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New fixed point theorems for mixed monotone operators and local existence–uniqueness of positive solutions for nonlinear boundary value problems ☆

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ABSTRACT

In this article we present a new fixed point theorem for a class of general mixed monotone operators, which extends the existing corresponding results. Moreover, we establish some pleasant properties of nonlinear eigenvalue problems for mixed monotone operators. Based on them the local existence–uniqueness of positive solutions for nonlinear boundary value problems which include Neumann boundary value problems, three-point boundary value problems and elliptic boundary value problems for Lane–Emden–Fowler equations is proved. The theorems for nonlinear boundary value problems obtained here are very general.

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1. Introduction

It is well known that nonlinear boundary value problems (BVPs for short) arise in a variety of different areas of applied mathematics, physics, chemistry and biology, which can be found in the theory of nonlinear diffusion generated by nonlinear sources, in thermal ignition of gases, in the vibrations of a guy wire of a uniform cross-section and composed of N parts of different densities, and in the theory of elastics stability, in chemical or biological problems (see, for instance, [20,37,38,54,57,63]). Therefore, nonlinear BVPs have attracted much attention and have been widely studied, see [2–8,10–19,21–25,28–36,41–45,47–53,55–62,64–66] for some references along this line. The results of these papers are based on the Leray–Schauder continuation theorem, the nonlinear alternative of Leray–Schauder, the coincidence degree theory of Mawhin, Krasnosel'skii's fixed point theorem, Schauder fixed point theorem, fixed point theorems in cones and so on. Different from these finite methods, in this article we first state and prove new fixed point theorems for mixed monotone operators. And then we establish some criterions for the local existence–uniqueness of positive solutions to BVPs which include the Neumann BVPs, three-point BVPs and nonlinear elliptic BVPs for the Lane–Emden–Fowler equations. Let $\mathbf{R}^+ = [0,\infty)$, $\mathbf{R}^{++} = (0,\infty)$, $J = [0,1]$ and Ω be a bounded domain with smooth boundary in $\mathbf{R}^N$ $(N \geqslant 1)$. Our basic assumptions on a nonlinear function $f(t,u,v)$ here are:

(H_1) $f : J \times \mathbf{R}^+ \times \mathbf{R}^+ \to \mathbf{R}^+$;
$(H_1)'$ $f : \Omega \times \mathbf{R}^{++} \times \mathbf{R}^{++} \to \mathbf{R}^{++}$;

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(H_2) $f(t,u,v)$ is nondecreasing in u for each $t \in J$ and $v \in \mathbf{R}^+$, nonincreasing in v for each $t \in J$ and $u \in \mathbf{R}^+$, and for any $\gamma \in (0,1)$, there exists a constant $\varphi(\gamma) \in (\gamma, 1]$ such that

$$f(t, \gamma u, \gamma^{-1} v) \geqslant \varphi(\gamma) f(t,u,v) \quad \text{for any } u, v \in \mathbf{R}^+;$$

$(H_2)'$ $f(x,u,v)$ is nondecreasing in u for each $x \in \Omega$ and $v \in \mathbf{R}^{++}$, nonincreasing in v for each $x \in \Omega$ and $u \in \mathbf{R}^{++}$, and for any $\gamma \in (0,1)$, there exists a constant $\varphi(\gamma) \in (\gamma, 1]$ such that

$$f(x, \gamma u, \gamma^{-1} v) \geqslant \varphi(\gamma) f(x,u,v) \quad \text{for any } u, v \in \mathbf{R}^{++}.$$

In the next section, we state and prove a new existence–uniqueness result of positive fixed points for mixed monotone operators. Moreover, we establish some pleasant properties of nonlinear eigenvalue problems for mixed monotone operators. In Section 3, using the main results obtained in Section 2, we give the local existence–uniqueness results of positive solutions for nonlinear BVPs which include the Neumann BVPs, three-point BVPs and nonlinear elliptic BVPs for the Lane–Emden–Fowler equations. It must be pointed out that the method used in this article can be applied to many nonlinear BVPs.

2. Fixed points and eigenvalue problems for mixed monotone operators

Mixed monotone operators were introduced by Guo and Lakshmikantham [27] in 1987. Thereafter many authors have investigated these kinds of operators in Banach spaces and obtained a lot of interesting and important results (see [26,46,74] and the references therein). They are used extensively in nonlinear differential and integral equations. In this section, we modify the methods in [26,46] to obtain a new existence and uniqueness result of positive fixed point for mixed monotone operators.

Suppose that $(E, \|\cdot\|)$ is a real Banach space which is partially ordered by a cone $P \subset E$, i.e., $x \leqslant y$ if and only if $y - x \in P$. If $x \leqslant y$ and $x \neq y$, then we denote $x < y$ or $y > x$. By θ we denote the zero element of E. Recall that a nonempty closed convex set $P \subset E$ is a cone if it satisfies (i) $x \in P$, $\lambda \geqslant 0 \Rightarrow \lambda x \in P$; (ii) $x \in P$, $-x \in P \Rightarrow x = \theta$.

Putting $\mathring{P} = \{x \in P \mid x \text{ is an interior point of } P\}$, a cone P is said to be solid if its interior $\mathring{P}$ is nonempty. Moreover, P is called normal if there exists a constant $M > 0$ such that, for all $x, y \in E$, $\theta \leqslant x \leqslant y$ implies $\|x\| \leqslant M\|y\|$; in this case M is called the normality constant of P. If $x_1, x_2 \in E$, the set $[x_1, x_2] = \{x \in E \mid x_1 \leqslant x \leqslant x_2\}$ is called the order interval between x_1 and x_2.

For all $x, y \in E$, the notation $x \sim y$ means that there exist $\lambda > 0$ and $\mu > 0$ such that $\lambda x \leqslant y \leqslant \mu x$. Clearly, $\sim$ is an equivalence relation. Given $h > \theta$ (i.e., $h \geqslant \theta$ and $h \neq \theta$), we denote by P_h the set $P_h = \{x \in E \mid x \sim h\}$. It is easy to see that $P_h \subset P$ is convex and $\lambda P_h = P_h$ for all $\lambda > 0$. If $\mathring{P} \neq \emptyset$ and $h \in \mathring{P}$, it is clear that $P_h = \mathring{P}$.

Definition 2.1. (See [26,27].) $A : P \times P \to P$ is said to be a mixed monotone operator if $A(x,y)$ is increasing in x and decreasing in y, i.e., u_i, v_i $(i = 1,2) \in P$, $u_1 \leqslant u_2$, $v_1 \geqslant v_2$ imply $A(u_1, v_1) \leqslant A(u_2, v_2)$. Element $x \in P$ is called a fixed point of A if $A(x,x) = x$.

2.1. Fixed point theorems

Now we consider the mixed monotone operator $A : P \times P \to P$. The following conditions will be assumed:

(A_1) there exists $h \in P$ with $h \neq \theta$ such that $A(h,h) \in P_h$,
(A_2) for any $u, v \in P$ and $t \in (0,1)$, there exists $\varphi(t) \in (t, 1]$ such that $A(tu, t^{-1}v) \geqslant \varphi(t) A(u,v)$.

Lemma 2.1. *Assume (A_1), (A_2) hold. Then $A : P_h \times P_h \to P_h$; and there exist $u_0, v_0 \in P_h$ and $r \in (0,1)$ such that $r v_0 \leqslant u_0 < v_0$, $u_0 \leqslant A(u_0, v_0) \leqslant A(v_0, u_0) \leqslant v_0$.*

Proof. Firstly, from condition (A_2) we get

$$A(t^{-1}x, ty) \leqslant \frac{1}{\varphi(t)} A(x,y), \quad \forall t \in (0,1),\ x, y \in P. \tag{2.1}$$

For any $u, v \in P_h$, there exist $\mu_1, \mu_2 \in (0,1)$ such that

$$\mu_1 h \leqslant u \leqslant \frac{1}{\mu_1} h, \qquad \mu_2 h \leqslant v \leqslant \frac{1}{\mu_2} h.$$

Let $\mu = \min\{\mu_1, \mu_2\}$. Then $\mu \in (0,1)$. From (2.1) and the mixed monotone properties of operator A, we have

$$A(u,v) \leqslant A\left(\frac{1}{\mu_1} h, \mu_2 h\right) \leqslant A\left(\frac{1}{\mu} h, \mu h\right) \leqslant \frac{1}{\varphi(\mu)} A(h,h),$$

$$A(u,v) \geqslant A\left(\mu_1 h, \frac{1}{\mu_2} h\right) \geqslant A\left(\mu h, \frac{1}{\mu} h\right) \geqslant \varphi(\mu) A(h,h).$$

It follows from $A(h,h) \in P_h$ that $A(u,v) \in P_h$. Hence we have $A: P_h \times P_h \to P_h$. Since $A(h,h) \in P_h$, we can choose a sufficiently small number $t_0 \in (0,1)$ such that

$$t_0 h \leqslant A(h,h) \leqslant \frac{1}{t_0} h. \tag{2.2}$$

Noting that $t_0 < \varphi(t_0) \leqslant 1$, we can take a positive integer k such that

$$\left(\frac{\varphi(t_0)}{t_0}\right)^k \geqslant \frac{1}{t_0}. \tag{2.3}$$

Put $u_0 = {t_0}^k h$, $v_0 = \frac{1}{{t_0}^k} h$. Evidently, $u_0, v_0 \in P_h$ and $u_0 = {t_0}^{2k} v_0 < v_0$. Take any $r \in (0, {t_0}^{2k}]$, then $r \in (0,1)$ and $u_0 \geqslant r v_0$. By the mixed monotone properties of A, we have $A(u_0, v_0) \leqslant A(v_0, u_0)$. Further, combining condition (A_2) with (2.2), (2.3), we have

$$\begin{aligned}
A(u_0, v_0) &= A\left({t_0}^k h, \frac{1}{{t_0}^k} h\right) = A\left(t_0 \cdot {t_0}^{k-1} h, \frac{1}{t_0} \cdot \frac{1}{{t_0}^{k-1}} h\right) \geqslant \varphi(t_0) A\left({t_0}^{k-1} h, \frac{1}{{t_0}^{k-1}} h\right) \\
&= \varphi(t_0) A\left(t_0 \cdot {t_0}^{k-2} h, \frac{1}{t_0} \cdot \frac{1}{{t_0}^{k-2}} h\right) \geqslant \varphi(t_0) \cdot \varphi(t_0) A\left({t_0}^{k-2} h, \frac{1}{{t_0}^{k-2}} h\right) \geqslant \cdots \\
&\geqslant \big(\varphi(t_0)\big)^k A(h,h) \geqslant \big(\varphi(t_0)\big)^k t_0 h \geqslant {t_0}^k h = u_0.
\end{aligned}$$

From (2.1) we get

$$\begin{aligned}
A(v_0, u_0) &= A\left(\frac{1}{{t_0}^k} h, {t_0}^k h\right) = A\left(\frac{1}{t_0} \cdot \frac{1}{{t_0}^{k-1}} h, t_0 \cdot {t_0}^{k-1} h\right) \\
&\leqslant \frac{1}{\varphi(t_0)} A\left(\frac{1}{{t_0}^{k-1}} h, {t_0}^{k-1} h\right) = \frac{1}{\varphi(t_0)} A\left(\frac{1}{t_0} \cdot \frac{1}{{t_0}^{k-2}} h, t_0 \cdot {t_0}^{k-2} h\right) \\
&\leqslant \frac{1}{\varphi(t_0)} \cdot \frac{1}{\varphi(t_0)} A\left(\frac{1}{{t_0}^{k-2}} h, {t_0}^{k-2} h\right) \leqslant \cdots \\
&\leqslant \frac{1}{\big(\varphi(t_0)\big)^k} A(h,h) \leqslant \frac{1}{t_0 \big(\varphi(t_0)\big)^k} h.
\end{aligned}$$

An application of (2.3) implies that

$$A(v_0, u_0) \leqslant \frac{1}{t_0 \big(\varphi(t_0)\big)^k} h \leqslant \frac{1}{{t_0}^k} h = v_0.$$

Thus we have $u_0 \leqslant A(u_0, v_0) \leqslant A(v_0, u_0) \leqslant v_0$. $\square$

Theorem 2.1. *Suppose that P is a normal cone of E, and (A_1), (A_2) hold. Then operator A has a unique fixed point x^* in P_h. Moreover, for any initial $x_0, y_0 \in P_h$, constructing successively the sequences*

$$x_n = A(x_{n-1}, y_{n-1}), \qquad y_n = A(y_{n-1}, x_{n-1}), \quad n = 1, 2, \ldots,$$

we have $\|x_n - x^*\| \to 0$ and $\|y_n - x^*\| \to 0$ as $n \to \infty$.

Proof. From Lemma 2.1, there exist $u_0, v_0 \in P_h$ and $r \in (0,1)$ such that $r v_0 \leqslant u_0 < v_0$, $u_0 \leqslant A(u_0, v_0) \leqslant A(v_0, u_0) \leqslant v_0$. Construct successively the sequences

$$u_n = A(u_{n-1}, v_{n-1}), \qquad v_n = A(v_{n-1}, u_{n-1}), \quad n = 1, 2, \ldots.$$

Evidently, $u_1 \leqslant v_1$. By the mixed monotone properties of A, we obtain $u_n \leqslant v_n$, $n = 1, 2, \ldots.$ It also follows from Lemma 2.1 and the mixed monotone properties of A that

$$u_0 \leqslant u_1 \leqslant \cdots \leqslant u_n \leqslant \cdots \leqslant v_n \leqslant \cdots \leqslant v_1 \leqslant v_0. \tag{2.4}$$

Noting that $u_0 \geqslant r v_0$, we can get $u_n \geqslant u_0 \geqslant r v_0 \geqslant r v_n$, $n = 1, 2, \ldots.$ Let

$$t_n = \sup\{t > 0 \mid u_n \geqslant t v_n\}, \quad n = 1, 2, \ldots.$$

Thus we have $u_n \geqslant t_n v_n$, $n = 1, 2, \ldots,$ and then $u_{n+1} \geqslant u_n \geqslant t_n v_n \geqslant t_n v_{n+1}$, $n = 1, 2, \ldots.$ Therefore, $t_{n+1} \geqslant t_n$, i.e., $\{t_n\}$ is increasing with $\{t_n\} \subset (0,1]$. Suppose $t_n \to t^*$ as $n \to \infty$, then $t^* = 1$. Otherwise, $0 < t^* < 1$. Then from condition (A_2) and $t_n \leqslant t^*$, we have

$$u_{n+1} = A(u_n, v_n) \geqslant A\left(t_n v_n, \frac{1}{t_n} u_n\right) = A\left(\frac{t_n}{t^*} t^* v_n, \frac{t^*}{t_n}\frac{1}{t^*} u_n\right)$$
$$\geqslant \frac{t_n}{t^*} A\left(t^* v_n, \frac{1}{t^*} u_n\right) \geqslant \frac{t_n}{t^*}\varphi\big(t^*\big) A(v_n, u_n) = \frac{t_n}{t^*}\varphi\big(t^*\big) v_{n+1}.$$

By the definition of t_n, $t_{n+1} \geqslant \frac{t_n}{t^*} \cdot \varphi(t^*)$. Let $n \to \infty$, we get $t^* \geqslant \varphi(t^*) > t^*$, which is a contradiction. Thus, $\lim_{n\to\infty} t_n = 1$. For any natural number p we have

$$\theta \leqslant u_{n+p} - u_n \leqslant v_n - u_n \leqslant v_n - t_n v_n = (1 - t_n) v_n \leqslant (1 - t_n) v_0,$$
$$\theta \leqslant v_n - v_{n+p} \leqslant v_n - u_n \leqslant (1 - t_n) v_0.$$

Since the cone P is normal, we have

$$\|u_{n+p} - u_n\| \leqslant M(1 - t_n)\|v_0\| \to 0, \qquad \|v_n - v_{n+p}\| \leqslant M(1 - t_n)\|v_0\| \to 0 \quad (n \to \infty),$$

where M is the normality constant of P. So we can claim that $\{u_n\}$ and $\{v_n\}$ are Cauchy sequences. Because E is complete, there exist u^*, v^* such that $u_n \to u^*$, $v_n \to v^*$ as $n \to \infty$. By (2.4), we know that $u_n \leqslant u^* \leqslant v^* \leqslant v_n$ with $u^*, v^* \in P_h$ and $\theta \leqslant v^* - u^* \leqslant v_n - u_n \leqslant (1 - t_n) v_0$. Further

$$\big\|v^* - u^*\big\| \leqslant M(1 - t_n)\|v_0\| \to 0 \quad (n \to \infty),$$

and thus $u^* = v^*$. Let $x^* := u^* = v^*$ and then we obtain $u_{n+1} = A(u_n, v_n) \leqslant A(x^*, x^*) \leqslant A(v_n, u_n) = v_{n+1}$. Let $n \to \infty$, then we get $x^* = A(x^*, x^*)$. That is, x^* is a fixed point of A in P_h.

In the following, we prove that x^* is the unique fixed point of A in P_h. In fact, suppose $\bar{x}$ is a fixed point of A in P_h. Since $x^*, \bar{x} \in P_h$, there exist positive numbers $\bar{\mu}_1, \bar{\mu}_2, \bar{\lambda}_1, \bar{\lambda}_2 > 0$ such that

$$\bar{\mu}_1 h \leqslant x^* \leqslant \bar{\lambda}_1 h, \qquad \bar{\mu}_2 h \leqslant \bar{x} \leqslant \bar{\lambda}_2 h.$$

Then we obtain

$$\bar{x} \leqslant \bar{\lambda}_2 h = \frac{\bar{\lambda}_2}{\bar{\mu}_1}\bar{\mu}_1 h \leqslant \frac{\bar{\lambda}_2}{\bar{\mu}_1} x^*, \qquad \bar{x} \geqslant \bar{\mu}_2 h = \frac{\bar{\mu}_2}{\bar{\lambda}_1}\bar{\lambda}_1 h \geqslant \frac{\bar{\mu}_2}{\bar{\lambda}_1} x^*.$$

Let $e_1 = \sup\{t > 0 \mid t x^* \leqslant \bar{x} \leqslant t^{-1} x^*\}$. Evidently, $0 < e_1 \leqslant 1$, $e_1 x^* \leqslant \bar{x} \leqslant \frac{1}{e_1} x^*$. Next we prove $e_1 = 1$. If $0 < e_1 < 1$, then

$$\bar{x} = A(\bar{x}, \bar{x}) \geqslant A\left(e_1 x^*, \frac{1}{e_1} x^*\right) \geqslant \varphi(e_1) A\big(x^*, x^*\big) = \varphi(e_1) x^*.$$

Since $\varphi(e_1) > e_1$, this contradicts the definition of e_1. Hence $e_1 = 1$, and we get $\bar{x} = x^*$. Therefore, A has a unique fixed point x^* in P_h. Note that $[u_0, v_0] \subset P_h$, then we know that x^* is the unique fixed point of A in $[u_0, v_0]$.

Now we construct successively the sequences $x_n = A(x_{n-1}, y_{n-1})$, $y_n = A(y_{n-1}, x_{n-1})$, $n = 1, 2, \ldots,$ for any initial points $x_0, y_0 \in P_h$. Since $x_0, y_0 \in P_h$, we can choose small numbers $e_2, e_3 \in (0, 1)$ such that

$$e_2 h \leqslant x_0 \leqslant \frac{1}{e_2} h, \qquad e_3 h \leqslant y_0 \leqslant \frac{1}{e_3} h.$$

Let $e^* = \min\{e_2, e_3\}$. Then $e^* \in (0, 1)$ and

$$e^* h \leqslant x_0, \qquad y_0 \leqslant \frac{1}{e^*} h.$$

We can choose a sufficiently large positive integer m such that

$$\left[\frac{\varphi(e^*)}{e^*}\right]^m \geqslant \frac{1}{e^*}.$$

Put $\bar{u}_0 = e^{*m} h$, $\bar{v}_0 = \frac{1}{e^{*m}} h$. It is easy to see that $\bar{u}_0, \bar{v}_0 \in P_h$ and $\bar{u}_0 < x_0$, $y_0 < \bar{v}_0$. Let

$$\bar{u}_n = A(\bar{u}_{n-1}, \bar{v}_{n-1}), \qquad \bar{v}_n = A(\bar{v}_{n-1}, \bar{u}_{n-1}), \quad n = 1, 2, \ldots.$$

Similarly, it follows that there exists $y^* \in P_h$ such that $A(y^*, y^*) = y^*$, $\lim_{n\to\infty} \bar{u}_n = \lim_{n\to\infty} \bar{v}_n = y^*$. By the uniqueness of fixed points of operator A in P_h, we get $x^* = y^*$. And by induction, $\bar{u}_n \leqslant x_n$, $y_n \leqslant \bar{v}_n$, $n = 1, 2, \ldots.$ Since cone P is normal, we have $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = x^*$. □

Remark 2.1. Compared with the corresponding results in [46, Theorems 2.1, 2.2], we remove the conditions: there exist $u_0, v_0 \in P_h$ such that $u_0 \leqslant A(u_0, v_0) \leqslant A(v_0, u_0) \leqslant v_0$. If we suppose that operator $A : P_h \times P_h \to P_h$ or $A : \mathring{P} \times \mathring{P} \to \mathring{P}$ with P is a solid cone, then $A(h, h) \in P_h$ is automatically satisfied. When $\varphi(t) = t^\alpha$ with $\alpha \in (0, 1)$ for $t \in (0, 1)$, the following result in [26] turns out to be a special case of Theorem 2.1.

Corollary 2.2. *(See [26].) Let P be a normal, solid cone of E, and let $A : \mathring{P} \times \mathring{P} \to \mathring{P}$ be a mixed monotone operator; suppose that: there exists $\alpha \in (0, 1)$ such that*

$$A\big(tu, t^{-1}v\big) \geqslant t^\alpha A(u, v), \quad \forall u, v \in \mathring{P},\ t \in (0, 1).$$

Then operator A has a unique fixed point x^* in $\mathring{P}$. Moreover, for any initial $x_0, y_0 \in \mathring{P}$, constructing successively the sequences $x_n = A(x_{n-1}, y_{n-1})$, $y_n = A(y_{n-1}, x_{n-1})$, $n = 1, 2, \ldots,$ we have $\|x_n - x^*\| \to 0$ and $\|y_n - x^*\| \to 0$ as $n \to \infty$.

2.2. Eigenvalue problems

Motivated by the idea of work [72, Theorem 2.4], we study the nonlinear eigenvalue problem $A(x, x) = \lambda x$. The next theorem shows that the solution has some pleasant properties.

Theorem 2.3. *Assume that operator A satisfies the conditions of Theorem 2.1. Let x_λ $(\lambda > 0)$ denote the unique solution of nonlinear eigenvalue equation $A(x, x) = \lambda x$ in P_h. Then we have the following conclusions:*

- (R_1) *If $\varphi(t) > t^{\frac{1}{2}}$ for $t \in (0, 1)$, then x_λ is strictly decreasing in λ, that is, $0 < \lambda_1 < \lambda_2$ implies $x_{\lambda_1} > x_{\lambda_2}$;*
- (R_2) *If there exists $\beta \in (0, 1)$ such that $\varphi(t) \geqslant t^\beta$ for $t \in (0, 1)$, then x_λ is continuous in λ, that is, $\lambda \to \lambda_0$ $(\lambda_0 > 0)$ implies $\|x_\lambda - x_{\lambda_0}\| \to 0$;*
- (R_3) *If there exists $\beta \in (0, \frac{1}{2})$ such that $\varphi(t) \geqslant t^\beta$ for $t \in (0, 1)$, then $\lim_{\lambda\to\infty} \|x_\lambda\| = 0$, $\lim_{\lambda\to 0^+} \|x_\lambda\| = \infty$.*

Proof. Fix $\lambda > 0$ and by Lemma 2.1, $\frac{1}{\lambda}A : P_h \times P_h \to P_h$ is mixed monotone and satisfies

$$\left(\frac{1}{\lambda}A\right)\big(tx, t^{-1}y\big) = \frac{1}{\lambda}A\big(tx, t^{-1}y\big) \geqslant \frac{1}{\lambda}\varphi(t)A(x, y) = \varphi(t)\left(\frac{1}{\lambda}A\right)(x, y), \quad \forall x, y \in P_h,\ t \in (0, 1).$$

So it follows from Theorem 2.1 that $\frac{1}{\lambda}A$ has a unique fixed point x_λ in P_h. That is, $A(x_\lambda, x_\lambda) = \lambda x_\lambda$. For convenience of proof, we let

$$\alpha(t) = \frac{\ln\varphi(t)}{\ln t}, \quad \forall t \in (0, 1).$$

Then $\alpha(t) \in [0, 1)$ and $\varphi(t) = t^{\alpha(t)}$. Thus $A(tx, t^{-1}y) \geqslant t^{\alpha(t)}A(x, y)$, $\forall x, y \in P_h$, $t \in (0, 1)$.

(1) *Proof of* (R_1). Suppose $0 < \lambda_1 < \lambda_2$ and let $t_0 = \sup\{t > 0 \mid x_{\lambda_1} \geqslant tx_{\lambda_2},\ x_{\lambda_2} \geqslant tx_{\lambda_1}\}$, then we have $0 < t_0 < 1$ and

$$x_{\lambda_1} \geqslant t_0 x_{\lambda_2}, \qquad x_{\lambda_2} \geqslant t_0 x_{\lambda_1}. \tag{2.5}$$

By the mixed monotone properties of A,

$$\lambda_1 x_{\lambda_1} = A(x_{\lambda_1}, x_{\lambda_1}) \geqslant A\big(t_0 x_{\lambda_2}, {t_0}^{-1}x_{\lambda_2}\big) \geqslant {t_0}^{\alpha(t_0)}A(x_{\lambda_2}, x_{\lambda_2}) = {t_0}^{\alpha(t_0)}\lambda_2 x_{\lambda_2},$$
$$\lambda_2 x_{\lambda_2} = A(x_{\lambda_2}, x_{\lambda_2}) \geqslant A\big(t_0 x_{\lambda_1}, {t_0}^{-1}x_{\lambda_1}\big) \geqslant {t_0}^{\alpha(t_0)}A(x_{\lambda_1}, x_{\lambda_1}) = {t_0}^{\alpha(t_0)}\lambda_1 x_{\lambda_1}.$$

Further

$$x_{\lambda_1} \geqslant {\lambda_1}^{-1}\lambda_2 {t_0}^{\alpha(t_0)}x_{\lambda_2}, \qquad x_{\lambda_2} \geqslant {\lambda_2}^{-1}\lambda_1 {t_0}^{\alpha(t_0)}x_{\lambda_1}. \tag{2.6}$$

Noting that ${\lambda_1}^{-1}\lambda_2 {t_0}^{\alpha(t_0)} > t_0$, from the definition of t_0 and (2.6), we know that ${\lambda_2}^{-1}\lambda_1 {t_0}^{\alpha(t_0)} \leqslant t_0$, which in turn yields

$$t_0 \geqslant \left(\frac{\lambda_1}{\lambda_2}\right)^{\frac{1}{1-\alpha(t_0)}}. \tag{2.7}$$

Hence

$$x_{\lambda_1} \geqslant {\lambda_1}^{-1}\lambda_2\left(\frac{\lambda_1}{\lambda_2}\right)^{\frac{\alpha(t_0)}{1-\alpha(t_0)}}x_{\lambda_2} = \left(\frac{\lambda_2}{\lambda_1}\right)^{\frac{1-2\alpha(t_0)}{1-\alpha(t_0)}}x_{\lambda_2}. \tag{2.8}$$

Noting that $\varphi(t_0) > {t_0}^{\frac{1}{2}}$, we have $\alpha(t_0) < \frac{1}{2}$ and in consequence,

$$\left(\frac{\lambda_2}{\lambda_1}\right)^{\frac{1-2\alpha(t_0)}{1-\alpha(t_0)}} > 1.$$

Thus, $x_{\lambda_1} > x_{\lambda_2}$.

(2) *Proof of* (R_2). Since $\varphi(t) \geqslant t^\beta$ for $t \in (0,1)$, we have $\alpha(t) \leqslant \beta$ for $t \in (0,1)$. By (2.5), (2.7),

$$\left(\frac{\lambda_1}{\lambda_2}\right)^{\frac{1}{1-\beta}} x_{\lambda_2} \leqslant \left(\frac{\lambda_1}{\lambda_2}\right)^{\frac{1}{1-\alpha(t_0)}} x_{\lambda_2} \leqslant x_{\lambda_1} \leqslant \frac{1}{t_0} x_{\lambda_2} \leqslant \left(\frac{\lambda_2}{\lambda_1}\right)^{\frac{1}{1-\alpha(t_0)}} x_{\lambda_2} \leqslant \left(\frac{\lambda_2}{\lambda_1}\right)^{\frac{1}{1-\beta}} x_{\lambda_2}, \tag{2.9}$$

$$\left(\frac{\lambda_1}{\lambda_2}\right)^{\frac{1}{1-\beta}} x_{\lambda_1} \leqslant \left(\frac{\lambda_1}{\lambda_2}\right)^{\frac{1}{1-\alpha(t_0)}} x_{\lambda_1} \leqslant x_{\lambda_2} \leqslant \frac{1}{t_0} x_{\lambda_1} \leqslant \left(\frac{\lambda_2}{\lambda_1}\right)^{\frac{1}{1-\alpha(t_0)}} x_{\lambda_1} \leqslant \left(\frac{\lambda_2}{\lambda_1}\right)^{\frac{1}{1-\beta}} x_{\lambda_1}. \tag{2.10}$$

Further

$$\theta \leqslant x_{\lambda_1} - \left(\frac{\lambda_1}{\lambda_2}\right)^{\frac{1}{1-\beta}} x_{\lambda_2} \leqslant \left[\left(\frac{\lambda_2}{\lambda_1}\right)^{\frac{1}{1-\beta}} - \left(\frac{\lambda_1}{\lambda_2}\right)^{\frac{1}{1-\beta}}\right] x_{\lambda_2}.$$

Consequently, from the normality of cone P,

$$\begin{aligned} \|x_{\lambda_1} - x_{\lambda_2}\| &\leqslant \left\| x_{\lambda_1} - \left(\frac{\lambda_1}{\lambda_2}\right)^{\frac{1}{1-\beta}} x_{\lambda_2} \right\| + \left\| \left(\frac{\lambda_1}{\lambda_2}\right)^{\frac{1}{1-\beta}} x_{\lambda_2} - x_{\lambda_2} \right\| \\ &\leqslant M\left[\left(\frac{\lambda_2}{\lambda_1}\right)^{\frac{1}{1-\beta}} - \left(\frac{\lambda_1}{\lambda_2}\right)^{\frac{1}{1-\beta}}\right] \|x_{\lambda_2}\| + \left| \left(\frac{\lambda_1}{\lambda_2}\right)^{\frac{1}{1-\beta}} - 1 \right| \|x_{\lambda_2}\|, \end{aligned}$$

where M is the normality constant. Let $\lambda_1 \to {\lambda_2}^-$, we have $\|x_{\lambda_1} - x_{\lambda_2}\| \to 0$. Similarly, let $\lambda_2 \to {\lambda_1}^+$, from (2.10) we can also prove $\|x_{\lambda_2} - x_{\lambda_1}\| \to 0$. So the conclusion (R_2) holds.

(3) *Proof of* (R_3). Since $\varphi(t) \geqslant t^\beta$ for $t \in (0,1)$, we have $\alpha(t) \leqslant \beta < \frac{1}{2}$ for $t \in (0,1)$. Let $\lambda_1 = 1$, $\lambda_2 = \lambda$ in (2.8), then we have

$$x_1 \geqslant \lambda^{\frac{1-2\alpha(t_0)}{1-\alpha(t_0)}} x_\lambda \geqslant \lambda^{\frac{1-2\beta}{1-\beta}} x_\lambda, \quad \forall \lambda > 1.$$

Thus we can easily obtain

$$\|x_\lambda\| \leqslant \frac{M}{\lambda^{\frac{1-2\beta}{1-\beta}}} \|x_1\|, \quad \forall \lambda > 1,$$

where M is the normality constant. Let $\lambda \to \infty$, then $\|x_\lambda\| \to 0$. Similarly, let $\lambda_1 = \lambda$, $\lambda_2 = 1$ in (2.8), then

$$x_\lambda \geqslant \lambda^{-\frac{1-2\alpha(t_0)}{1-\alpha(t_0)}} x_1 \geqslant \lambda^{-\frac{1-2\beta}{1-\beta}} x_1, \quad \forall 0 < \lambda < 1.$$

Thus

$$\|x_\lambda\| \geqslant M^{-1} \lambda^{-\frac{1-2\beta}{1-\beta}} \|x_1\|, \quad \forall 0 < \lambda < 1,$$

where M is the normality constant. Let $\lambda \to 0^+$, then we have $\|x_\lambda\| \to \infty$. □

3. Local existence–uniqueness of positive solutions for nonlinear BVPs

In this section, we will apply Theorem 2.1 and Theorem 2.3 to study nonlinear BVPs which include the Neumann BVPs, three-point BVPs and nonlinear elliptic BVPs for the Lane–Emden–Fowler equations. And then we will obtain new results on the local existence–uniqueness of positive solutions for these problems, which are not the consequences of the corresponding fixed point theorems in [26,46,74].

3.1. Two-point BVPs

First we are interested in the local existence–uniqueness of positive solutions for the following nonlinear Neumann boundary value problems (NBVPs for short)

$$\begin{cases} -u''(t) + m^2 u(t) = \lambda f\big(t, u(t), u(t)\big), & 0 < t < 1, \\ u'(0) = u'(1) = 0, \end{cases} \tag{3.1}$$

and

$$\begin{cases} u''(t) + m^2 u(t) = \lambda f\big(t, u(t), u(t)\big), & 0 < t < 1, \\ u'(0) = u'(1) = 0, \end{cases} \tag{3.2}$$

where m is a positive constant, λ is a positive parameter, $f(t,u,v)$ is continuous.

It is well known that NBVPs for the ordinary differential equations and elliptic equations is an important kind of boundary value problems. During the last two decades, NBVPs have deserved the attention of many researchers [8,10–12,35,43, 60–62,70]. By using fixed point theorems in cone, in [8,35,60–62], the authors discussed the existence of positive solutions to ordinary differential equation NBVPs.

Recently, the authors [12] discussed second-order superlinear repulsive singular NBVPs by using a nonlinear alternative of Leray–Schauder and Krasnosel'skii's fixed point theorem on compression and expansion of cones, and obtained the existence of at least two positive solutions under reasonable conditions. In [43], the authors established the existence of sign-changing solutions and positive solutions for fourth-order NBVPs by using the fixed point index and the critical group. Besides the above mentioned methods, the method of upper and lower solutions is also used in the literature [10,11,70]. However, to the best of our knowledge, few papers can be found in the literature on the existence–uniqueness of positive solutions for the NBVPs (3.1) and (3.2) by mixed monotone method. The objective here is to fill this gap.

By a positive solution of (3.1) (or (3.2)) we understand a function $u(t) \in C^2[0,1]$ which is positive on $0 < t < 1$ and satisfies the differential equation and the boundary conditions in (3.1) (or (3.2)).

In the following we will work in the Banach space $C[0,1]$ and only the sup-norm is used. Set $P = \{x \in C[0,1] \mid x(t) \geqslant 0, t \in [0,1]\}$, the standard cone. It is easy to see that P is a normal cone of which the normality constant is 1. Let $G(t,s)$ be the Green function for the boundary value problem

$$\begin{cases} -u''(t) + m^2 u(t) = 0, & 0 < t < 1, \\ u'(0) = u'(1) = 0. \end{cases} \tag{3.3}$$

Then

$$G(t,s) = \frac{1}{\rho} \begin{cases} \psi(s)\psi(1-t), & 0 \leqslant s \leqslant t \leqslant 1, \\ \psi(t)\psi(1-s), & 0 \leqslant t \leqslant s \leqslant 1, \end{cases}$$

where $\rho = \frac{1}{2}m(e^m - e^{-m})$, $\psi(t) = \frac{1}{2}(e^{mt} + e^{-mt})$. It is obvious that $\psi(t)$ is increasing on $[0,1]$, and

$$0 < G(t,s) \leqslant G(t,t), \quad 0 \leqslant t, s \leqslant 1. \tag{3.4}$$

Lemma 3.1. *(See [62].) Let $G(t,s)$ be the Green function for the NBVP* (3.3)*. Then*

$$G(t,s) \geqslant C\psi(t)\psi(1-t)G(t_0,s), \quad t, t_0, s \in [0,1],$$

where $C = 1/\psi^2(1)$.

Theorem 3.1. *Assume that the function $f(t,u,v)$ satisfies (H_1), (H_2) and*

(H_3) for any $t \in [0,1]$, $f(t,a,b) > 0$, where

$$a = \frac{1}{4}\big(e^m + e^{-m} + 2\big), \qquad b = \frac{1}{2}\big(e^m + e^{-m}\big).$$

Then the NBVP (3.1) *has a unique positive solution u_λ^* in P_h, where $h(t) = \psi(t)\psi(1-t)$, $t \in [0,1]$. Moreover, if $\varphi(t) > t^{\frac{1}{2}}$ for $t \in (0,1)$, then u_λ^* is strictly decreasing in λ, that is, $0 < \lambda_1 < \lambda_2$ implies $u_{\lambda_1}^* \geqslant u_{\lambda_2}^*$, $u_{\lambda_1}^* \neq u_{\lambda_2}^*$. If there exists $\beta \in (0,1)$ such that $\varphi(t) \geqslant t^\beta$ for $t \in (0,1)$, then u_λ^* is continuous in λ, that is, $\lambda \to \lambda_0$ $(\lambda_0 > 0)$ implies $\|u_\lambda^* - u_{\lambda_0}^*\| \to 0$. If there exists $\beta \in (0, \frac{1}{2})$ such that $\varphi(t) \geqslant t^\beta$ for $t \in (0,1)$, then $\lim_{\lambda\to\infty} \|u_\lambda^*\| = 0$, $\lim_{\lambda\to 0^+} \|u_\lambda^*\| = \infty$.*

Remark 3.1. It is easy to check that $a = \min\{h(t){:}\ t \in [0,1]\}$, $b = \max\{h(t){:}\ t \in [0,1]\}$, where a, b are given as in (H_3).

Proof of Theorem 3.1. It is well known that u is a solution of the NBVP (3.1) if and only if

$$u(t) = \lambda \int_0^1 G(t,s) f\big(s, u(s), u(s)\big)\, ds,$$

where $G(t,s)$ is the Green function for the NBVP (3.3). For any $u, v \in P$, we define

$$A_\lambda(u,v)(t) = \lambda \int_0^1 G(t,s) f\big(s, u(s), v(s)\big)\, ds.$$

From (H_1), it is easy to check that $A_\lambda : P \times P \to P$. From (H_2), we know that $A_\lambda : P \times P \to P$ is a mixed monotone operator. Next we show that A_λ satisfies the conditions in Theorem 2.1. From (H_2), for any $\gamma \in (0,1)$ and $u, v \in P$, we obtain

$$\begin{aligned} A_\lambda\big(\gamma u, \gamma^{-1} v\big)(t) &= \lambda \int_0^1 G(t,s) f\big(s, \gamma u(s), \gamma^{-1} v(s)\big)\, ds \\ &\geqslant \lambda \int_0^1 G(t,s)\varphi(\gamma) f\big(s, u(s), v(s)\big)\, ds \\ &= \varphi(\gamma) A_\lambda(u,v)(t), \quad t \in [0,1]. \end{aligned}$$

That is, $A_\lambda(\gamma u, \gamma^{-1} v) \geqslant \varphi(\gamma) A_\lambda(u,v)$, $\forall u, v \in P$, $\gamma \in (0,1)$. So the condition (A_2) in Theorem 2.1 is satisfied. On the one hand, it follows from (H_2), (H_3), Lemma 3.1 and Remark 3.1 that

$$\begin{aligned} A_\lambda(h,h)(t) &= \lambda \int_0^1 G(t,s) f\big(s, h(s), h(s)\big)\, ds \\ &\geqslant \lambda \int_0^1 C\psi(t)\psi(1-t) G(t_0, s) f(s,a,b)\, ds \\ &= \lambda C h(t) \int_0^1 G(t_0,s) f(s,a,b)\, ds, \quad t \in [0,1]. \end{aligned}$$

On the other hand, from (3.4), (H_2) and Remark 3.1, we obtain

$$\begin{aligned} A_\lambda(h,h)(t) &= \lambda \int_0^1 G(t,s) f\big(s, h(s), h(s)\big)\, ds \\ &\leqslant \lambda \int_0^1 G(t,t) f(s,b,a)\, ds \\ &= \lambda \frac{1}{\rho} h(t) \int_0^1 f(s,b,a)\, ds, \quad t \in [0,1]. \end{aligned}$$

Let

$$r_1 = \min_{t\in[0,1]} f(t,a,b), \qquad r_2 = \max_{t\in[0,1]} f(t,b,a).$$

Then $0 < r_1 \leqslant r_2$. Consequently,

$$A_\lambda(h,h)(t) \geqslant r_1 \lambda C \int_0^1 G(t_0,s)\, ds \cdot h(t), \qquad A_\lambda(h,h)(t) \leqslant r_2 \lambda \frac{1}{\rho} h(t), \quad t \in [0,1].$$

Note that

$$\int_0^1 G(t_0,s)\, ds = \frac{1}{\rho} \int_0^{t_0} \psi(s)\psi(1-t_0)\, ds + \frac{1}{\rho} \int_{t_0}^1 \psi(t_0)\psi(1-s)\, ds = \frac{1}{m^2},$$

then we have $r_1 \lambda C \int_0^1 G(t_0,s)\, ds > 0$. Hence $A_\lambda(h,h) \in P_h$, the condition (A_1) in Theorem 2.1 is satisfied. Therefore, by Theorem 2.1, there exists a unique $u_\lambda^* \in P_h$ such that $A_\lambda(u_\lambda^*, u_\lambda^*) = u_\lambda^*$. It is easy to check that u_λ^* is a unique positive solution of the NBVP (3.1) for given $\lambda > 0$. Moreover, if $\varphi(t) > t^{\frac{1}{2}}$ for $t \in (0,1)$, then Theorem 2.3(R_1) means that u_λ^* is strictly decreasing in λ, that is, $0 < \lambda_1 < \lambda_2$ implies $u_{\lambda_1}^* \geqslant u_{\lambda_2}^*$, $u_{\lambda_1}^* \neq u_{\lambda_2}^*$. If there exists $\beta \in (0,1)$ such that $\varphi(t) \geqslant t^\beta$ for $t \in (0,1)$, then Theorem 2.3(R_2) means that u_λ^* is continuous in λ, that is, $\lambda \to \lambda_0$ $(\lambda_0 > 0)$ implies $\|u_\lambda^* - u_{\lambda_0}^*\| \to 0$. If there exists $\beta \in (0, \frac{1}{2})$ such that $\varphi(t) \geqslant t^\beta$ for $t \in (0,1)$, then Theorem 2.3(R_3) means $\lim_{\lambda\to\infty} \|u_\lambda^*\| = 0$, $\lim_{\lambda\to 0^+} \|u_\lambda^*\| = \infty$. □

Example 3.1. Consider the following NBVP

$$\begin{cases} -u''(t)+(\ln 2)^2u(t)=\lambda\big[u^{\frac{1}{3}}(t)+u^{-\frac{1}{4}}(t)\big], & 0<t<1,\\ u'(0)=u'(1)=0, \end{cases} \tag{3.5}$$

where λ is a positive parameter. In this example, we let $m=\ln 2$, $f(t,x,y):=f(x,y)=x^{\frac{1}{3}}+y^{-\frac{1}{4}}$. After a simple calculation, we get $a=\frac{9}{8}$, $b=\frac{5}{4}$ and

$$h(t)=\frac{5}{8}+\frac{1}{4}\big(2^{1-2t}+2^{2t-1}\big),\quad t\in[0,1].$$

Evidently, $f(x,y)$ is increasing in x for $y\geqslant 0$, decreasing in y for $x\geqslant 0$.

$$f(a,b)=\left(\frac{9}{8}\right)^{\frac{1}{3}}+\left(\frac{5}{4}\right)^{-\frac{1}{4}}>0.$$

Moreover, set $\varphi(\gamma)=\gamma^{\frac{5}{12}}$, $\gamma\in(0,1)$. Then

$$f\big(\gamma x,\gamma^{-1}y\big)=\gamma^{\frac{1}{3}}x^{\frac{1}{3}}+\gamma^{\frac{1}{4}}y^{-\frac{1}{4}}\geqslant\varphi(\gamma)f(x,y),\quad x,y\geqslant 0.$$

Hence, all the conditions of Theorem 3.1 are satisfied. An application of Theorem 3.1 implies that the NBVP (3.5) has a unique positive solution u_λ^* in P_h. Moreover, note that $\varphi(t)>t^{\frac{1}{2}}$ for $t\in(0,1)$, then from Theorem 3.1, u_λ^* is strictly decreasing in λ, that is, $0<\lambda_1<\lambda_2$ implies $u_{\lambda_1}^*\geqslant u_{\lambda_2}^*$, $u_{\lambda_1}^*\neq u_{\lambda_2}^*$. Taking $\beta\in[\frac{5}{12},\frac{1}{2})$ and applying Theorem 3.1, we know that u_λ^* is continuous in λ and $\lim_{\lambda\to\infty}\|u_\lambda^*\|=0$, $\lim_{\lambda\to 0^+}\|u_\lambda^*\|=\infty$. $\square$

In the following, using the same technique, we study general NBVP (3.2) with $m\in(0,\frac{\pi}{2})$. Let $G(t,s)$ be the Green function for the boundary value problem

$$\begin{cases} u''(t)+m^2u(t)=0, & 0<t<1,\\ u'(0)=u'(1)=0. \end{cases} \tag{3.6}$$

Then

$$G(t,s)=\frac{1}{m\sin m}\begin{cases} \cos ms\cos m(1-t), & 0\leqslant s\leqslant t\leqslant 1,\\ \cos mt\cos m(1-s), & 0\leqslant t\leqslant s\leqslant 1. \end{cases}$$

It is obvious that $\cos mt$ is decreasing on $[0,1]$, and

$$G(t,s)\geqslant G(t,t),\quad 0\leqslant t,s\leqslant 1. \tag{3.7}$$

Lemma 3.2. *Let $G(t,s)$ be the Green function for the NBVP* (3.6)*. Then*

$$G(t,s)\leqslant C\cdot\cos mt\cos m(1-t)\cdot G(t_0,s),\quad t,t_0,s\in[0,1],$$

where $C=1/\cos^2 m$.

Proof. When $t,t_0\leqslant s$,

$$\begin{aligned}\frac{G(t,s)}{G(t_0,s)}&=\frac{\cos m(1-s)\cos mt}{\cos m(1-s)\cos mt_0}=\frac{\cos m(1-t)\cos mt}{\cos m(1-t)\cos mt_0}\\ &\leqslant\frac{1}{\cos^2 m}\cos m(1-t)\cos mt=C\cos m(1-t)\cos mt.\end{aligned}$$

If $t\leqslant s\leqslant t_0$,

$$\begin{aligned}\frac{G(t,s)}{G(t_0,s)}&=\frac{\cos m(1-s)\cos mt}{\cos m(1-t_0)\cos ms}=\frac{\cos m(1-t)\cos mt}{\cos m(1-t)\cos ms}\cdot\frac{\cos m(1-s)}{\cos m(1-t_0)}\\ &\leqslant\frac{1}{\cos^2 m}\cos m(1-t)\cos mt=C\cos m(1-t)\cos mt.\end{aligned}$$

If $t_0\leqslant s\leqslant t$,

$$\begin{aligned}\frac{G(t,s)}{G(t_0,s)}&=\frac{\cos m(1-t)\cos ms}{\cos m(1-s)\cos mt_0}=\frac{\cos m(1-t)\cos mt}{\cos m(1-s)\cos mt}\cdot\frac{\cos ms}{\cos mt_0}\\ &\leqslant\frac{1}{\cos^2 m}\cos m(1-t)\cos mt=C\cos m(1-t)\cos mt.\end{aligned}$$

For $s \leqslant t, t_0$,

$$\frac{G(t,s)}{G(t_0,s)} = \frac{\cos m(1-t)\cos ms}{\cos m(1-t_0)\cos ms} = \frac{\cos m(1-t)\cos mt}{\cos m(1-t_0)\cos mt}$$
$$\leqslant \frac{1}{\cos^2 m}\cos m(1-t)\cos mt = C\cos m(1-t)\cos mt.$$

Therefore,

$$G(t,s) \leqslant C\cdot\cos m(1-t)\cos mt\cdot G(t_0,s), \quad t, t_0, s \in [0,1].$$

This completes the proof. □

Theorem 3.2. *Assume (H_1), (H_2) hold and $f(t,\cos^2 m, 1) > 0$ for any $t \in [0,1]$. Then the NBVP (3.2) has a unique positive solution u_λ^* in P_h, where $h(t) = \cos m(1-t)\cos mt$, $t \in [0,1]$. Moreover, if $\varphi(t) > t^{\frac{1}{2}}$ for $t \in (0,1)$, then u_λ^* is strictly decreasing in λ, that is, $0 < \lambda_1 < \lambda_2$ implies $u_{\lambda_1}^* \geqslant u_{\lambda_2}^*$, $u_{\lambda_1}^* \neq u_{\lambda_2}^*$. If there exists $\beta \in (0,1)$ such that $\varphi(t) \geqslant t^\beta$ for $t \in (0,1)$, then u_λ^* is continuous in λ, that is, $\lambda \to \lambda_0$ $(\lambda_0 > 0)$ implies $\|u_\lambda^* - u_{\lambda_0}^*\| \to 0$. If there exists $\beta \in (0,\frac{1}{2})$ such that $\varphi(t) \geqslant t^\beta$ for $t \in (0,1)$, then $\lim_{\lambda\to\infty}\|u_\lambda^*\| = 0$, $\lim_{\lambda\to 0^+}\|u_\lambda^*\| = \infty$.*

Remark 3.2. It is easy to check that $cos^2 m \leqslant h(t) \leqslant 1$ for $\forall t \in [0,1]$.

Proof of Theorem 3.2. It is well known that u is a solution of the NBVP (3.2) if and only if

$$u(t) = \lambda\int_0^1 G(t,s) f\bigl(s, u(s), u(s)\bigr)\,ds,$$

where $G(t,s)$ is the Green function for the NBVP (3.6). For any $u, v \in P$, we define

$$A_\lambda(u,v)(t) = \lambda\int_0^1 G(t,s) f\bigl(s, u(s), v(s)\bigr)\,ds.$$

Similarly to the proof of Theorem 3.1, we know that $A_\lambda : P \times P \to P$ is a mixed monotone operator and satisfies the condition (A_2) in Theorem 2.1. That is,

$$A_\lambda\bigl(\gamma u, \gamma^{-1}v\bigr) \geqslant \varphi(\gamma)A_\lambda(u,v), \quad \forall u, v \in P,\ \gamma \in (0,1).$$

It follows from condition (H_2), Lemma 3.2 and Remark 3.2 that

$$\begin{aligned} A_\lambda(h,h)(t) &= \lambda\int_0^1 G(t,s) f\bigl(s, h(s), h(s)\bigr)\,ds \\ &\leqslant \lambda\int_0^1 C\cdot\cos mt\cos m(1-t)\cdot G(t_0,s) f\bigl(s, 1, \cos^2 m\bigr)\,ds \\ &= \lambda C h(t)\int_0^1 G(t_0,s) f\bigl(s, 1, \cos^2 m\bigr)\,ds, \quad t \in [0,1]. \end{aligned}$$

From (3.7), (H_2) and Remark 3.2, we obtain

$$\begin{aligned} A_\lambda(h,h)(t) &= \lambda\int_0^1 G(t,s) f\bigl(s, h(s), h(s)\bigr)\,ds \\ &\geqslant \lambda\int_0^1 G(t,t) f\bigl(s, \cos^2 m, 1\bigr)\,ds \\ &= \lambda\frac{1}{m\sin m}h(t)\int_0^1 f\bigl(s, \cos^2 m, 1\bigr)\,ds, \quad t \in [0,1]. \end{aligned}$$

Let

$$r_1 = \min_{t\in[0,1]} f\big(t, \cos^2 m, 1\big), \qquad r_2 = \max_{t\in[0,1]} f\big(t, 1, \cos^2 m\big).$$

Then $0 < r_1 \leqslant r_2$. Consequently,

$$A_\lambda(h,h)(t) \leqslant r_2\lambda C \int_0^1 G(t_0,s)\,ds \cdot h(t), \qquad A_\lambda(h,h)(t) \geqslant r_1\lambda \frac{1}{m\sin m} h(t), \quad t\in[0,1].$$

Note that

$$\int_0^1 G(t_0,s)\,ds = \frac{1}{m\sin m}\int_0^{t_0} \cos m(1-t_0)\cos ms\,ds + \frac{1}{m\sin m}\int_{t_0}^1 \cos m(1-s)\cos mt_0\,ds = \frac{1}{m^2},$$

then we have $r_2\lambda C\int_0^1 G(t_0,s)\,ds > 0$. Hence $A_\lambda(h,h)\in P_h$, the condition (A_1) in Theorem 2.1 is satisfied. Therefore, by Theorem 2.1, there exists a unique $u_\lambda^*\in P_h$ such that $A_\lambda(u_\lambda^*,u_\lambda^*) = u_\lambda^*$. It is easy to check that u_λ^* is a unique positive solution of the NBVP (3.2) for given $\lambda>0$. Moreover, if $\varphi(t) > t^{\frac{1}{2}}$ for $t\in(0,1)$, then Theorem 2.3(R_1) means that u_λ^* is strictly decreasing in λ, that is, $0<\lambda_1<\lambda_2$ implies $u_{\lambda_1}^* \geqslant u_{\lambda_2}^*$, $u_{\lambda_1}^* \neq u_{\lambda_2}^*$. If there exists $\beta\in(0,1)$ such that $\varphi(t)\geqslant t^\beta$ for $t\in(0,1)$, then Theorem 2.3(R_2) means that u_λ^* is continuous in λ, that is, $\lambda\to\lambda_0$ $(\lambda_0>0)$ implies $\|u_\lambda^* - u_{\lambda_0}^*\|\to 0$. If there exists $\beta\in(0,\frac{1}{2})$ such that $\varphi(t)\geqslant t^\beta$ for $t\in(0,1)$, then Theorem 2.3(R_3) means $\lim_{\lambda\to\infty}\|u_\lambda^*\| = 0$, $\lim_{\lambda\to 0^+}\|u_\lambda^*\| = \infty$. □

Example 3.2. Consider the following NBVP

$$\begin{cases} u''(t) + \left(\dfrac{\pi}{3}\right)^2 u(t) = \lambda\big[u^{\frac{1}{3}}(t) + u^{-\frac{1}{4}}(t)\big], & 0<t<1,\\ u'(0) = u'(1) = 0, \end{cases} \tag{3.8}$$

where λ is a positive parameter. In this example, we let $m = \frac{\pi}{3}$, $f(t,x,y) := f(x,y) = x^{\frac{1}{3}} + y^{-\frac{1}{4}}$. Then $m\in(0,\frac{\pi}{2})$ and

$$h(t) = \cos\frac{\pi}{3}t\cos\frac{\pi}{3}(1-t), \quad t\in[0,1].$$

Evidently, $f(x,y)$ is increasing in x for $y\geqslant 0$, decreasing in y for $x\geqslant 0$.

$$f\left(\cos^2\frac{\pi}{3}, 1\right) = \left(\frac{1}{4}\right)^{\frac{1}{3}} + 1 > 0.$$

Moreover, set $\varphi(\gamma) = \gamma^{\frac{5}{12}}$, $\gamma\in(0,1)$. Then

$$f\big(\gamma x, \gamma^{-1}y\big) = \gamma^{\frac{1}{3}}x^{\frac{1}{3}} + \gamma^{\frac{1}{4}}y^{-\frac{1}{4}} \geqslant \varphi(\gamma)f(x,y), \quad x,y\geqslant 0.$$

Hence, all the conditions of Theorem 3.2 are satisfied. An application of Theorem 3.2 implies that the NBVP (3.8) has a unique positive solution u_λ^* in P_h. Moreover, note that $\varphi(t) > t^{\frac{1}{2}}$ for $t\in(0,1)$, then from Theorem 3.2, u_λ^* is strictly decreasing in λ, that is, $0<\lambda_1<\lambda_2$ implies $u_{\lambda_1}^*\geqslant u_{\lambda_2}^*$, $u_{\lambda_1}^*\neq u_{\lambda_2}^*$. Taking $\beta\in[\frac{5}{12},\frac{1}{2})$ and applying Theorem 3.2, we know that u_λ^* is continuous in λ and $\lim_{\lambda\to\infty}\|u_\lambda^*\| = 0$, $\lim_{\lambda\to 0^+}\|u_\lambda^*\| = \infty$. □

Next we consider the following two-point BVPs:

$$\begin{cases} u'' + \lambda f(t,u,u) = 0, & t\in(0,1),\\ u(0) = u(1) = 0, \end{cases} \tag{ς_1}$$

$$\begin{cases} u'' + \lambda f(t,u,u) = 0, & t\in(0,1),\\ u(0) = u'(1) = 0, \end{cases} \tag{ς_2}$$

$$\begin{cases} u'' + \lambda f(t,u,u) = 0, & t\in(0,1),\\ u'(0) = u(1) = 0, \end{cases} \tag{ς_3}$$

$$\begin{cases} u''' + \lambda f(t,u,u) = 0, & t\in(0,1),\\ u(0) = u'(0) = u''(1) = 0, \end{cases} \tag{ς_4}$$

where λ is a positive parameter and $f(t,u,v)$ is continuous.

It is well known that u is the solution of the problem (ς_i), $i=1,2,3,4$, if and only if

$$u(t)=\lambda\int_0^1 G_i(t,s)f\big(s,u(s),u(s)\big)\,ds,\quad t\in[0,1],\ i=1,2,3,4,$$

where

$$G_1(t,s)=\begin{cases}t(1-s), & 0\leqslant t\leqslant s\leqslant 1,\\ s(1-t), & 0\leqslant s\leqslant t\leqslant 1,\end{cases}\qquad G_2(t,s)=\begin{cases}t, & 0\leqslant t\leqslant s\leqslant 1,\\ s, & 0\leqslant s\leqslant t\leqslant 1,\end{cases}$$
$$G_3(t,s)=\begin{cases}1-s, & 0\leqslant t\leqslant s\leqslant 1,\\ 1-t, & 0\leqslant s\leqslant t\leqslant 1,\end{cases}\qquad G_4(t,s)=\begin{cases}\frac12 t^2, & 0\leqslant t\leqslant s\leqslant 1,\\ \frac12 t^2-\frac12(t-s)^2, & 0\leqslant s\leqslant t\leqslant 1.\end{cases}$$

Theorem 3.3. *Assume that the function $f(t,u,v)$ satisfies (H_1), (H_2) and*

(H_4) *for any $t\in[0,1]$, $f(t,0,b_i)>0$, $i=1,2,3,4$, where*

$$b_1=\frac18,\qquad b_2=b_3=b_4=\frac12.$$

Then the BVP (ς_i), $i=1,2,3,4$, has a unique positive solution u_λ^* in P_{h_i}, where

$$h_1(t)=\frac12 t(1-t),\qquad h_2(t)=\frac12 t(2-t),\qquad h_3(t)=\frac12\big(1-t^2\big),\qquad h_4(t)=\frac12 t^2,\quad t\in[0,1].$$

Moreover, if $\varphi(t)>t^{\frac12}$ for $t\in(0,1)$, then u_λ^* is strictly decreasing in λ, that is, $0<\lambda_1<\lambda_2$ implies $u_{\lambda_1}^*\geqslant u_{\lambda_2}^*$, $u_{\lambda_1}^*\neq u_{\lambda_2}^*$. If there exists $\beta\in(0,1)$ such that $\varphi(t)\geqslant t^\beta$ for $t\in(0,1)$, then u_λ^* is continuous in λ, that is, $\lambda\to\lambda_0$ $(\lambda_0>0)$ implies $\|u_\lambda^*-u_{\lambda_0}^*\|\to 0$. If there exists $\beta\in(0,\frac12)$ such that $\varphi(t)\geqslant t^\beta$ for $t\in(0,1)$, then $\lim_{\lambda\to\infty}\|u_\lambda^*\|=0$, $\lim_{\lambda\to 0^+}\|u_\lambda^*\|=\infty$.

Sketch of the proof. For any $u,v\in P$, we define

$$A_\lambda(u,v)(t)=\lambda\int_0^1 G_i(t,s)f\big(s,u(s),v(s)\big)\,ds,\quad i=1,2,3,4.$$

Similarly to the proof of Theorem 3.1, we know that $A_\lambda: P\times P\to P$ is a mixed monotone operator and satisfies the condition (A_2) in Theorem 2.1. Using the same argument as in Lemma 3.2, we can easily prove that $G_4(t,s)\geqslant h_4(t)G_4(t_0,s)$, $t,s\in[0,1]$, $t_0\in(0,1]$. Moreover, note that $h_i(t)=\int_0^1 G_i(t,s)\,ds$, $i=1,2,3$, and $G_4(t,s)\leqslant h_4(t)$, $t,s\in[0,1]$; then we can prove that the condition (A_1) in Theorem 2.1 is satisfied. Therefore, the conclusion follows from Theorems 2.1 and 2.3. □

3.2. Three-point BVPs

Three-point BVPs for differential equations or difference equations arise in a variety of different areas of applied mathematics and physics. The study of multi-point BVPs for linear second-order ordinary differential equations was initiated by Il'in and Moiseev [33,34]. Motivated by the study of Il'in and Moiseev, Gupta [28] studied certain three-point BVPs for nonlinear ordinary differential equations. Since then, more general nonlinear three-point BVPs have been studied by many authors with much of the attention given to positive solutions. For a small sample of such work, we refer the reader to works by Ahmad and Nieto [6], Gupta and Trofimchuk [29], Karaca [36], Ma [51], Raffoul [56], Xu [68], Yang, Zhai and Yan [69] and Zhai [71]. However, few papers have been reported on the existence–uniqueness for three-point BVPs. In this subsection we consider the following two classes of three-point BVPs for second-order differential equation:

$$\begin{cases}u''+\lambda f(t,u,u)=0, & t\in(0,1),\\ u(0)=0, & u(1)-\beta u(\eta)=0,\end{cases}\tag{3.9}$$

where $\eta\in(0,1)$, $\beta>0$, $1-\beta\eta>0$;

$$\begin{cases}u''+\lambda f(t,u,u)=0, & t\in(0,1),\\ u'(0)=0, & u(1)-\beta u(\eta)=0,\end{cases}\tag{3.10}$$

where $\eta\in(0,1)$, $0<\beta<1$; λ is a positive parameter and $f(t,u,v)$ is continuous.

Different from the above mentioned works, here we will use Theorems 2.1 and 2.3 to show the existence–uniqueness of positive solutions for the problems (3.9) and (3.10).

By a positive solution of (3.9) or (3.10) we understand a function $u(t)$ which is positive on $0 < t < 1$ and satisfies differential equation and boundary conditions.

We also work in the space $C[0,1]$. $P = \{u \in C[0,1] \mid u(t) \geqslant 0,\ t \in [0,1]\}$, the standard cone.

Theorem 3.4. *Assume* (H_1), (H_2) *hold and*

(H_5) *for any* $t \in [0,1]$, $f(t, 0, h(t_0)) > 0$, *where*

$$t_0 = \frac{1-\beta\eta^2}{2(1-\beta\eta)}, \qquad h(t) = -\frac{1}{2}t^2 + \frac{1-\beta\eta^2}{2(1-\beta\eta)}t, \quad t \in [0,1].$$

Then the three-point BVP (3.9) *has a unique positive solution* u_λ^* *in* P_h. *Moreover, if* $\varphi(t) > t^{\frac{1}{2}}$ *for* $t \in (0,1)$, *then* u_λ^* *is strictly decreasing in* λ, *that is,* $0 < \lambda_1 < \lambda_2$ *implies* $u_{\lambda_1}^* \geqslant u_{\lambda_2}^*$, $u_{\lambda_1}^* \neq u_{\lambda_2}^*$. *If there exists* $\beta \in (0,1)$ *such that* $\varphi(t) \geqslant t^\beta$ *for* $t \in (0,1)$, *then* u_λ^* *is continuous in* λ, *that is,* $\lambda \to \lambda_0$ ($\lambda_0 > 0$) *implies* $\|u_\lambda^* - u_{\lambda_0}^*\| \to 0$. *If there exists* $\beta \in (0, \frac{1}{2})$ *such that* $\varphi(t) \geqslant t^\beta$ *for* $t \in (0,1)$, *then* $\lim_{\lambda\to\infty} \|u_\lambda^*\| = 0$, $\lim_{\lambda\to 0^+} \|u_\lambda^*\| = \infty$.

Remark 3.3. Function $h(t)$ satisfies $h(0) = 0$, $\beta h(\eta) = h(1)$, $h''(t) \equiv -1$ and for $t \in [0,1]$,

$$h(t) = -\int_0^t (t-s)\,ds - \frac{\beta t}{1-\beta\eta}\int_0^\eta (\eta - s)\,ds + \frac{t}{1-\beta\eta}\int_0^1 (1-s)\,ds.$$

It is easy to prove that $h(t) \geqslant 0$, $h(t) \not\equiv 0$ and $0 \leqslant h(t) \leqslant h(t_0)$ for $t \in [0,1]$.

Proof of Theorem 3.4. It is well known that u is the solution of the problem (3.9) if and only if $u = A_\lambda(u,u)$, where

$$\begin{aligned} A_\lambda(u,v)(t) = &-\int_0^t (t-s)\lambda f\big(s, u(s), v(s)\big)\,ds \\ &- \frac{\beta t}{1-\beta\eta}\int_0^\eta (\eta - s)\lambda f\big(s, u(s), v(s)\big)\,ds + \frac{t}{1-\beta\eta}\int_0^1 (1-s)\lambda f\big(s, u(s), v(s)\big)\,ds. \end{aligned}$$

Next we show that A_λ is mixed monotone and satisfies (A_1), (A_2). To illuminate this, we divide into two cases: (i) for any $t \in [0,\eta]$, we have

$$\begin{aligned} A_\lambda(u,v)(t) = &-\int_0^t (t-s)\lambda f\big(s, u(s), v(s)\big)\,ds \\ &- \frac{\beta t}{1-\beta\eta}\int_0^\eta (\eta - s)\lambda f\big(s, u(s), v(s)\big)\,ds + \frac{t}{1-\beta\eta}\int_0^1 (1-s)\lambda f\big(s, u(s), v(s)\big)\,ds \\ = &\ \frac{t}{1-\beta\eta}\int_\eta^1 (1-s)\lambda f\big(s, u(s), v(s)\big)\,ds \\ &+ \frac{t}{1-\beta\eta}\int_t^\eta (1 - s - \beta\eta + \beta s)\lambda f\big(s, u(s), v(s)\big)\,ds \\ &+ \frac{1}{1-\beta\eta}\int_0^t (s - ts + \beta ts - \beta s\eta)\lambda f\big(s, u(s), v(s)\big)\,ds, \end{aligned}$$

(ii) for any $t \in (\eta, 1]$, we have

$$A_\lambda(u,v)(t) = -\int_0^t (t-s)\lambda f\big(s, u(s), v(s)\big)\,ds$$

$$
\begin{aligned}
&- \frac{\beta t}{1-\beta\eta}\int_0^\eta (\eta - s)\lambda f\big(s,u(s),v(s)\big)\,ds + \frac{t}{1-\beta\eta}\int_0^1 (1-s)\lambda f\big(s,u(s),v(s)\big)\,ds \\
&= \frac{t}{1-\beta\eta}\int_t^1 (1-s)\lambda f\big(s,u(s),v(s)\big)\,ds \\
&+ \frac{1}{1-\beta\eta}\int_\eta^t (s - st + \beta\eta t - \beta\eta s)\lambda f\big(s,u(s),v(s)\big)\,ds \\
&+ \frac{1}{1-\beta\eta}\int_0^\eta (s - st + s\beta t - s\beta\eta)\lambda f\big(s,u(s),v(s)\big)\,ds.
\end{aligned}
$$

For case (i), we can easily get $1 - s - \beta\eta + \beta s \geqslant 0$ for $s \in [t,\eta]$ and $s - ts + \beta ts - \beta s\eta \geqslant 0$ for $s \in [0,t]$. For case (ii), we can easily get $s - st + \beta\eta t - \beta\eta s \geqslant 0$ for $s \in [\eta,t]$ and $s - st + s\beta t - s\beta\eta \geqslant 0$ for $s \in [0,\eta]$. Note that $1 - \beta\eta > 0$ and from (H_1), we obtain $A_\lambda(u,v)(t) \geqslant 0$, for $u, v \in P$, $t \in [0,1]$. Further, also from the above two cases (i), (ii) and that $f(t,x,y)$ is increasing in x, decreasing in y, we can easily prove that $A_\lambda : P \times P \to P$ is mixed monotone. For any $\gamma \in (0,1)$ and $u, v \in P$, we have

$$
\begin{aligned}
A_\lambda\big(\gamma u, \gamma^{-1}v\big)(t) = &-\int_0^t (t-s)\lambda f\big(s,\gamma u(s),\gamma^{-1}v(s)\big)\,ds \\
&- \frac{\beta t}{1-\beta\eta}\int_0^\eta (\eta - s)\lambda f\big(s,\gamma u(s),\gamma^{-1}v(s)\big)\,ds \\
&+ \frac{t}{1-\beta\eta}\int_0^1 (1-s)\lambda f\big(s,\gamma u(s),\gamma^{-1}v(s)\big)\,ds.
\end{aligned}
$$

It follows from the above two cases (i), (ii) and (H_2) that

$$
\begin{aligned}
A_\lambda\big(\gamma u, \gamma^{-1}v\big)(t) \geqslant \varphi(\gamma)\Bigg[&-\int_0^t (t-s)\lambda f\big(s,u(s),v(s)\big)\,ds - \frac{\beta t}{1-\beta\eta}\int_0^\eta (\eta - s)\lambda f\big(s,u(s),v(s)\big)\,ds \\
&+ \frac{t}{1-\beta\eta}\int_0^1 (1-s)\lambda f\big(s,u(s),v(s)\big)\,ds\Bigg] = \varphi(\gamma)A_\lambda(u,v)(t).
\end{aligned}
$$

In the following we show that $A_\lambda(h,h) \in P_h$. Let

$$
r_1 = \min_{t\in[0,1]} f\big(t,0,h(t_0)\big), \qquad r_2 = \max_{t\in[0,1]} f\big(t,h(t_0),0\big),
$$

then $0 < r_1 \leqslant r_2$. From the above two cases (i), (ii), we have

$$
\begin{aligned}
A_\lambda(h,h)(t) = &-\int_0^t (t-s)\lambda f\big(s,h(s),h(s)\big)\,ds \\
&- \frac{\beta t}{1-\beta\eta}\int_0^\eta (\eta - s)\lambda f\big(s,h(s),h(s)\big)\,ds + \frac{t}{1-\beta\eta}\int_0^1 (1-s)\lambda f\big(s,h(s),h(s)\big)\,ds \\
\geqslant r_1\lambda\Bigg[&-\int_0^t (t-s)\,ds - \frac{\beta t}{1-\beta\eta}\int_0^\eta (\eta - s)\,ds + \frac{t}{1-\beta\eta}\int_0^1 (1-s)\,ds\Bigg] = r_1\lambda h(t),
\end{aligned}
$$

and

$$\begin{aligned}A_\lambda(h,h)(t) &= -\int_0^t (t-s)\lambda f\big(s,h(s),h(s)\big)\,ds \\ &\quad - \frac{\beta t}{1-\beta\eta}\int_0^\eta (\eta-s)\lambda f\big(s,h(s),h(s)\big)\,ds + \frac{t}{1-\beta\eta}\int_0^1 (1-s)\lambda f\big(s,h(s),h(s)\big)\,ds \\ &\leqslant r_2\lambda\Bigg[-\int_0^t (t-s)\,ds - \frac{\beta t}{1-\beta\eta}\int_0^\eta (\eta-s)\,ds + \frac{t}{1-\beta\eta}\int_0^1 (1-s)\,ds\Bigg] = r_2\lambda h(t).\end{aligned}$$

Hence $A_\lambda(h,h) \in P_h$. Therefore, the conclusion follows from Theorems 2.1 and 2.3. □

Theorem 3.5. *Assume* (H_1), (H_2) *hold and*

(H_6) *for any* $t \in [0,1]$, $f(t,h(1),h(0)) > 0$, *where*

$$h(t) = -\frac{1}{2}t^2 + \frac{1-\beta\eta^2}{2(1-\beta)}, \quad t \in [0,1].$$

Then the three-point BVP (3.10) *has a unique positive solution* u_λ^* *in* P_h. *Moreover, if* $\varphi(t) > t^{\frac{1}{2}}$ *for* $t \in (0,1)$, *then* u_λ^* *is strictly decreasing in* λ, *that is,* $0 < \lambda_1 < \lambda_2$ *implies* $u_{\lambda_1}^* \geqslant u_{\lambda_2}^*$, $u_{\lambda_1}^* \neq u_{\lambda_2}^*$. *If there exists* $\beta \in (0,1)$ *such that* $\varphi(t) \geqslant t^\beta$ *for* $t \in (0,1)$, *then* u_λ^* *is continuous in* λ, *that is,* $\lambda \to \lambda_0$ $(\lambda_0 > 0)$ *implies* $\|u_\lambda^* - u_{\lambda_0}^*\| \to 0$. *If there exists* $\beta \in (0,\frac{1}{2})$ *such that* $\varphi(t) \geqslant t^\beta$ *for* $t \in (0,1)$, *then* $\lim_{\lambda\to\infty}\|u_\lambda^*\| = 0$, $\lim_{\lambda\to 0^+}\|u_\lambda^*\| = \infty$.

Remark 3.4. Function $h(t)$ satisfies $h'(0) = 0$, $\beta h(\eta) = h(1)$, $h''(t) \equiv -1$ and for $t \in [0,1]$,

$$h(t) = \frac{1}{1-\beta}\int_0^1 (1-s)\,ds - \frac{\beta}{1-\beta}\int_0^\eta (\eta-s)\,ds - \int_0^t (t-s)\,ds.$$

It is easy to prove that $h(t) \geqslant 0$, $h(t) \not\equiv 0$ and $h(1) \leqslant h(t) \leqslant h(0)$ for $t \in [0,1]$.

Proof of Theorem 3.5. It is easy to see that u is the solution of the problem (3.10) if and only if u is a solution of the operator equation

$$\begin{aligned}A_\lambda(u,v)(t) &= \frac{1}{1-\beta}\int_0^1 (1-s)\lambda f\big(s,u(s),v(s)\big)\,ds - \frac{\beta}{1-\beta}\int_0^\eta (\eta-s)\lambda f\big(s,u(s),v(s)\big)\,ds \\ &\quad - \int_0^t (t-s)\lambda f\big(s,u(s),v(s)\big)\,ds.\end{aligned}$$

Next we show that A_λ is mixed monotone and satisfies (A_1), (A_2). Firstly, we also divide into two cases: (i) for any $t \in [0,\eta]$, we have

$$\begin{aligned}A_\lambda(u,v)(t) &= \frac{1}{1-\beta}\int_0^1 (1-s)\lambda f\big(s,u(s),v(s)\big)\,ds - \frac{\beta}{1-\beta}\int_0^\eta (\eta-s)\lambda f\big(s,u(s),v(s)\big)\,ds \\ &\quad - \int_0^t (t-s)\lambda f\big(s,u(s),v(s)\big)\,ds \\ &= \frac{1}{1-\beta}\Bigg[\int_0^t (1-s)\lambda f\big(s,u(s),v(s)\big)\,ds + \int_t^\eta (1-s)\lambda f\big(s,u(s),v(s)\big)\,ds \\ &\quad + \int_\eta^1 (1-s)\lambda f\big(s,u(s),v(s)\big)\,ds\Bigg]\end{aligned}$$

$$
\begin{aligned}
&\quad - \frac{\beta}{1-\beta}\left[\int_0^t (\eta - t)\lambda f\big(s, u(s), v(s)\big)\,ds + \int_t^\eta (\eta - s)\lambda f\big(s, u(s), v(s)\big)\,ds\right] \\
&\quad - \int_0^t (t-s)\lambda f\big(s, u(s), v(s)\big)\,ds \\
&= \frac{1}{1-\beta}\left[\int_0^t (1 - t - \beta\eta + \beta t)\lambda f\big(s, u(s), v(s)\big)\,ds + \int_t^\eta \big(1 - s - \beta(\eta - s)\big)\lambda f\big(s, u(s), v(s)\big)\,ds\right. \\
&\quad \left. + \int_\eta^1 (1-s)\lambda f\big(s, u(s), v(s)\big)\,ds\right],
\end{aligned}
$$

(ii) for any $t \in (\eta, 1]$, we have

$$
\begin{aligned}
A_\lambda(u,v)(t) &= \frac{1}{1-\beta}\int_0^1 (1-s)\lambda f\big(s, u(s), v(s)\big)\,ds - \frac{\beta}{1-\beta}\int_0^\eta (\eta - s)\lambda f\big(s, u(s), v(s)\big)\,ds \\
&\quad - \int_0^t (t-s)\lambda f\big(s, u(s), v(s)\big)\,ds \\
&= \frac{1}{1-\beta}\left[\int_0^\eta (1-s)\lambda f\big(s, u(s), v(s)\big)\,ds + \int_\eta^t (1-s)\lambda f\big(s, u(s), v(s)\big)\,ds\right. \\
&\quad \left. + \int_t^1 (1-s)\lambda f\big(s, u(s), v(s)\big)\,ds\right] - \frac{\beta}{1-\beta}\int_0^\eta (\eta - s)\lambda f\big(s, u(s), v(s)\big)\,ds \\
&\quad - \left[\int_0^\eta (t-s)\lambda f\big(s, u(s), v(s)\big)\,ds + \int_\eta^t (t-s)\lambda f\big(s, u(s), v(s)\big)\,ds\right] \\
&= \frac{1}{1-\beta}\left[\int_0^\eta (1 - t - \beta\eta + \beta t)\lambda f\big(s, u(s), v(s)\big)\,ds + \int_\eta^t \big(1 - t + \beta(t - s)\big)\lambda f\big(s, u(s), v(s)\big)\,ds\right. \\
&\quad \left. + \int_t^1 (1-s)\lambda f\big(s, u(s), v(s)\big)\,ds\right].
\end{aligned}
$$

From $\eta \in (0,1)$, $0 < \beta < 1$, the condition (H_1) and the above two cases (i), (ii), we have $A_\lambda(u,v)(t) \geqslant 0$ for $u, v \in P$, $t \in [0,1]$. Secondly, from (H_2), we know that $A_\lambda : P \times P \to P$ is mixed monotone. Finally, using the same reasoning as in Theorem 3.4, we have $A_\lambda(h,h) \in P_h$. The conclusion follows from Theorems 2.1 and 2.3. □

Example 3.3. Consider the following three-point BVP

$$
\begin{cases}
u'' + \lambda\big[u^\alpha + u^{-\tau}\big] = 0, \quad t \in (0,1), \\
u(0) = 0, \quad u(1) - \dfrac{1}{2}u\left(\dfrac{1}{4}\right) = 0,
\end{cases}
\tag{3.11}
$$

where $\alpha, \tau \in (0,1)$ and λ is a positive parameter.

In this example, $\eta = \frac{1}{4}$, $\beta = \frac{1}{2}$. Evidently, $1 - \beta\eta > 0$. Set $f(t,u,v) = u^\alpha + v^{-\tau}$ and $\varphi(\gamma) = \gamma^{\min\{\alpha,\tau\}}$, then $f(t,u,v)$ satisfies (H_1) and (H_2). In addition,

$$
t_0 = \frac{31}{56}, \qquad h(t) = -\frac{1}{2}t^2 + \frac{1 - \beta\eta^2}{2(1 - \beta\eta)}t = -\frac{1}{2}t^2 + \frac{31}{56}t \geqslant 0, \quad t \in [0,1].
$$

For $t \in [0,1]$, we have $f(t,0,h(t_0)) = [h(t_0)]^{-\tau} > 0$. Hence, all the conditions of Theorem 3.4 are satisfied. An application of Theorem 3.4 implies that the BVP (3.11) has a unique positive solution in P_h. $\square$

Example 3.4. Consider the following three-point BVP

$$\begin{cases} u'' + \lambda\big[u^{\alpha} + u^{-\tau}\big] = 0, & t \in (0,1), \\ u'(0) = 0, \quad u(1) - \dfrac{1}{2}u\left(\dfrac{1}{4}\right) = 0, \end{cases} \tag{3.12}$$

where $\alpha, \tau \in (0,1)$ and λ is a positive parameter.

In this example, $\eta = \frac{1}{4}$, $\beta = \frac{1}{2}$. Set $f(t,u,v) = u^{\alpha} + v^{-\tau}$ and $\varphi(\gamma) = \gamma^{\min\{\alpha,\tau\}}$. In addition,

$$h(t) = -\frac{1}{2}t^2 + \frac{1-\beta\eta^2}{2(1-\beta)} = -\frac{1}{2}t^2 + \frac{31}{32} \geqslant 0, \quad t \in [0,1].$$

For $t \in [0,1]$, we have $f(t,h(1),h(0)) > 0$. An application of Theorem 3.5 implies that the BVP (3.12) has a unique positive solution in P_h.

3.3. Nonlinear elliptic BVPs for the Lane–Emden–Fowler equations

Let Ω be a bounded domain with smooth boundary in $\mathbf{R}^N$ $(N \geqslant 1)$. Consider the following singular Dirichlet problem for the Lane–Emden–Fowler equation:

$$\begin{cases} -\Delta u = \lambda f(x,u,u), & x \in \Omega, \\ u(x) > 0, & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega, \end{cases} \tag{3.13}$$

where $\lambda > 0$ and the nonlinear term $f(x,u,u)$ is allowed to be singular on $\partial\Omega$.

The problem (3.13) arises in the study of non-Newtonian fluids, boundary layer phenomena for viscous fluids, chemical heterogeneous catalysts, as well as in the theory of heat conduction in electrically materials (see [15,21,22,55,59]). The theory of singular BVPs has become an important area of investigation in the past three decades, see [2–5,13,15–17,21–24,31, 32,40,41,48,52,55,58,64,67,73] and references therein. Among these singular elliptic boundary value problems for partial differential equations, papers [13,15,16,22,24,31,32,41,52,58,73] established some existence and nonexistence results, a unique positive solution by means of sub-supersolutions and various techniques related to the maximum principle for elliptic equations. For one-dimensional case, the corresponding singular boundary problems for second-order ordinary differential equations have been studied extensively in the literature (see for instance [2–5,48] and references therein). However, to our knowledge, the results on the existence–uniqueness of positive solutions for the general singular elliptic equation are still very few. The purpose here is to establish the existence–uniqueness of positive solutions to the singular Dirichlet problem for the Lane–Emden–Fowler equation (3.13). Different from the works mentioned above, we will use Theorem 2.1 and 2.3 to show the existence–uniqueness of positive solutions for the problem (3.13).

Throughout this subsection, denote by $W^{k,l}(\Omega)$ the Sobolev space (see [1]), where $l > 1$ and k is a nonnegative integer. And denote by h_1 the first eigenfunction of the following eigenvalue problem $-\Delta\varphi = \lambda\varphi$ in Ω, and $\varphi|_{\partial\Omega} = 0$. For convenience, we assume that $h_1(x) \geqslant 0$ in $\bar{\Omega}$. Moreover, it is well known that (see for instance [67]) there exist two positive constants C_2, C_3 such that the first eigenvalue function satisfies

$$0 < C_2 \leqslant h_1(x)\big[d(x)\big]^{-1} \leqslant C_3, \quad x \in \Omega, \tag{3.14}$$

where $d(x) = dist(x, \partial\Omega)$.

Lemma 3.3. *(See [9].) Let Ω be a bounded domain with smooth boundary in $\mathbf{R}^N$ $(N \geqslant 1)$. If the operator $-\Delta + k$ is coercive and $u \in L^1_{loc}(\Omega)$ satisfies*

$$\begin{cases} -\Delta u + ku \geqslant 0, & x \in \Omega, \\ u(x) \geqslant 0, & x \in \bar{\Omega}, \end{cases}$$

then either $u(x) \equiv 0$ or $u(x) \geqslant C_0 d(x)$, $x \in \Omega$, where $d(x) = dist(x, \partial\Omega)$ and C_0 is a positive constant depending only upon N, Ω and k.

The above result is originally due to G. Stampacchia, which plays an important role in the proof of the following main result.

Lemma 3.4 *(From the proof of Theorem 3.1 in [42]). Let Ω be a bounded domain with smooth boundary in $\mathbf{R}^N$ $(N \geqslant 1)$. If $w \in W^{2,l}(\Omega)$ and $w(x) = 0$ for $x \in \partial\Omega$, then there exists a constant $M_1 > 0$ such that*

$$\left|w(x)\right| \leqslant M_1 h_1(x), \quad x \in \Omega,$$

where M_1 depends only upon N and Ω.

Theorem 3.6. *Assume that $(H_1)'$, $(H_2)'$ hold and*

(H_7) *$f(x,u,v)$ is Hölder continuous in the variable x with the Hölder exponent $\gamma \in (0,1)$ for each $u, v \in \mathbf{R}^{++}$ and is continuous in the variables u, v for each $x \in \Omega$;*
(H_8) *$f(x,u,v)$ satisfies the condition of integrability, i.e.,*

$$\int_\Omega f\big(x, h_1(x), h_1(x)\big)^l \, dx < +\infty \quad \text{for some } l > N.$$

Then the problem (3.13) has a unique positive solution $u_\lambda^* \in C^{1,\beta}(\bar{\Omega})$ with respect to $\lambda > 0$, where $\beta = 1 - \frac{N}{l}$. Moreover, if $\varphi(t) > t^{\frac{1}{2}}$ for $t \in (0,1)$, then u_λ^* is strictly decreasing in λ, that is, $0 < \lambda_1 < \lambda_2$ implies $u_{\lambda_1}^* \geqslant u_{\lambda_2}^*$, $u_{\lambda_1}^* \neq u_{\lambda_2}^*$. If there exists $\beta^* \in (0,1)$ such that $\varphi(t) \geqslant t^{\beta^*}$ for $t \in (0,1)$, then u_λ^* is continuous in λ, that is, $\lambda \to \lambda_0$ $(\lambda_0 > 0)$ implies $\|u_\lambda^* - u_{\lambda_0}^*\| \to 0$. If there exists $\beta^* \in (0, \frac{1}{2})$ such that $\varphi(t) \geqslant t^{\beta^*}$ for $t \in (0,1)$, then $\lim_{\lambda\to\infty} \|u_\lambda^*\| = 0$, $\lim_{\lambda\to 0^+} \|u_\lambda^*\| = \infty$.

Remark 3.5. Compared with the corresponding result in [42, Theorem 3.1], the above result is very general. Some examples of the functions which satisfy the conditions $(H_1)'$, $(H_2)'$, (H_7), (H_8) are:

(1) $f(x,u,v) = g(x)[r(u) + \zeta(v)]$, where $r : \mathbf{R}^{++} \to \mathbf{R}^{++}$ is increasing, $\zeta : \mathbf{R}^{++} \to \mathbf{R}^{++}$ is decreasing, $g(x) \geqslant 0$, and $g(x) \in C^\gamma(\bar{\Omega})$ with $\gamma \in (0,1)$. r, ζ satisfy $\int_\Omega [r(h_1(x)) + \zeta(h_1(x))]^l \, dx < +\infty$ and for any $t \in (0,1)$, there exist constants $\varphi_1(t), \varphi_2(t) \in (t,1]$ such that $r(tu) \geqslant \varphi_1(t) r(u)$, $\zeta(tu) \geqslant \varphi_2(t)\zeta(u)$, $u \in \mathbf{R}^{++}$. Here we take $\varphi(t) = \min\{\varphi_1(t), \varphi_2(t)\}$, $t \in (0,1)$.
(2) $f(x,u,v) = a(x)[u^p + v^{-\tau}]$, where $p, \tau \in (0,1)$ and a is a Hölder continuous function in Ω such that $c_1 d(x)^q \leqslant a(x) \leqslant c_2 d(x)^q$ in Ω, here $c_1, c_2 > 0$, q is a real number and $d(x) = dist(x, \partial\Omega)$. Moreover, if $N(p+q) > -1$, then $\int_\Omega [{h_1}^p(x)]^l \, dx < +\infty$; if $0 < \tau < \frac{1}{N}$, then $\int_\Omega [{h_1}^{-\tau}(x)]^l \, dx < +\infty$, where $l > N$. Here we take $\varphi(t) = \min\{t^p, t^\tau\}$, $t \in (0,1)$.

Proof of Theorem 3.6. For the sake of convenience, set $E = C(\bar{\Omega})$, the Banach space of continuous functions on $\bar{\Omega}$ with the norm $\|u\| = \max\{|u(x)|\colon x \in \bar{\Omega}\}$. Set $P = \{u \in C(\bar{\Omega}) \mid u(x) \geqslant 0,\ x \in \bar{\Omega}\}$, the standard cone. It is clear that P is a normal cone in E and the normality constant is 1.

Firstly, we show that, for any $u, v \in P_{h_1}$, the following linear elliptic boundary value problem

$$\begin{cases} -\Delta w = \lambda f(x,u,v), & x \in \Omega, \\ w(x) > 0, & x \in \Omega, \\ w(x) = 0, & x \in \partial\Omega, \end{cases} \tag{3.15}$$

admits a unique strong solution. Since $u, v \in P_{h_1}$, we can choose sufficiently small numbers $r_u, r_v \in (0,1)$ such that

$$r_u h_1(x) \leqslant u(x) \leqslant \frac{1}{r_u} h_1(x), \qquad r_v h_1(x) \leqslant v(x) \leqslant \frac{1}{r_v} h_1(x), \quad x \in \bar{\Omega}.$$

Let $r_0 = \min\{r_u, r_v\}$. Then from $(H_2)'$, there exists $\varphi(r_0) \in (r_0, 1]$ such that

$$f\big(x, u(x), v(x)\big) \geqslant f\left(x, r_0 h_1(x), \frac{1}{r_0} h_1(x)\right) \geqslant \varphi(r_0) f\big(x, h_1(x), h_1(x)\big), \quad x \in \Omega,$$
$$f\big(x, u(x), v(x)\big) \leqslant f\left(x, \frac{1}{r_0} h_1(x), r_0 h_1(x)\right) \leqslant \frac{1}{\varphi(r_0)} f\big(x, h_1(x), h_1(x)\big), \quad x \in \Omega.$$

Thus we get by applying the integrability condition (H_8) that

$$\int_\Omega \left[f\big(x, u(x), v(x)\big)\right]^l dx < +\infty,$$

namely, $f(x,u,v) \in L^l(\Omega)$. By the classical theory of linear elliptic equations (see [39]), the problem (3.15) admits a unique strong solution $w_{u,v} \in W^{2,l}(\Omega) \cap W_0^{1,l}(\Omega)$. Recall that $l > N$. Using the Sobolev imbedding theory, $w_{u,v} \in C^{1,\beta}(\bar{\Omega})$ with $\beta = 1 - \frac{N}{l}$. Now we define an operator $A_\lambda : P_{h_1} \times P_{h_1} \to E$ by

$$A_\lambda(u,v)(x) = w_{u,v}(x), \quad u,v \in P_{h_1},$$

where $w_{u,v}$ is the unique strong solution of (3.15) for $u \in P_{h_1}$. Evidently, $A_\lambda : P_{h_1} \times P_{h_1} \to P$. Next we prove that $A_\lambda(h_1,h_1) \in P_{h_1}$. Suppose that ϕ is the solution of (3.15) with $u = v = h_1$, then $A_\lambda(h_1,h_1) = \phi \in C^{1,\beta}(\bar{\Omega})$. Then from Lemma 3.4, there exists a positive constant C_ϕ such that

$$\phi(x) \leqslant C_\phi h_1(x), \quad x \in \bar{\Omega}.$$

Note that $f(x,h_1(x),h_1(x)) \geqslant 0$. By the maximal principle, $\phi(x) \geqslant 0$. Since $\phi(x) > 0$ for $x \in \Omega$, an application of Lemma 3.3 implies that

$$\phi(x) \geqslant C_0 d(x), \quad x \in \bar{\Omega}. \tag{3.16}$$

Combining (3.14) and (3.16), there exists a positive constant c_ϕ such that

$$\phi(x) \geqslant c_\phi h_1(x), \quad x \in \bar{\Omega}.$$

Hence, $\phi = A_\lambda(h_1,h_1) \in P_{h_1}$. From $(H_2)'$ and the comparison principle, we can easily prove that $A_\lambda : P_{h_1} \times P_{h_1} \to P$ is mixed monotone.

Secondly, we prove that A_λ satisfies (A_2). For any $u,v \in P_{h_1}$ and $t \in (0,1)$, we have

$$\begin{cases} -\Delta A_\lambda\big(tu,t^{-1}v\big) = \lambda f\big(x,tu,t^{-1}v\big), & x \in \Omega, \\ A_\lambda\big(tu,t^{-1}v\big)(x) = 0, & x \in \partial\Omega, \end{cases}$$

and

$$\begin{cases} -\Delta \varphi(t) A_\lambda(u,v) = \lambda \varphi(t) f(x,u,v), & x \in \Omega, \\ \varphi(t) A_\lambda(u,v)(x) = 0, & x \in \partial\Omega. \end{cases}$$

From $(H_2)'$ we get $f(x,tu(x),t^{-1}v(x)) - \varphi(t) f(x,u(x),v(x)) \geqslant 0$ for any $x \in \bar{\Omega}$. Therefore,

$$\begin{cases} -\Delta\big(A_\lambda\big(tu,t^{-1}v\big) - \varphi(t) A_\lambda(u,v)\big) \geqslant 0, & x \in \Omega, \\ A_\lambda\big(tu,t^{-1}v\big)(x) - \varphi(t) A_\lambda(u,v)(x) = 0, & x \in \partial\Omega. \end{cases}$$

Using the comparison principle again, we can obtain $A_\lambda(tu,t^{-1}v) \geqslant \varphi(t) A_\lambda(u,v)$ immediately. Finally, using Theorem 2.1, operator A_λ has a unique fixed point u_λ^* in P_{h_1}, i.e., $A_\lambda(u_\lambda^*,u_\lambda^*) = u_\lambda^*$. This implies that the problem (3.13) admits a unique solution $u_\lambda^* \in P_{h_1}$. By the theory of the linear elliptic equation, for fixed $u = v = u_\lambda^*$, the problem (3.15) admits a unique solution $\bar{u_\lambda^*} \in W^{2,l}(\Omega) \cap W_0^{1,l}(\Omega)$, and hence $\bar{u_\lambda^*} \in C^{1,\beta}(\Omega)$. Recalling the uniqueness of the solution of (3.13), one can see that $\bar{u_\lambda^*} = u_\lambda^*$. Thus the problem (3.13) has a unique classical solution $u_\lambda^* \in C^{1,\beta}(\bar{\Omega})$. Moreover, by using Theorem 2.3 and the theory of the linear elliptic equation, if $\varphi(t) > t^{\frac{1}{2}}$ for $t \in (0,1)$, then u_λ^* is strictly decreasing in λ, that is, $0 < \lambda_1 < \lambda_2$ implies $u_{\lambda_1}^* \geqslant u_{\lambda_2}^*$, $u_{\lambda_1}^* \neq u_{\lambda_2}^*$. If there exists $\beta^* \in (0,1)$ such that $\varphi(t) \geqslant t^{\beta^*}$ for $t \in (0,1)$, then u_λ^* is continuous in λ, that is, $\lambda \to \lambda_0$ $(\lambda_0 > 0)$ implies $\|u_\lambda^* - u_{\lambda_0}^*\| \to 0$. If there exists $\beta^* \in (0,\frac{1}{2})$ such that $\varphi(t) \geqslant t^{\beta^*}$ for $t \in (0,1)$, then $\lim_{\lambda\to\infty} \|u_\lambda^*\| = 0$, $\lim_{\lambda\to 0^+} \|u_\lambda^*\| = \infty$. □

Remark 3.6. The method used here is new to the literature and so is the existence–uniqueness result to the singular Dirichlet problem for the Lane–Emden–Fowler equation. This is also the main motivation for the study of (3.13) in the present work.

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