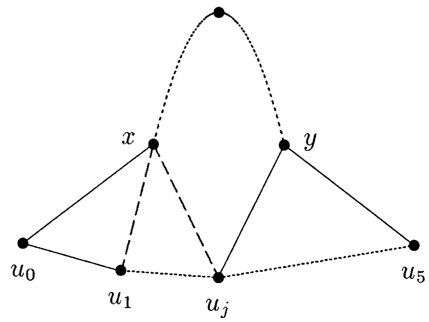


Fig. 12.

Proof. By contradiction, let G be a graph such that the chords of every cycle of length greater than or equal to 6 are not adjacent to a single vertex and for a certain vertex v, $N_k(v)$ contains a path $P=\{u_0,u_1,u_2,u_3,u_4,u_5\}$ of length 5. Since every vertex in P has a neighbor in $N_{k-1}(v)$, let x (respectively, y) be such a neighbor of u_0 (respectively, u_5) (Fig. 12).

Note that neither u_0 (respectively, u_5) nor u_1 (respectively, u_4) can be adjacent to y (respectively, x). Indeed, if, for instance, u_1 is adjacent to y, then we obtain a cycle $\{y,u_1,u_2,u_3,u_4,u_5\}$ of length 6 with all its chords adjacent to y. The same result arises in the other cases.

Let u_j be the first vertex in P adjacent to y (obviously, $j\geqslant 2$). There is certainly a minimal $\{x,y\}$-path P' of length greater than or equal to 2 and consisting of vertices having levels at most $k-1$.

Thus, we obtain a cycle containing the vertices $u_0,u_1,u_2,\ldots$ and u_j and those in P' (Fig. 11). Since $j\geqslant 2$ and the cardinality of P' is at least 3, then the length of this cycle is at least 6. Moreover, all its chords are adjacent to x, a contradiction. □

Corollary 7. *Let G be an almost distance-hereditary graph, and let v be any vertex of G. Then there is no induced path of length* 4 *in* $N_1(v)$ *and for any* $k\geqslant 2$, *there is no induced path of length* 5 *in* $N_k(v)$.

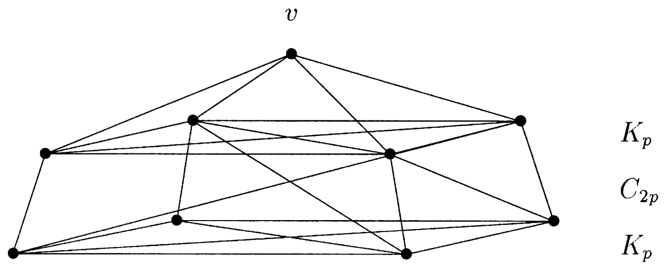


Fig. 13.

D'Atri and Moscarini [2] have shown that the vertical part of any distance-hereditary graph with respect to a vertex v is a bipartite graph such that any cycle of length at least 6 has two chords. Recall that the horizontal part $H(G,v)$ of a graph $G=(V,E)$, with respect to the vertex v is the partial graph (V,F) of G that contains all the edges between vertices equidistant from v and the vertical part of G with respect to v, denoted by $V(G,v)$ is the complement of $H(G,v)$ in G.

In almost distance-hereditary graphs, this property is false.

Proposition 8. *For every integer $k \geqslant 2$, there exists an almost distance-hereditary graph such that its vertical part contains an induced cycle on $2k$ vertices.*

Proof. Let us consider the graph defined on $(2k+1)$ vertices as follows. First, $2k$ vertices are linked in a cycle (which is bipartite). Then we add all possible edges between vertices in the same stable of the bipartition. We obtain two cliques each over k vertices linked by a cycle of length $2k$. Finally, we connect the remaining vertex, say v, to all vertices in one of the cliques (Fig. 13).

It is easy to see that the graph in Fig. 13 is almost distance-hereditary. Moreover, its vertical part contains a cycle of length $2k$ without chords. □

Proposition 9. *Let G be a graph such that the chords of every cycle of length greater than or equal to 6 do not have the same endpoint, and let v be any vertex of G. If x and y are two non-adjacent vertices in $N_k(v)$, connected by a path whose vertices have level greater than k, then x and y have the same neighborhood in $N_{k-1}(v)$.*

Proof. If $k=1$, there is nothing to prove.

If $k \geqslant 2$, let P be an induced $\{x,y\}$-path whose internal vertices are in levels greater than k and let x' (respectively, y') be a neighbor of x (respectively, y) in $N_{k-1}(v)$. Assume that y and x' are not adjacent. Let P' be an $\{x',y'\}$-path such that

- all vertices in P' are in levels lower than or equal to $k-2$, and
- the only possible chord in P' is $\{x',y'\}$.

Thus, the vertices of P and those of P' constitute a cycle of length at least six. Now, note that all the chords of this cycle are adjacent to y' (Fig. 14), a contradiction.

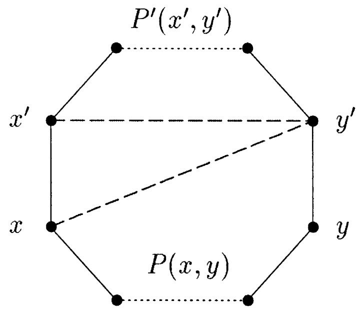


Fig. 14.

Corollary 10. *Let $G=(V,E)$ be an almost distance-hereditary graph, and let v be any vertex of G. If x and y are two non-adjacent vertices in $N_k(v)$, connected by a path whose vertices have levels greater than k, then x and y have the same neighborhood in $N_{k-1}(v)$.*

5. Structure of almost distance-hereditary graphs

The vertices of an almost distance-hereditary graph can be partitioned in two subsets. The first one consists of good vertices, those vertices v such that for every vertex x, any induced $\{v,x\}$-path is minimum. In the second one, we have the vertices v for which there is at least a vertex x such that there exists an induced non-minimum $\{v,x\}$-path. Such a vertex will be called defective (its defectivity is not identically null). The defectivity of a vertex v in a graph G of order n, can be defined to be the n-integer vector def_v, where

$$\mathrm{def}_v(x) = \mathrm{Max}\{|Q(v,x)| - d(v,x)/Q(v,x) \text{ is an induced } \{v,x\}\text{-path}\}.$$

Of course, in a distance-hereditary graph, every vertex is good and its defectivity is identically null. An almost distance-hereditary graph, which is not distance-hereditary necessarily contains a defective vertex, and also a C_5-configuration as an induced subgraph. Bandelt and Mulder [1] have shown that if G is a distance-hereditary graph, then G contains either two pendant vertices or a pair of twins. The following result immediately follows.

Proposition 11. *Let G be an almost distance-hereditary graph with at least two vertices. Then G contains*

(a) *either two pendant vertices,*
(b) *or a pair of twins,*
(c) *or an induced C_5-configuration.*

In fact, any distance-hereditary graph can be obtained from a simple vertex by a sequence of attaching pendant vertices and splitting vertices. These two operations obviously preserve almost distance-heredity. One can define an other operation, which also preserves almost distance-heredity. It consists to identify any vertex of an almost distance-hereditary graph G with any good vertex of an other almost distance-hereditary graph G'. Unhappily, if the first two operations characterize connected distance-hereditary graphs [1], we cannot characterize, in the same way, almost distance-hereditary graphs, even if we add the third operation. The graphs in Fig. 4 cannot be obtained in this way.

6. A metric result

When studying some classes of graphs involving the distance notion, a sort of 'four-point condition' is often used. Indeed, for any four vertices u, v, w, x, one compares the distance sums:

$$d(u,v)+d(w,x), \quad d(u,w)+d(v,x), \quad d(u,x)+d(v,w).$$

Distance-hereditary graphs have been characterized in this way [1]. They are precisely those graphs which always at least two of the above distance sums are equal. For almost distance-hereditary graphs, we have the following.

Proposition 12. *Let G be graph with no C_n-configuration, $n \geqslant 6$, as induced subgraph, and such that for any four vertices u, v, w, x, the difference between any two of the distance sums $d(u,v)+d(w,x)$, $d(u,w)+d(v,x)$ and $d(u,x)+d(v,w)$ is at most equal to* 2. *Then G is almost distance-hereditary.*

Proof. Let G be a connected graph with no induced C_n-configurations, for $n \geqslant 6$, such that for any four vertices u, v, w, x, the difference between any two of the distance sums $S_1 = d(u,v)+d(w,x)$, $S_2 = d(u,w)+d(v,x)$ and $S_3 = d(u,x)+d(v,w)$ is at most equal to 2. We have to show that the length of every induced $\{u,x\}$-path is equal to either $d(u,x)$ or to $d(u,x)+1$ as well.

Let P be an induced $\{u,x\}$-path in G. We proceed by induction on the length of P.

The result is obvious if the length of P is either equal to 2 or 3.

Let us assume that any induced $\{u,x\}$-path in G with length q less than or equal to k is such that $q = d(u,x)$ or $q = d(u,x)+1$. Let $P = \{u_0, u_1, \ldots, u_k, u_{k+1}\}$ be an induced path in G with length $k+1$ ($k+1 \geqslant 4$). We have to show that, if we set $u = u_0$, $v = u_1$, $w = u_k$ and $x = u_{k+1}$ (Fig. 15), then either $d(u,x) = k$ or $d(u,x) = k+1$.

From the choice of the path P, we have

(i) $d(u,x) \geqslant 2$, $d(u,w) \geqslant 2$ and $d(v,x) \geqslant 2$ (P is an induced path).

(ii) $S_1 = 2$, $S_3 \leqslant S_1 + 2 = 4$ and also $S_2 \leqslant 4$. Then from (i), we have $S_2 = 4$, and $3 \leqslant S_3 \leqslant 4$, and consequently, $d(u,w) = 2$ and $d(v,x) = 2$.

(iii) $k-1 \leqslant d(u,w) \leqslant k$, by induction hypothesis, since the length of $P(u,w)$ is k. It follows that $2 \leqslant k \leqslant 3$.

Consequently, we have $k = 3$.

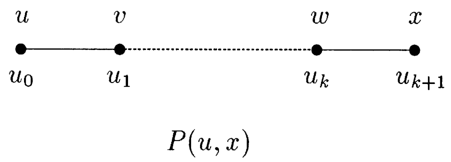


Fig. 15.

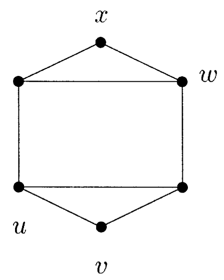


Fig. 16.

Now, observe that if $d(u,x) = 2$, then G should contain a C_6-configuration, a contradiction. □

Note that the converse of this proposition is false. We can verify this with the graph in Fig. 16.

7. Some open questions

Bandelt and Mulder [1] have completely characterized distance-hereditary graphs in every way. They have obtained combinatorial, algorithmic and metric characterizations. For almost distance-hereditary graphs, we have just obtained a combinatorial characterization and some partial results about metric and algorithmic approaches. It will be interesting to complete these results by obtaining an efficient recognition algorithm for this class of graphs, if such exists (may be this recognition problem is not polynomial). Also, one can try to set a necessary and sufficient condition for almost distance-hereditary graphs, like the four-point condition (in the same way that sufficient condition given in Proposition 12).

The notion of almost distance-heredity can also be adapted to bipartite graphs, where it is allowed to the distance between any pair of non-adjacent vertices of any connected subgraph increase by two (instead of one unit). Similar properties and characterizations as for almost distance-hereditary can be easily derived.

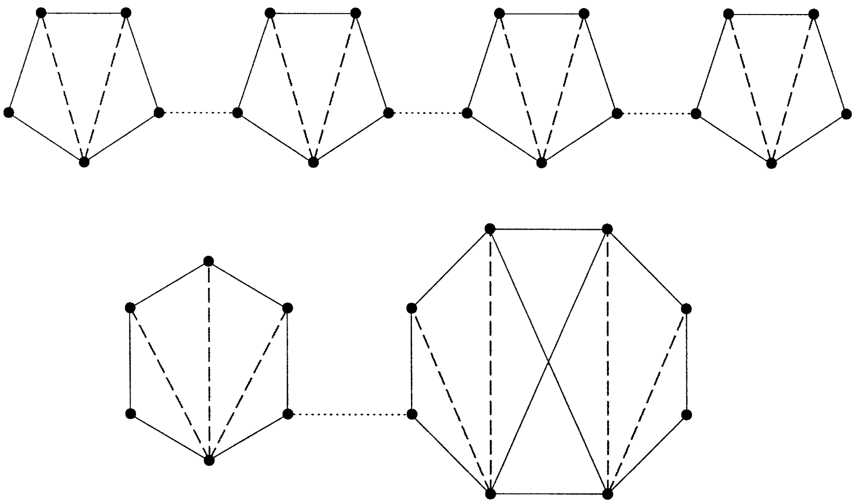


Fig. 17.

Note that, the characterization in Theorem 3 can be extended and adapted to the larger class of graphs defined in a similar way that almost distance-hereditary graphs, but where distance constraint is further relaxed so that for all connected induced subgraphs $H=(Y,F)$ of $G=(V,E)$, we have

$$\forall u, \quad v \in Y, \quad d_H(u,v) \leqslant d_G(u,v) + r \ \text{ (where } r \geqslant 2)$$

Let $\mathscr{D}(r)$ denotes the class of such graphs. For example, forbidden induced subgraphs for $\mathscr{D}(1)$ are cycles of length at least 7 and those configurations obtained by combining (in the same way that in the definition of $2C_5$-configurations of type (a)) either two forbidden subgraphs of $\mathscr{D}(1)$ (almost distance-hereditary graphs) or a forbidden configuration of $\mathscr{D}(0)$ (distance-hereditary graphs) with a forbidden configuration of $\mathscr{D}(2)$ (some of forbidden configurations of $\mathscr{D}(3)$ are represented in Fig. 17. But, there are other forbidden configurations, defined in a similar manner that $2C_5$-configurations of type (b) which cannot be obtained in this way.

- How to characterize graphs in $\mathscr{D}(r)$?
- How to determine the smallest integer r such that a given graph G belongs to $\mathscr{D}(r)$?

An other extension of the notion of distance-heredity can be the defectivity approach. One can associate to any graph on n vertices, a $(n \times n)$-integer matrix D, where $D_{v,x} = \mathrm{def}_v(x)$.

Obviously, the defectivity matrix of a distance-hereditary graph has all its entries zero.

What are the properties of this matrix for almost distance-hereditary graphs and most generally for graphs in $\mathscr{D}(r)$?

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