Available online at www.sciencedirect.com

SCIENCE DIRECT

ELSEVIER

Journal of Functional Analysis 216 (2004) 172–190

JOURNAL OF Functional Analysis

http://www.elsevier.com/locate/jfa

Asymptotics for a resonance-counting function for potential scattering on cylinders ☆

T. Christiansen

Department of Mathematics, University of Missouri, Math Sciences Building, Columbia, MO 65211 USA

Received 3 October 2003; accepted 19 February 2004

Communicated by Richard. B. Melrose

Abstract

We study certain resonance-counting functions for potential scattering on infinite cylinders or half-cylinders. Under certain conditions on the potential, we obtain asymptotics of the counting functions, with an explicit formula for the constant appearing in the leading term.

Keywords: Resonances; Infinite cylinders; Potential scattering

1. Introduction

We study potential scattering on infinite cylinders and half-cylinders. In particular, we give some sharp upper bounds and some asymptotics for resonance-counting functions in this setting.

Let $X = (-\infty, \infty) \times Y$, or $[0, \infty) \times Y$, where Y is a smooth, compact, connected manifold, with or without boundary. We consider the product metric

$$(dx)^2 + g_Y,$$

where g_Y is a smooth metric on Y. Let Δ be the Laplacian on X, with Dirichlet or Neumann boundary conditions if X has a boundary. We consider operators $\Delta + V$, where $V \in L^\infty_{\mathrm{comp}}(X; \mathbb{C})$.

☆ Partially supported by NSF Grant DMS 0088922.
E-mail address: tjc@math.missouri.edu.

0022-1236/$ - see front matter
doi:10.1016/j.jfa.2004.02.008

Let Δ_Y be the Laplacian on Y, with boundary conditions if necessary, and let $\{\sigma_j^2\}$, $\sigma_0^2 \leqslant \sigma_1^2 \leqslant \sigma_2^2 \leqslant \cdots$ be the set of all eigenvalues of Δ_Y, repeated according to their multiplicity, and let $v_0^2 < v_1^2 < v_2^2 < \cdots$ be the *distinct* eigenvalues of Δ_Y. Then the resolvent of the Laplacian Δ on X, or of $\Delta + V$, for $V \in L^\infty_{\mathrm{comp}}(X)$, has a meromorphic continuation to the Riemann surface $\hat{Z}$ on which $r_j(z) = (z - v_j^2)^{1/2}$ is a single-valued function for all j [10,12]. Thus the resonances, poles of the meromorphic continuation of the resolvent, are points in $\hat{Z}$. In many settings, resonances correspond to waves which eventually decay. Additionally, they are in many ways analogous to eigenvalues. Because of this, they have been widely studied—see [15,17,18] for an introduction to resonances and for further references.

Here, we study a simple case of scattering on manifolds with infinite cylindrical ends. The spectral and scattering theory of such manifolds exhibits some characteristics one expects both from one-dimensional scattering and from n-dimensional spectral theory (if $\dim X = n$). The resonance-counting functions we consider here demonstrate the one-dimensional nature of the scattering. Evidence of the n-dimensional nature can be seen, for example, in the Weyl asymptotics or in the maximal rate of growth of the eigenvalue-counting function [5,13]. It also appears in some resonance-counting functions, e.g. [3].

For $z \in \mathbb{C} \backslash [v_0^2, \infty)$, $R_V(z) = (\Delta + V - z)^{-1}$ is bounded on $L^2(X)$ except, perhaps, for a (perhaps infinite, if V is complex-valued) set of points corresponding to eigenvalues. Considered as a map from $L^2_{\mathrm{comp}}(X)$ to $H^2_{\mathrm{loc}}(X)$, R_V has a meromorphic continuation to the Riemann surface $\hat{Z}$ described earlier. Let $r_j(z) = (z - v_j^2)^{1/2}$ and let $\tilde{r}_k(z) = r_j(z)$ if $\sigma_k^2 = v_j^2$. We use the convention that $\operatorname{Im} r_j(z) > 0$ for all j in the region in which $R_V(z)$ is bounded on $L^2(X)$.

Let

$$P_x : \mathbb{R}_x \times Y_y \ni (x, y) \mapsto x \in \mathbb{R}$$

and for $V \in L^\infty(\mathbb{R} \times Y)$, let $\mathrm{conv}_x(\mathrm{supp}(V))$ be the convex hull of $P_x(\mathrm{supp}(V))$ (cf. [16]).

Theorem 1.1. *Let $X = (-\infty, \infty) \times Y$ and let $V \in L^\infty_{\mathrm{comp}}(X; \mathbb{C})$. Fix a sheet of $\hat{Z}$, and suppose that $\operatorname{Im} r_{j_0}(z) < 0$ on this sheet. Then, there is a constant $c_{V,\mathscr{E}} \geqslant 0$ such that for any $\alpha > 0$,*

$$\#\{z_k : z_k \text{ is a pole of } R_V(z) \text{ on this sheet},$$

$$|r_{j_0}(z_k)| < r,\ \operatorname{Im} r_{j_0}(z_k) < -\alpha\} = c_{V,\mathscr{E}}\, r + o_\alpha(r).$$

The constant $c_{V,\mathscr{E}}$ depends on the potential V and the sheet (indicated by $\mathscr{E}$). Moreover, if $\mathrm{conv}_x(\mathrm{supp}(V)) = [-\beta, \gamma]$, then

$$c_{V,\mathscr{E}} \leqslant \frac{2}{\pi}(\gamma + \beta)\#\{l : \operatorname{Im} \tilde{r}_l(z) < 0 \text{ when } z \text{ lies on this sheet}\}.$$

Here, as everywhere, we count resonances with multiplicities. The error term $o_\alpha(r)$ depends on V and on the sheet as well as on α, of course.

We remark that this bound on the constant $c_{V,\mathscr{E}}$ is sharp, as can easily be seen by considering a potential that depends only on x, and using the results of [16] or [9] for potential scattering on the line.

Although Theorem 1.1 gives, in some sense, asymptotics of a resonance-counting function, it does not give meaningful lower bounds on the size of $c_{V,\mathscr{E}}$. In some settings we are able to actually determine $c_{V,\mathscr{E}}$, but we need some additional conditions on V.

Let $\{\phi_j\}$ be an orthonormal set of eigenfunctions of Δ_Y associated with σ_j^2. By translating if necessary, we can, in the case of the full cylinder, arrange that for some $b \in \mathbb{R}$, the support of V is contained in $[-b, b] \times Y$, but is not contained in the product of any smaller interval with Y.

Theorem 1.2. *Let* $X = (-\infty, \infty) \times Y$ *and suppose that* $\operatorname{conv}_x(\operatorname{supp}(V)) = [-b, b]$. *Restrict ourselves to a sheet of* $\hat{Z}$ *with* $\operatorname{Im} r_j(z) < 0$ *if and only if* $j = j_0$. *Suppose that* $v_{j_0}^2$ *is a simple eigenvalue of* Δ_Y, *with* $v_{j_0}^2 = \sigma_{l_0}^2$, *and that*

$$C|V_{l_0 l_0}(x)| = C\left|\int_Y V(x,y)|\phi_{l_0}(y)|^2 d\,\mathrm{vol}_Y\right| \geqslant |V(x,y)|, \text{ for } |x-b|<\varepsilon,\ |x+b|<\varepsilon \tag{1}$$

for some $C, \varepsilon > 0$. *Then, for any* $\alpha > 0$,

$$\#\{z_k : z_k \text{ is a pole of } R_V(z) \text{ on this sheet}, |r_{j_0}(z_k)| < r,\ \operatorname{Im} r_{j_0}(z_k) < -\alpha\}$$
$$= \frac{4}{\pi} br + o_\alpha(r).$$

In Section 4, we give an example of a nontrivial complex-valued potential for which (1) is not satisfied and for which the conclusion of the theorem does not hold. In fact, this particular potential has no resonances away from the ramification points of $\widehat{Z}$ so that $c_{V,\mathscr{E}} = 0$ for all sheets. This given an example of some behaviour which is even asymptotically truly different from that demonstrated by scattering by the family of potentials $V(x)$. Moreover, this means that potential scattering on cylinders provides an example of a setting in which even the order of growth of a resonance-counting function may vary depending on the potential.

In Section 4, we prove a theorem which gives another situation in which we can determine $c_{V,\mathscr{E}}$. In Section 5, we give some analogous results for potential scattering on half-cylinders.

Scattering on cylinders bears some resemblance to potential scattering on the line. On the line, the distribution of resonances has been studied in [9,14,16]. The complicated nature of $\hat{Z}$ makes more difficult the question of bounding the number of resonances in the cylindrical end setting. Earlier results on resonances for manifolds with cylindrical ends include [1–3,6–8], and references therein. For general scattering theory on manifolds with cylindrical ends, references include [10,12].

2. Preliminaries and notation

Let $r_j(z) = (z - v_j^2)^{1/2}$ and identify the physical sheet of $\hat{Z}$ as being the part of $\hat{Z}$ on which $\operatorname{Im} r_j(z) > 0$ for all j and all z and on which $R_V(z)$ is bounded on $L^2(X)$ for all but a discrete set of z. Other sheets will be identified, when necessary, by indicating for which values of j $\operatorname{Im} r_j(z) < 0$. Each sheet can be identified with $\mathbb{C} \setminus [v_0^2, \infty)$. With this language, there are points in $\hat{Z}$ which belong to no sheet but which belong to the boundary of the closure of two sheets, and the ramification points, which correspond to $\{v_j^2\}$ and belong to the closure of four sheets (except for ramification points corresponding to v_0^2). We note that sheets that meet the physical sheet are characterized by the existence of a $J \in \mathbb{N}$ such that

$$\operatorname{Im} r_j(z) < 0 \text{ for all } z \text{ on that sheet if and only if } j \leqslant J.$$

We can associate to a fixed sheet of $\hat{Z}$ a set $\mathscr{E} \subset \mathbb{N} \cup \{0\} = \mathbb{N}_0$,

$$\mathscr{E} = \{j : \operatorname{Im} r_j(z) < 0 \text{ on this sheet}\}.$$

We shall call $\mathscr{E}$ the *labeling set* and denote the associated sheet by $Z_{\mathscr{E}}$. Let

$$\tilde{\mathscr{E}} = \{l \in \mathbb{N}_0 : \sigma_l^2 = v_j^2 \text{ for some } j \in \mathscr{E}\}.$$

Let $\{\phi_j\}$ be an orthonormal set of eigenfunctions of Δ_Y associated with $\{\sigma_j^2\}$.

In general, we shall use z to stand for a point in $\hat{Z}$ and $\Pi(z)$ to represent its projection to $\mathbb{C}$. For $w \in \mathbb{R}^m$, $\langle w \rangle = (1 + |w|^2)^{1/2}$. We will denote by C a constant whose value may change from line to line.

Next, we recall some results and language of complex analysis, e.g. [11], and recall a theorem we shall need on the distribution of zeros of functions which are "good" in a half-plane.

We shall often work with functions that are holomorphic not in the whole plane but are holomorphic within an angle (θ_1, θ_2). A function F holomorphic in an angle (θ_1, θ_2) is of order ρ there if

$$\overline{\lim}_{r \to \infty} \frac{\ln \ln (\sup_{\theta \in (\theta_1, \theta_2)} |F(re^{i\theta})|)}{\ln r} = \rho.$$

A function of order ρ in the angle (θ_1, θ_2) is of type τ there if

$$\overline{\lim}_{r \to \infty} \frac{\ln \sup_{\theta \in (\theta_1, \theta_2)} |F(re^{i\theta})|}{r^\rho} = \tau.$$

A function of order 1 and type $\tau < \infty$ (in an angle (θ_1, θ_2)) is said to be of exponential type there. Of course, ρ and τ can depend on θ_1 and θ_2.

The indicator of a function F holomorphic in an angle $\theta_1 < \arg\zeta < \theta_2$ and of order ρ is

$$h_F(\theta) = \overline{\lim}_{r\to\infty} \frac{\ln|F(re^{i\theta})|}{r^\rho}.$$

A function F is of completely regular growth within the angle (θ_1, θ_2) if

$$\lim_{\substack{r\to\infty \\ r\notin E}} \frac{\ln|F(re^{i\theta})|}{r^\rho} = h_F(\theta),$$

where the set $E\subset\mathbb{R}_+$ is of zero relative measure and the convergence is uniform for $\theta\in(\theta_1,\theta_2)$.

We shall abuse notation slightly and also use the language above for a function that is holomorphic for $\theta_1 < \arg\zeta < \theta_2$ and ζ outside of a compact set.

For a function f defined in the lower half plane, let $n_{f-}(r)$ be the number of zeros of f, counted with multiplicity, that lie in the lower half-plane and have norm less than r.

Theorem 2.1. *Suppose* $f(\zeta)$ *is holomorphic in a neighborhood of the closed lower half-plane* $\operatorname{Im}\zeta \leqslant 0$,

$$|f(\zeta)| \leqslant Ce^{C|\zeta|}$$

there, $f(0) = 1$,

$$\left| \int_{-\infty}^{\infty} \frac{d[\arg f(t)]}{dt}\, dt \right| < \infty$$

and

$$\left| \int_{-\infty}^{\infty} \frac{\ln|f(t)|}{1+t^2}\, dt \right| < \infty.$$

Then

$$\lim_{r\to\infty} \frac{n_{f-}(r)}{r} = \frac{1}{2\pi}\int_{\pi}^{2\pi} h_f(\varphi)\, d\varphi.$$

The proof of this theorem can be found in [4]. It is an adaptation of arguments of [11, Chapter III, Section 2] and [11, Theorem 3, Chapter III, Section 3].

We note, moreover, that the assumptions of Theorem 2.1 mean that f is a function of completely regular growth in the lower half-plane and that $h_f(\theta) = c_f|\sin\theta|$ for $\pi < \theta < 2\pi$.

3. Proof of Theorem 1.1

As in [9], here we find a matrix B so that the poles of the resolvent in the region in question are included in the zeros of $\det(I+B)$. We study the properties of the matrix B, and then apply Theorem 2.1. Recall that here $X=(-\infty,\infty)\times Y$.

Let

$$R_0(z)=(\Delta-z)^{-1}=\sum_{j=1}^{\infty}\frac{i}{2r_j(z)}e^{i|x-x'|r_j(z)}\sum_{\sigma_l^2=v_j^2}\phi_l(y)\bar{\phi}_l(y'). \tag{2}$$

Then

$$(\Delta+V-z)R_0(z)=I+VR_0(z). \tag{3}$$

Since $R_0(z)$ has no null space, away from the ramification points of $\hat{Z}$, $R_V(z)$ has a pole if and only if $I+VR_0(z)$ has nontrivial null space (and the multiplicities agree).

If $\mathscr{E}\subset\mathbb{N}_0$ is a finite set, define $w_{\mathscr{E}}:\hat{Z}\to\hat{Z}$ as follows. To z we may associate the set of square roots $\{r_j(z)\}$. Then $w_{\mathscr{E}}(z)$ may be determined by saying it is the element of $\hat{Z}$ associated to the set $\{r_j(w_{\mathscr{E}}(z))\}$, with

$$r_j(w_{\mathscr{E}}(z))=\begin{cases}-r_j(z) & \text{if } j\in\mathscr{E},\\ r_j(z) & \text{if } j\notin\mathscr{E}.\end{cases}$$

Suppose we now restrict ourselves to consider only $z\in Z_{\mathscr{E}}$. Then $w_{\mathscr{E}}(z)$ lies in the physical sheet. Moreover,

$$\begin{aligned}I+VR_0(z)&=(I+VR_0(w_{\mathscr{E}}(z)))\Big[I+[I+VR_0(w_{\mathscr{E}}(z))]^{-1}V[R_0(z)-R_0(w_{\mathscr{E}}(z))]\Big]\\ &=(I+VR_0(w_{\mathscr{E}}(z)))\Big[I+[I+VR_0(w_{\mathscr{E}}(z))]^{-1}A_1(z)\Big],\end{aligned} \tag{4}$$

where $A_1(z)$ has Schwartz kernel

$$V(x,y)\sum_{l\in\tilde{\mathscr{E}}}\frac{i}{2\tilde{r}_l(z)}(e^{i\tilde{r}_l(z)(x-x')}+e^{-i\tilde{r}_l(z)(x-x')})\phi_l(y)\overline{\phi_l}(y').$$

If $|\operatorname{Im}\Pi(w_{\mathscr{E}}(z))|>\|V(x,y)\|_{L^\infty}$, then $I+VR_0(w_{\mathscr{E}}(z))$ is invertible. If we restrict ourselves to such z, then, the poles of the resolvent of $\Delta+V$ are given by the zeros of

$$\det(I+A_2(z)),$$

where $A_2(z)$ is

$$A_2(z)=\sum_{l\in\tilde{\mathscr{E}}}\frac{i}{2\tilde{r}_l(z)}(\varphi_{l+}\otimes\Psi_{l-}+\varphi_{l-}\otimes\Psi_{l+})$$

with

$$\Phi_{l\pm}(x,y,z) = e^{\pm i\tilde{r}_l(z)x}\phi_l(y)$$

$$\varphi_{l\pm}(x,y,z) = \Big((I+VR_0(w_{\mathscr{E}}(z)))^{-1}(V\Phi_{l\pm})(\bullet,z)\Big)(x,y)$$

$$\Psi_{l,\pm}(x,y,z) = e^{\pm i\tilde{r}_l(z)x}\overline{\phi_l}(y).$$

Here we use the notation

$$(f\otimes g)h(x,y) = f(x,y)\int_X g(x',y')h(x',y')d\,\mathrm{vol}_X.$$

One can then see that the zeros of $(I+A_2(z))$ are the same as the zeros of $(I+A_2(z)\chi)$, where $\chi\in L^\infty_{\mathrm{comp}}(X)$ is one on the support of V. The zeros of $(I+A_2(z)\chi)$ are the same as the zeros of $\det(I+B(z))$, where

$$B(z) = \begin{pmatrix} B_{+-}(z) & B_{--}(z) \\ B_{++}(z) & B_{-+}(z)\end{pmatrix}, \tag{5}$$

$B_{+\pm} = (b_{+\pm lj})_{l,j\in\tilde{\mathscr{E}}}$, $B_{-\pm} = (b_{-\pm lj})_{l,j\in\tilde{\mathscr{E}}}$, and

$$\begin{aligned} b_{+\mp lj}(z) &= \frac{i}{2\tilde{r}_l(z)}\int_X \varphi_{j+}(x,y,z)\chi(x,y)\Psi_{l\mp}(x,y,z)\,d\,\mathrm{vol}_X,\\ b_{-\mp lj}(z) &= \frac{i}{2\tilde{r}_l(z)}\int_X \varphi_{j-}(x,y,z)\chi(x,y)\Psi_{l\mp}(x,y,z)\,d\,\mathrm{vol}_X. \end{aligned}\tag{6}$$

We shall first obtain upper bounds on the entries in the matrix B, and thus on $\det(I+B(z))$. To do so, we will use the following lemma.

Lemma 3.1. *Let* $f_\pm(x,z) = e^{\pm i\tilde{r}_j(z)x}$, and let $\chi_1,\ \chi_2\in C_c^\infty(X)$. *If* z *lies on the physical sheet of* $\hat{Z}$ *and* $\mathrm{Im}\,\tilde{r}_j(z) = t_0 > 0$, *then*

$$\left\|\chi_1\frac{1}{f_\pm}R_0(z)f_\pm\chi_2\right\|_{L^2(X)\to L^2(X)} \leqslant \frac{C}{|\mathrm{Re}\ \tilde{r}_j(z)|^{7/12}}$$

when $|\tilde{r}_j(z)|$ *is sufficiently large. Moreover, for* $\mathrm{Im}\,\tilde{r}_j(z)\geqslant t_0>0$,

$$\left\|\chi_1\frac{1}{f_\pm}R_0(z)f_\pm\chi_2\right\|_{L^2(X)\to L^2(X)} \leqslant \frac{C}{|\tilde{r}_j(z)|^{5/12}}$$

when $|\tilde{r}_j(z)|$ *is sufficiently large.*

Proof. Without loss of generality we can assume χ_1 and χ_2 are independent of y and thus it is suffices to consider, for $l \in \mathbb{N}$,

$$\left\| \chi_1 \frac{1}{f_\pm} R_{0l}(z) f_\pm \chi_2 \right\|_{L^2(X) \to L^2(X)},$$

where $R_{0l}(z)$ has Schwartz kernel

$$\frac{i}{2\tilde{r}_l(z)} e^{i\tilde{r}_l(z)|x-x'|} \phi_l(y) \bar{\phi}_l(y').$$

The Schwartz kernel of $\chi_1 (f_\pm)^{-1} R_{0l}(z) f_\pm \chi_2$ is

$$K_{l\pm}(x,y,x',y',z) = \begin{cases} \dfrac{i}{2\tilde{r}_l(z)} e^{i(\tilde{r}_l(z) \mp \tilde{r}_j(z))(x-x')} \phi_l(y) \bar{\phi}_l(y') \chi_1(x) \chi_2(x') & \text{if } x > x' \\ \dfrac{i}{2\tilde{r}_l(z)} e^{i(-\tilde{r}_l(z) \mp \tilde{r}_j(z))(x-x')} \phi_l(y) \bar{\phi}_l(y') \chi_1(x) \chi_2(x') & \text{if } x < x'. \end{cases}$$

We shall show that when $\operatorname{Im} \tilde{r}_j(z) = t_0$

$$\int_X \int_X |K_{l\pm}(x,y,x',y',z)|^2 \, d\,\mathrm{vol}_X \, d\,\mathrm{vol}_X \leqslant \frac{C}{|\operatorname{Re}\ \tilde{r}_j(z)|^{7/6}}$$

with constant C independent of l, which will prove the first part of the lemma.

First, notice that on the support of $\chi_1(x)\chi_2(x')$, the exponential function in $K_{l\pm}$ is bounded independent of l. This is because $\operatorname{Im} \tilde{r}_l(z) > 0$ and $|\operatorname{Im} \tilde{r}_j(z)(x-x')|$ is bounded for $x \in \operatorname{supp}\ \chi_1,\ x' \in \operatorname{supp}\ \chi_2$. Thus,

$$\|K_{l\pm}(z)\|_{L^2}^2 \leqslant \frac{C}{|\tilde{r}_l(z)|^2}. \tag{7}$$

When $\tilde{r}_l \neq \tilde{r}_j$, we may integrate by parts to see that

$$\|K_{l\pm}(z)\|_{L^2}^2 \leqslant \frac{C}{|\operatorname{Im}(\tilde{r}_j(z) - \tilde{r}_l(z))|} \frac{1}{|\tilde{r}_l(z)|^2}$$

so that

$$\|K_{l\pm}(z)\|_{L^2}^2 \leqslant \frac{C}{|\tilde{r}_l(z)|^2} \min(1, (|\operatorname{Im}(\tilde{r}_j(z) - \tilde{r}_l(z))|)^{-1}).$$

Let $\tilde{r}_j = s + it_0$. Then if $\tilde{r}_l(z) = u + iv$, a computation shows that, with $g = \sigma_j^2 + s^2 - t_0^2 - \sigma_l^2$, $u^2 = \frac{1}{2}(g + \sqrt{g^2 + 4s^2 t_0^2})$, and $v^2 = \frac{1}{2}(-g + \sqrt{g^2 + 4s^2 t_0^2})$.

If $g \leqslant (|s|t_0)^{7/6}$, then

$$v^2 \geqslant \frac{1}{2}\left(-(|s|t_0)^{7/6} + \sqrt{(|s|t_0)^{7/3} + 4(|s|t_0)^2}\right)$$
$$= (|s|t_0)^{5/6} + O((|s|t_0)^{1/2}). \tag{8}$$

Then

$$||K_{l\pm}(z)||^2_{L^2} \leqslant \frac{C}{|\tilde{r}_l(z)|^2|v - t_0|} \leqslant \frac{C}{(|s|t_0)^{5/6}(|s|t_0)^{5/12}} \leqslant \frac{C}{|s|^{5/4}}$$

when $|s|$ is sufficiently large and $\operatorname{Im} \tilde{r}_j(z) = t_0$.

If, on the other hand, $g \geqslant (|s|t_0)^{7/6}$, then we use

$$u^2 = \frac{1}{2}\left(g + \sqrt{g^2 + 4(st_0)^2}\right) \geqslant g \geqslant (|s|t_0)^{7/6} \tag{9}$$

and

$$||K_{l\pm}(z)||^2_{L^2} \leqslant \frac{C}{|\tilde{r}_l(z)|^2} \leqslant \frac{C}{u^2} \leqslant \frac{C}{(|s|t_0)^{7/6}}.$$

This finishes the proof of the first part of the lemma.

To prove the second part of the lemma, first notice that if $\tilde{r}_j(z) = s + it$ and $|s| < 1$, then $\frac{1}{|\tilde{r}_l(z)|^2} \leqslant \frac{C}{t^2}$ when t is sufficiently large, so that

$$||K_{l\pm}(z)||^2_{L^2} \leqslant \frac{C}{t^2}$$

in this region. On the other hand, if $|s| \geqslant 1$, inequalities (7)–(9) together show that when $t \geqslant t_0$,

$$||K_{l\pm}(z)||^2_{L^2} \leqslant \frac{C}{|s + it|^{5/6}}. \qquad \square$$

Fix $j_0 \in \mathscr{E}$. We shall eventually use $k = r_{j_0}(z)$ to identify our fixed sheet $Z_{\mathscr{E}}$ of $\hat{Z}$ with the lower half-plane. However, we shall continue to use z as a coordinate as well, when it is more convenient. In any case, we restrict ourselves to $Z_{\mathscr{E}}$.

Lemma 3.2. *Fix a sheet $Z_{\mathscr{E}}$ of $\hat{Z}$ and let $j_0 \in \mathscr{E}, l, j \in \tilde{\mathscr{E}}$. If $z \in Z_{\mathscr{E}}$, $-\operatorname{Im} r_{j_0}(z) \geqslant \alpha > 0$, then for $|r_{j_0}(z)|$ sufficiently large (depending on α),*

$$|b_{+-lj}(z)| \leqslant \frac{C}{|\tilde{r}_l(z)|}, \quad |b_{-+lj}(z)| \leqslant \frac{C}{|\tilde{r}_l(z)|}.$$

Proof. First we show that in this region, for $j\in\tilde{\mathscr{E}}$ and $\chi\in L^{\infty}_{\text{comp}}(X)$,

$$||e^{\mp i\tilde{r}_j(z)x}(I+VR_0(w_{\mathscr{E}}(z)))^{-1}\chi e^{\pm i\tilde{r}_j(z)x}||\leqslant C \tag{10}$$

when $|r_{j_0}(z)|$ is sufficiently large.

When $|r_{j_0}(z)|$ is sufficiently large, and $\tilde{\chi}\in L^{\infty}_{\text{comp}}(X)$ is one on the support of V,

$$\begin{aligned}
&||e^{\mp i\tilde{r}_j(z)x}(I+VR_0(w_{\mathscr{E}}(z)))^{-1}\chi e^{\pm i\tilde{r}_j(z)x}||\\
&\quad=||\sum_{m=0}^{\infty} e^{\mp i\tilde{r}_j(z)x}(-1)^m(VR_0(w_{\mathscr{E}}(z))\tilde{\chi})^m\chi e^{\pm i\tilde{r}_j(z)x}||\\
&\quad=||\sum_{m=0}^{\infty}(-1)^m(e^{\mp i\tilde{r}_j(z)x}VR_0(w_{\mathscr{E}}(z))\tilde{\chi}e^{\pm i\tilde{r}_j(z)x})^m\chi||\\
&\leqslant C,
\end{aligned} \tag{11}$$

where we are using Lemma 3.1. Using this estimate and the definition of b_{+-lj}, b_{-+lj}, we obtain the desired estimates. □

We shall need the following bound on the $b_{++lj}(z)$ and $b_{--lj}(z)$.

Lemma 3.3. *Fix a sheet $Z_{\mathscr{E}}$ of $\hat{Z}$, and let $j_0\in\mathscr{E}$. If $z\in Z_{\mathscr{E}}$,* $\operatorname{Im} r_{j_0}(z)\leqslant -\alpha<0$, $l,j\in\tilde{\mathscr{E}}$, *and* $\operatorname{conv}_x(\operatorname{supp}(V))=[-\beta,\gamma]$, *then for $|r_{j_0}(z)|$ sufficiently large* (*depending on* α),

$$|b_{++lj}(z)|\leqslant\frac{Ce^{2\gamma|\operatorname{Im}\tilde{r}_j(z)|}}{|\tilde{r}_l(z)|},\quad |b_{--lj}(z)|\leqslant\frac{Ce^{2\beta|\operatorname{Im}\tilde{r}_j(z)|}}{|\tilde{r}_l(z)|}.$$

Proof. We give the proof for b_{++lj}. Recall that $\Phi_{j\pm}(x,y,z)=e^{\pm i\tilde{r}_j(z)x}\phi_j(y)$. Then we obtain that

$$||\varphi_{j+}||=||(I+VR_0(w_{\mathscr{E}}(z)))^{-1}V\Phi_{j+}||\leqslant Ce^{\gamma|\operatorname{Im}\tilde{r}_j(z)|}.$$

Note that if $\operatorname{supp} f\subset\operatorname{supp}V$, then $\operatorname{supp}(I+VR_0(w_{\mathscr{E}}(z)))^{-1}f\subset\operatorname{supp}\ V$. We use this fact and the bound above to obtain

$$\begin{aligned}
|b_{++lj}(z)|&=\left|\frac{1}{2\tilde{r}_l(z)}\int_X \varphi_{j+}(x,y,z)\chi(x,y)\Psi_{l+}(x,y,z)\,d\operatorname{vol}_X\right|\\
&\leqslant\frac{C}{|\tilde{r}_l(z)|}e^{2\gamma|\operatorname{Im}\tilde{r}_j(z)|}
\end{aligned}$$

for $|r_{j_0}(z)|$ sufficiently large, $\operatorname{Im} r_{j_0}(z)\leqslant-\alpha$. For the last inequality we have also used that $\tilde{r}_i(z)\to\tilde{r}_j(z)$ as $\Pi(z)\to\infty$.

A similar argument yields the proof of the bound for $b_{--lj}(z)$. □

Proof of Theorem 1.1. We use the coordinate $k = r_{j_0}(z)$ to identify our fixed sheet with the lower half-plane. Let $g_1(k) = \det(I + B(z(k)))$, where $\Pi(z(k)) = k^2 + v_{j_0}^2$ and z lies on our sheet. Here $B(z)$ is as defined in (5) and (6). Then $g_1(k)$ has at most a finite number of poles, $k_1, k_2, \ldots, k_{m_\alpha}$, listed with multiplicity, in $\operatorname{Im} k \leqslant -\alpha$. Let

$$g_2(k) = g_1(k)(k - k_1)(k - k_2)\cdots(k - k_{m_\alpha})$$

and, if $g_2(-i\alpha) \neq 0$, let

$$g_3(k) = \frac{g_2(k)}{g_2(-i\alpha)}.$$

If $g_2(-i\alpha) = 0$, let

$$g_3(k) = \frac{g_2(k)l!}{(k + i\alpha)^l g_2^{(l)}(-i\alpha)},$$

where l is chosen so that $g_2^{(m)}(-i\alpha) = 0$ if $m < l$ but $g_2^{(l)}(-i\alpha) \neq 0$. Then Lemmas 3.2 and 3.3 show that the hypotheses of Theorem 2.1 are satisfied for $g_4(k) = g_3(k - i\alpha)$, with

$$|h_{g_4}(\varphi)| \leqslant 2(\gamma + \beta)\operatorname{card}(\tilde{\mathscr{E}})|\sin\varphi|.$$

Recalling that, except possibly for a finite number, the zeros of $g_3(k)$ correspond to the poles of $R_V(z)$ in this region, an application of Theorem 2.1 finishes the proof. $\square$

4. Determining $c_{V,\mathscr{E}}$ and a counterexample

In this section we prove Theorem 1.2, give a counterexample, and give another example of a setting in which $c_{V,\mathscr{E}}$ can be determined.

We shall need the following lemma, which is Lemma 4.1 of [9].

Lemma 4.1. *Suppose $v \in L^\infty(\mathbb{R})$ has compact support contained in $[-1, 1]$, but in no smaller interval. Suppose $f(x, k)$ is analytic for k in the lower half-plane, and for real k we have $f(x, k) \in L^2([-1, 1]\,dx, \mathbb{R}dk)$. Then $\int e^{\pm ikx}v(x)(1 - f(x, k))\,dx$ has exponential type at least 1 for k in the lower half-plane.*

In the next lemma, we use $k = \tilde{r}_l(z)$ as a coordinate, and, fixing a sheet of $\hat{Z}$, let $z(k)$ be the corresponding point on $\hat{Z}$.

Lemma 4.2. *Let $X = (-\infty, \infty) \times Y$ and suppose that $\operatorname{conv}_x(\operatorname{supp}(V)) = [-b, b]$. Suppose that*

$$C|V_{ll}(x)| = C\left|\int_Y V(x, y)|\phi_l(y)|^2\, d\operatorname{vol}_Y\right| \geqslant |V(x, y)|, \text{ for } |x - b| < \varepsilon,\ |x + b| < \varepsilon$$

for some $C, \varepsilon > 0$. *Fix a sheet* $Z_{\mathscr{E}}$ *of* $\hat{Z}$ *with* $l \in \tilde{\mathscr{E}}$, *and choose* α *so that there are no poles of* b_{++ll}, b_{--ll} *on* $Z_{\mathscr{E}}$ with $\operatorname{Im} r_l(z) \leqslant -\alpha$. *Then* $b_{++ll}(z(k))$, $b_{--ll}(z(k))$ *are functions of type at least* $2b$ *for the half-plane* $\operatorname{Im} k \leqslant -\alpha$, $k = \tilde{r}_l(z)$.

Proof. We give the proof for b_{++ll}, as the proof for b_{--ll} is similar.

Let $g(k,x) = e^{ikx}$ and let

$$\begin{aligned} f_1(x,y,k) &= \bar{\phi}_l(y)\frac{1}{V(x,y)}\left[\frac{1}{g}[I - [I + VR_0(w_{\mathscr{E}}(z(k)))]^{-1}]V\Phi_{l+}(\cdot, z(k))\right](x,y) \\ &= \bar{\phi}_l(y)\sum_{m=1}^{\infty}(-1)^{m+1}\left[((g)^{-1}R_0(w_{\mathscr{E}}(z(k)))Vg)^m\phi_l\right](x,y), \end{aligned}$$

where the second equality holds when $|k|$ is sufficiently large. Then

$$b_{++ll}(z(k)) = \frac{i}{2k}\int e^{2ikx}V(x,y)(|\phi_l|^2(y) - f_1(x,y,k))\, d\,\mathrm{vol}_X.$$

Let

$$\chi_\varepsilon(x) = \begin{cases} 0 & \text{if } |x| < b-\varepsilon \text{ or } |x| > b, \\ 1 & \text{if } b - \varepsilon \leqslant |x| \leqslant b. \end{cases}$$

Let

$$v(x) = \int_Y V(x,y)|\phi_l|^2(y)\, d\,\mathrm{vol}_Y = V_{ll}(x),$$

and

$$f(x,k) = \frac{1}{V_{ll}(x)}\chi_\varepsilon(x)\int_Y V(x,y)f_1(x,y,k)\, d\,\mathrm{vol}_Y.$$

Note that

$$\begin{aligned} b_{++ll}(z(k)) = &\frac{i}{2k}\int e^{2ikx}v(x)(1 - f(x,k))\, dx \\ &- \frac{i}{2k}\int_X e^{2ikx}(1-\chi_\varepsilon)V(x,y)f_1(x,y,k)\, d\,\mathrm{vol}_X. \end{aligned}$$

Using (10) and the support properties of $V(1-\chi_\varepsilon)$, the last term on the right is of type at most $2b - 2\varepsilon$, and so we need only show that the first integral on the right is of type at least $2b$. To do this, we will apply Lemma 4.1 to $b_{++ll}(z(k+i\alpha))$.

We must show that $f(x,k)\in L^2([-b,b]\,dx,\mathbb{R}\,dk)$ when $\operatorname{Im} k=-\alpha$. We have

$$\begin{aligned}\int |f(x,k)|^2\,dx &= \int_{|x|\leqslant b} |V_{ll}(x)|^{-2}\chi_\varepsilon(x)\left|\int_Y V(x,y)f_1(k,x,y)\,d\,\mathrm{vol}_Y\right|^2 dx\\ &\leqslant \int_{|x|\leqslant b}\int_Y |V_{ll}(x)|^{-2}\chi_\varepsilon(x)|V(x,y)|^2\,d\,\mathrm{vol}_Y\\ &\qquad\times\int_Y |f_1(k,x,y)|^2\,d\,\mathrm{vol}_Y\,dx\\ &\leqslant C\int_X |f_1(k,x,y)|^2\,d\,\mathrm{vol}_X.\end{aligned}$$

By Lemma 3.1, when $|\mathrm{Re}\ k|$ is sufficiently large, this is bounded by $C|\mathrm{Re}\ k|^{-7/6}$. When $|\mathrm{Re}\ k|$ is in a compact set (with $\operatorname{Im} k=-\alpha$), it is enough to note that $\int |f_1(k,x,y)|^2 d\mathrm{vol}_X$ is bounded, so that $f(x,k)\in L^2([-b,b]\,dx,\mathbb{R}\,dk)$. Then, applying Lemma 4.1 after appropriately rescaling, we finish the proof. $\square$

Proof of Theorem 1.2. We use $k=r_{j_0}(z)=\tilde{r}_{l_0}(z)$ as the coordinate. The simplicity of $v_{j_0}^2$ as an eigenvalue of Δ_Y means that the matrix B is a 2×2 matrix

$$B=\begin{pmatrix} b_{+-l_0l_0} & b_{--l_0l_0}\\ b_{++l_0l_0} & b_{-+l_0l_0}\end{pmatrix}.$$

Thus $\det(I+B)(z(k))=[(1+b_{+-l_0l_0})(1+b_{-+l_0l_0})-b_{--l_0l_0}b_{++l_0l_0}](z(k))=\varphi_1(k)$.

Suppose first that $\varphi_1(k)$ has no poles in the region $\operatorname{Im} k\leqslant-\alpha$. If $\varphi_1(-i\alpha)\neq 0$, let

$$\varphi_2(k)=\frac{\varphi_1(k)}{\varphi_1(-i\alpha)}.$$

If $\varphi_1(-i\alpha)=0$, let

$$\varphi_2(k)=\frac{\varphi_1(k)l!}{(k+i\alpha)^l\varphi_1^{(l)}(-i\alpha)},$$

where l is chosen so that $\varphi_1^{(m)}(-i\alpha)=0$ if $m<l$ but $\varphi_1^{(l)}(-i\alpha)\neq 0$.

Note that by Lemmas 3.2 and 3.3, for $s\in\mathbb{R}$, $\varphi_2(s-i\alpha)=c_0(1+O(|s|^{-1}))$ when $|s|\to\infty$, for some nonzero constant c_0. Moreover, by Lemmas 3.2, 3.3, and 4.2, $\varphi_2(k)$ is a function of type $4b$ in the half-plane $\operatorname{Im} k\leqslant-\alpha$. Then applying Theorem 2.1 to $\varphi_2(k)$ in the half-plane $\operatorname{Im} k\leqslant-\alpha$, we obtain the result.

If $\varphi_1(k)$ has poles in the region $\operatorname{Im} k\leqslant-\alpha$, they can be handled in the same manner as in the proof of Theorem 1.1. $\square$

We give a counterexample for Theorem 1.2. Let $X=\mathbb{R}\times\mathbb{S}^1$, and let $V(x,y)=V_1(x)e^{imy}$ with $V_1(x)\in L^\infty_{\mathrm{comp}}(\mathbb{R})$ nontrivial and $m>0$ an integer.

In fact, this potential has no resonances away from the ramification points of $\hat{Z}$. To see this, note that if z is resonance and not a ramification point, there is a nontrivial function $u(x,y) = \sum_{-\infty}^{\infty} u_j(x)e^{ijy} \in L^2(X)$, with

$$(I + V_1(x)e^{imy}R_0(z))u = 0$$

(see (3)). The function u is necessarily supported on the support of V. Then

$$u_{j+m}(x) = -V_1(x)(R_{0j}(z)u_j)(x), \tag{12}$$

where

$$R_{0j}(z) = \frac{i}{2r_j(z)} e^{i|x-x'|r_j(z)}.$$

The operator R_{0j} is bounded from $L^2_{\mathrm{comp}}(\mathbb{R})$ to $L^2_{\mathrm{comp}}(\mathbb{R})$. For a fixed z, when $|j|$ is sufficiently large we have

$$||V_1(x)R_{0j}(z)|| \leqslant C|j|^{-1}. \tag{13}$$

For u to be in $L^2(X)$, we must have $||u_j|| \to 0$ when $j \to \pm\infty$. But, using (12) and (13), when $|j|$ is sufficiently large,

$$||u_{j+m}|| \leqslant \frac{C}{|j|}||u_j||.$$

Thus, we cannot have $||u_j|| \to 0$ when $j \to -\infty$ unless there is a j_0 so that $u_j = 0$ whenever $j < j_0$. In this case, using (12), $u_j = 0$ for all j. Thus we have a contradiction.

In Theorem 1.2 we used some knowledge of the potential near the boundary of its support to allow us to find $c_{V,\mathscr{E}}$. In the following theorem we again make use of the fact that the potential is "controlled" near the boundary of its support.

Theorem 4.1. *Suppose for some potential* $V_0 \in L^\infty_{\mathrm{comp}}(X;\mathbb{C})$, *with* $\operatorname{supp} V_0 \subset [-b_0, b_0] \times Y$ *and for some sheet* $Z_{\mathscr{E}}$ *of* $\hat{Z}$ *with* $j_0 \in \mathscr{E}$ *we have*

$$\#\{z_k : z_k \in Z_{\mathscr{E}}, z_k \text{ is a pole of } R_{V_0}(z), |r_{j_0}(z_k)| < r,\ \operatorname{Im} r_{j_0}(z_k) < -\alpha\}$$
$$= \frac{4b_0}{\pi}\#\{l :\ l \in \tilde{\mathscr{E}}\}r + o_\alpha(r)$$

for some $\alpha > 0$. *Suppose in addition* $W \in L^\infty_{\mathrm{comp}}(X;\mathbb{C})$ *with* $\operatorname{supp} W \subset [-b_0 + \varepsilon, b_0 - \varepsilon] \times Y$ *for some* $\varepsilon > 0$. *Then*

$$\#\{z_k : z_k \in Z_{\mathscr{E}}, z_k \text{ is a pole of } R_{V_0+W}(z), |r_{j_0}(z_k)| < r,\ \operatorname{Im} r_{j_0}(z_k) < -\alpha\}$$
$$= \frac{4b_0}{\pi}\#\{l :\ l \in \tilde{\mathscr{E}}\}r + o_\alpha(r).$$

That is, if the resonance-counting function for $\Delta + V_0$ has maximal growth rate, so does that for $\Delta + V_0 + W$.

Proof. In the proof of this theorem, we will add a superscript to the matrix B from Section 3 and its entries to indicate to which potential it is associated. That is, when $|r_{j_0}(z)|$ is sufficiently large, the poles of the resolvent of $\Delta + V$ correspond to the zeros of $\det(I + B^V(z))$ and likewise for V_0. We shall also add a superscript to $\varphi_{l\pm}$.

In this proof, as previously, we shall sometimes use as coordinate on our sheet $k = r_{j_0}(z)$, and then $z(k)$ is the corresponding point on our sheet.

Let $V = V_0 + W$. We shall show that $B^V(z) = B^{V_0}(z) + D(z)$, with the entries $d_{lj}(z)$ of $D(z)$ satisfying

$$|d_{lj}(z)| \leqslant \frac{C}{|\tilde{r}_l(z)|} e^{(2b_0-\varepsilon)|\mathrm{Im}\, \tilde{r}_j(z)|}. \tag{14}$$

Because of the assumption on the distribution of resonances for $\Delta + V_0$, $\det(I + B^{V_0}(z(k)))$ is of type $4b_0\#\{l:\ l \in \tilde{\mathscr{E}}\}$ in $\mathrm{Im}\, k < -\alpha < 0$. Moreover, each entry of $B^{V_0}(z(k))$ has type at most $2b_0$ and is bounded by $Ce^{2b_0|\mathrm{Im}\, k|}$. Then

$$\begin{aligned}\det(I + B^V(z(k))) &= \det(I + B^{V_0}(z(k)) + D(z(k))) \\ &= \det(I + B^{V_0}(z(k))) + O\left(\frac{e^{|\mathrm{Im}\, k|(4b_0\#\{l:l\in\mathscr{E}\}-\varepsilon)}}{|k|}\right).\end{aligned}$$

Applying Theorem 2.1 as in the proof of Theorems 1.1 and 1.2 finishes the proof.

It remains to show (14). Note that we may write, when $|r_{j_0}(z)|$ is sufficiently large,

$$\begin{aligned}&[I + (V_0 + W)R_0(w_{\mathscr{E}}(z))]^{-1} \\ &\quad = \left(I + \sum_{m=1}^{\infty} (-1)^m \left[(I + V_0R_0(w_{\mathscr{E}}(z)))^{-1} WR_0(w_{\mathscr{E}}(z))\right]^m\right)[I + V_0R_0(w_{\mathscr{E}}(z))]^{-1}.\end{aligned}$$

Then

$$\begin{aligned}\varphi^V_{l+}(z) &= (I + VR_0(w_{\mathscr{E}}(z)))^{-1}(V_0 + W)\Phi_{l+}(\bullet, z)) \\ &= (I + V_0R_0(w_{\mathscr{E}}(z)))^{-1}(V_0\Phi_{l+}(\bullet, z)) + (I + V_0R_0(w_{\mathscr{E}}(z)))^{-1}(W\Phi_{l\pm}(\bullet, z)) \\ &\quad + \sum_{m=1}^{\infty} (-1)^m \left[(I + V_0R_0(w_{\mathscr{E}}(z)))^{-1} WR_0(w_{\mathscr{E}}(z))\right]^m \\ &\qquad [I + V_0R_0(w_{\mathscr{E}}(z))]^{-1}\Phi_{l+}(\bullet, z).\end{aligned}$$

The first term on the right is $\varphi^{V_0}_{l+}(z)$. The second term is, as in (11), bounded by $Ce^{(b_0-\varepsilon)|\mathrm{Im}\, r_l(z)|}$. Again as in (11), the third term is also bounded by $Ce^{(b_0-\varepsilon)|\mathrm{Im}\, r_l(z)|}$.

Putting all this into the definition of $b^V_{++lj}(z)$, we see that

$$b^V_{++lj}(z) = b^{V_0}_{++lj}(z) + O\left(\frac{e^{(2b_0-\varepsilon)|\mathrm{Im}\, r_{j_0}(z)|}}{|r_{j_0}(z)|}\right).$$

A similar argument works for the other entries of $B^V(z)$, proving (14). $\square$

Combining the previous theorem with the results for potential scattering in one dimension [9,16], we obtain the following corollary.

Corollary 4.1. *Let* $V(x,y) = V_0(x) + W(x,y) \in L^\infty_{\mathrm{comp}}(X;\mathbb{C})$, $\mathrm{conv}_x(\mathrm{supp}(V)) = [-b,b]$, *and* supp $W \subset [-b+\varepsilon, b-\varepsilon] \times Y$ *for some* $\varepsilon > 0$. *Then on any sheet* $Z_{\mathscr{E}}$ *of* $\hat{Z}$ *with* $j_0 \in \mathscr{E}$,

$$\#\{z_k : z_k \in Z_{\mathscr{E}},\ z_k \text{ is a pole of the resolvent of } \Delta + V, |r_{j_0}(z_k)| < r,\ \mathrm{Im}\, r_{j_0}(z_k) < -\alpha\}$$
$$= \frac{4b}{\pi} \#\{l : l \in \tilde{\mathscr{E}}\} r + o_\alpha(r)$$

for any $\alpha > 0$.

5. Results for half-cylinders

In this section, we consider half-cylinders $X = [0,\infty) \times Y$, with Δ either the Dirichlet or Neumann Laplacian on X. Let $V \in L^\infty_{\mathrm{comp}}(X;\mathbb{C})$. The resolvent $(\Delta + V - z)^{-1}$ has a meromorphic continuation to $\hat{Z}$ just as in the full cylinder case. We give several results analogous to the results for full cylinders. Since the proofs are so similar, we only sketch them.

Let $R_{0\pm}(z) = (\Delta - z)^{-1}$ be the resolvent for the Neumann $(+)$ or Dirichlet $(-)$ Laplacian on X, for $z \in \hat{Z}$. Restrict z to Z_ε. Then, following the same argument as in the beginning of Section 3, we can show that when $|\mathrm{Im}\, \Pi(w_{\mathscr{E}}(z))| > \|V\|_{L^\infty}$, the poles of the resolvent of $\Delta + V$ correspond to the zeros of $\det(I + B_\pm(z))$. Here we are again using "+" for the Neumann Laplacian and "−" for the Dirichlet Laplacian. To define $B_\pm(z)$, let

$$\Phi_{\pm l}(x,y,z) = (e^{i\tilde{r}_l(z)x} \pm e^{-i\tilde{r}_l(z)x})\phi_l(y),$$

$$\varphi_{\pm l}(x,y,z) = ((I + VR_{0\pm}(w_{\mathscr{E}}(z)))^{-1}(V\Phi_{\pm l})(\bullet, z))(x,y).$$

Then $B_\pm(z) = (b_{\pm jk}(z))_{j,k\in\tilde{\mathscr{E}}}$, with

$$b_{\pm jk}(z) = \frac{i}{2\tilde{r}_j(z)} \int_X \varphi_{\pm j}(x,y,z)\bar{\Phi}_{\pm k}(x,y,z)\, d\,\mathrm{vol}_X.$$

We obtain the following analog of Theorem 1.1.

Theorem 5.1. *Let $X = [0,\infty) \times Y$ and let $V \in L^\infty_{\mathrm{comp}}(X;\mathbb{C})$, with* $\operatorname{supp} V \subset [0,b] \times Y$. *Fix a sheet $Z_{\mathscr{E}}$* of $\hat{Z}$, *and suppose that $j_0 \in \mathscr{E}$. Then, there is a constant $c_{V,\mathscr{E}} \geqslant 0$ such that for any $\alpha > 0$,*

$$\begin{aligned}&\#\{z_k : z_k \in Z_{\mathscr{E}}, z_k \text{ is a pole of the resolvent}, |r_{j_0}(z_k)| < r,\ \operatorname{Im} r_{j_0}(z_k) < -\alpha\}\\ &\quad = c_{V,\mathscr{E}}\, r + o_\alpha(r).\end{aligned}$$

The constant $c_{V,\mathscr{E}}$ depends on the potential V and the sheet. Moreover,

$$c_{V,\mathscr{E}} \leqslant \frac{2b}{\pi}\#\{l : \operatorname{Im} \tilde{r}_l(z) < 0 \text{ when } z \in Z_{\mathscr{E}}\}.$$

Proof. Just as in the proof of Lemmas 3.2 and 3.3, we can show that on $Z_{\mathscr{E}}$

$$|b_{\pm jk}(z)| \leqslant \frac{C}{|\tilde{r}_j(z)|} e^{2b|\operatorname{Im} \tilde{r}_k(z)|}.$$

Then the proof follows just as the proof of Theorem 1.1. □

Theorem 5.2. *Let $X = [0,\infty) \times Y$ and suppose that the support of V is contained in $[0,b] \times Y$ and the number b cannot be replaced by a smaller one. Suppose that $v_{j_0}^2$ is a simple eigenvalue of Δ_Y, with $v_{j_0}^2 = \sigma_{l_0}^2$. Suppose, in addition, that*

$$C|V_{l_0 l_0}(x)| = C\left|\int_Y V(x,y)|\phi_{l_0}(y)|^2\, d\,\mathrm{vol}_Y\right| \geqslant |V(x,y)|, \quad \text{for } |x-b| < \varepsilon$$

for some $C, \varepsilon > 0$. Then, for any $\alpha > 0$,

$$\begin{aligned}&\#\{z_k : z_k \in Z_{\{j_0\}}, z_k \text{ is a pole of the resolvent}, |r_{j_0}(z_k)| < r, \operatorname{Im} r_{j_0}(z_k) < -\alpha\}\\ &\quad = \frac{2}{\pi} br + o_\alpha(r).\end{aligned}$$

Proof. Let $z \in Z_{\{j_0\}}$. In this case $B_\pm(z)$ is a single function, $b_{\pm l_0 l_0}$. Let $k = \tilde{r}_{l_0}(z)$ and let $z(k)$ be the corresponding point on $Z_{\{j_0\}}$. We have

$$\begin{aligned} b_{\pm l_0 l_0}(z(k)) &= \frac{i}{2k}\int_X (e^{ikx} \pm e^{-ikx})\bar{\phi}_{l_0}[[I + VR_0(w_{\mathscr{E}}(z(k)))]^{-1} V\Phi_{l\pm}(\bullet, z(k))]\, d\,\mathrm{vol}_X \\ &= \frac{i}{2k}\int_X e^{ikx}\bar{\phi}_{l_0}[I + VR_0(w_{\mathscr{E}}(z(k)))]^{-1} V f_{l_0}(\bullet, z(k))\, d\,\mathrm{vol}_X \\ &\quad + O(e^{b|\mathrm{Im}\, k|}). \end{aligned}$$

Here $f_{l_0}(x,y,z) = e^{i\tilde{r}_{l_0}(z)x}\phi_{l_0}(y)$, and we have used a bound similar to that of Lemma 3.1 to obtain the bound $O(e^{b|\mathrm{Im}\, k|})$ on the rest. Following the technique of Lemmas 3.2 and 4.2 shows that $b_{\pm l_0 l_0}(z(k))$ is an exponential function of type $2b$ for $\mathrm{Im}\, k \leqslant -\alpha$. The proof is completed as in the proof of Theorem 1.2. $\square$

Acknowledgments

1 would like to thank the referee for helpful comments.

References

[1] A. Aslanyan, L. Parnovski, D. Vassiliev, Complex resonances in acoustic waveguides, Quart. J. Mech. Appl. Math. 53 (3) (2000) 429–447.
[2] W. Bulla, F. Gesztesy, W. Renger, B. Simon, Weakly coupled bound states in quantum waveguides, Proc. Amer. Math. Soc. 125 (5) (1997) 1487–1495.
[3] T. Christiansen, Some upper bounds on the number of resonances for manifolds with infinite cylindrical ends, Ann. H. Poincaré 3 (5) (2002) 895–920.
[4] T. Christiansen, Resonances for steplike potentials: forward and inverse results, Trans. Amer. math. Soc., to appear.
[5] T. Christiansen, M. Zworski, Spectral asymptotics for manifolds with cylindrical ends, Ann. Inst. Fourier 45 (1) (1995) 251–263.
[6] E.B. Davies, L. Parnovski, Trapped modes in acoustic waveguides, Quart. J. Mech. Appl. Math. 51 (3) (1998) 477–492.
[7] P. Duclos, P. Exner, B. Meller, Exponential bounds on curvature-induced resonances in a two-dimensional Dirichlet tube, Helv. Phys. Acta 71 (2) (1998) 133–162.
[8] J. Edward, On the resonances of the Laplacian on waveguides, J. Math. Anal. Appl. 272 (1) (2002) 89–116.
[9] R. Froese, Asymptotic distribution of resonances in one dimension, J. Differential Equations 137 (2) (1997) 251–272.
[10] L. Guillopé, Théorie spectrale de quelques variétés à bouts, Ann. Scient. Ec. Norm. Sup. 22 (4) (1989) 137–160.
[11] B. Ja. Levin, Distribution of zeros of entire functions, American Mathematical Society, Providence, RI 1964 viii + 493pp.
[12] R.B. Melrose, The Atiyah–Patodi–Singer Index Theorem, A.K. Peters, Wellesley, MA, 1993.
[13] L. Parnovski, Spectral asymptotics of the Laplace operator on manifolds with cylindrical ends, Int. J. Math. 6 (1995) 911–920.

[14] T. Regge, Analytic properties of the scattering matrix, Nuovo Cimento 8 (5) (1958) 671–679.
[15] G. Vodev, Resonances in the Euclidean scattering, Cubo Matemática Educacional 3 (1) (2001) 317–360.
[16] M. Zworski, Distribution of poles for scattering on the real line, J. Funct. Anal. 73 (2) (1987) 277–296.
[17] M. Zworski, Counting scattering poles. in: Spectral and scattering theory (Sanda, 1992), Lecture Notes in Pure and Applied Mathematics, Vol. 161, Dekker, New York, 1994, pp. 301–331.
[18] M. Zworski, Quantum resonances and partial differential equations, Proceedings of the International Congress of Mathematicians, Vol. III (Beijing, 2002), Higher Ed. Press, Beijing, 2002, pp. 243–252.