Journal of Pure and Applied Algebra 35 (1985) 157-169
North-Holland

THE DIAGONAL OF A POINTED COALGEBRA AND INCIDENCE-LIKE STRUCTURE

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Communicated by A. Heller
Received 14 June 1976
Revised 10 November 1982 and 15 December 1983

0. Introduction

Vector space decompositions of a pointed coalgebra C over a field reflecting properties of its diagonal map are used by Sweedler [11] to classify the coalgebra, by Heyneman and Radford ([6], [10]) to discuss coreflexivity, and by Taft and Wilson (e.g. [13]) to obtain results about Hopf algebras from the underlying coalgebra. However, this type of structure has been classified in full only in specific cases, and the behavior of the diagonal on a general pointed coalgebra is known only to the extent obtained in the above. Using the structure of the first term of the coradical filtration (cf. [10], [11], [13]), Taft and Wilson [13] gave a vector-space decomposition yielding some information about the highest-weight terms in the diagonal (cf. 1.3 below). In addition, the complete structure of the diagonal is known for incidence coalgebras (on a partially-ordered set), and for those coalgebras which are a sum of their pointed irreducible components (PIC's). Subsequently, other papers have treated the structure of coalgebras from other points of view

In this paper, the result of Taft and Wilson is refined to obtain a generalization of the PIC case (2.4 and 2.6), which agrees with the natural structure in the case of incidence coalgebras. Further refinements (3.1 and 3.6) are obtained by the use of certain invariants of the coalgebra, yielding a characterization of PIC coalgebras (3.2).

Also, the class of generalized incidence coalgebras is characterized by use of these invariants, and the subclass of standard incidence coalgebras is isolated by homological arguments (4.6 and 4.8).

Feinberg [3] has another method of distinguishing certain incidence coalgebras, which appears to have a little in common with this approach. He also studies the significance of $H^1(P,k)$ and $H^1(P,k^{\#})$ in the incidence coalgebra. Graves [5] studies the (dual) Hochschild cohomology of incidence coalgebras, and obtains results which appear to be local analogues of 4.8. He also has results for other low-index cohomology groups. Ferrar and Allen [4] have results on (non-necessarily coassociative) incidence coalgebras which relate to Section 4.

1. Preliminaries

Let k be a given field with multiplicative group $k^{\#}$, and let all vector spaces and tensors be over k. Let Z denote the integers and P the positive integers. For a coalgebra C, Δ denotes its diagonal and ε its counit.

The coradical filtration and the wedge

The *wedge* $V \wedge_C W$ of two subspaces of C is defined by

$$V \wedge_C W = \Delta^{-1}(V \otimes C + C \otimes W).$$

If X is a subcoalgebra of C, then so are $\Lambda^{(0)}X = X$, $\Lambda^{(n+1)}X = (\Lambda^{(n)}X) \wedge X$, and $\Lambda^{(\infty)}X = \bigcup_{n=0}^{\infty} \Lambda^{(n)}X$. Note $\Lambda^{(n)}X \subset \Lambda^{(n+1)}X$. A subcoalgebra of C is *simple* if C has no proper subcoalgebras. The coradical C_0 of C is the sum of all simple subcoalgebras. The terms of the *coradical filtration* of C are given by $C_n = \Lambda^{(n)}C_0$. Then we have $C = C_\infty$.

For D and E subcoalgebras of C, D *generates* E iff $\Lambda_C^{(\infty)}D = E$. From [6], we have the following proposition:

Proposition 1.1. (1) *If D is a subcoalgebra of C, then $D_n = D \cap C_n$ for all n.*

(2) *If $\Lambda^{(\infty)}D = C$, then $C_0 \subset D$.*

(3) *If $\Lambda_C^{(\infty)}D = E$, then $\Lambda_E^{(n)}D = \Lambda_C^{(n)}D$, for $n \geq 0$.*

Grouplikes, nearly-primitives, and pseudo-primitives

$g \in C$ is *grouplike* if $\varepsilon(g) = 1$ and $\Delta g - g \otimes g = 0$. The set of all grouplikes of C is denoted $G(C)$, and there is a correspondence between $G(C)$ and the set of one-dimensional subcoalgebras of C. The *grouplike coalgebra* $C(S; k)$ on a set S has as basis the elements of S, with $\Delta s = s \otimes s$ and $\varepsilon(s) = 1$ for each $s \in S$. A coalgebra C is *pointed* if C_0 is a grouplike coalgebra. We henceforward assume all coalgebras are pointed. For a surjective map of pointed coalgebras, $f: C \to D$, we have $D_0 = f(C_0)$. $p \in C$ is a *nearly-primitive* if $\varepsilon(p) = 0$, and if there are grouplikes g and h with $\Delta p - g \otimes p - p \otimes h = 0$. A nearly-primitive with $g = h$ is called an h-primitive. $p \in C$ is a *pseudo-primitive of degree s* for $s \in P$, if $p \notin C_{s-1}$, if $\varepsilon(p) = 0$, and if there are grouplikes g and h with

$$\Delta p - g \otimes p - p \otimes h \in C_{s-1} \otimes C_{s-1}.$$

p is a *pseudo-primitive* if it is a pseudo-primitive of some degree.

For a coalgebra C, the *augmentation coideal* of C, C^+, is $\ker \varepsilon$. For any subcoalgebra $D, D^+ = \ker \varepsilon \,|\, D = C^+ \cap D$. All nearly- and pseudo-primitives are in C^+. p is a *(g,h)-nearly-primitive* (resp. pseudo-primitive) if p is nearly-primitive (resp. pseudo-primitive) with respect to grouplikes g and h.

We have directly:

Proposition 1.2. (1) *If p is nearly-primitive, then $p \in C_1$.*

(2) *If p is a (g,h)-nearly-primitive and $p \in C_0$, then $p \in k(g-h)$.*

(3) *If p is pseudo-primitive of degree s, then $p \in C_s$. Thus, s is unique.*

(4) *If p is a pseudo-primitive of degree of $s \geq 2$, then*

$$\Delta p - g \otimes p - p \otimes h \in \sum_{i=1}^{s-1} C_i \otimes C_{s-i}.$$

(5) *p is a pseudo-primitive for at most one pair of group-likes.*

(6) *If p is nearly primitive, either $p \in C_0^+$, or p is a pseudo-primitive of degree* 1.

Gradings and filtrations

A vector space V is *graded* by $\{V_i\}_{i=0}^{\infty}$ if $V = \bigoplus V_i$ and *filtered* by $\{W_i\}_{i=0}^{\infty}$ if $W_i \subseteq W_{i+1}$ for all i and $V = \bigcup_{i=0}^{\infty} W_i$. The partial sums, $\bigoplus_{j=0}^{i} V_j$, of a grading give an *associated filtration* FV; and complementary subspaces, W_i/W_{i-1}, of a filtration give an *associated grading* GV. All vector spaces have a trivial grading $G_0D = D$, $G_iD = \{0\}$ for $i \in P$, and a corresponding trivial filtration $F_iD = D$ for all $i \geq 0$.

The direct sum of two graded (filtered) vector spaces has the grading (filtration) $(A \oplus B)_i = A_i \oplus B_i$, and the intersection of two filtered vector spaces behaves similarly. Also, the tensor product is graded (filtered) by $(A \otimes B)_i = \sum_{j+r=i} A_j \otimes B_r$, where the sum in the graded case is direct.

A coalgebra C with a grading GC is a graded coalgebra if, for each $c \in G_iC$, $\Delta c \in (GC \otimes GC)_i$ (and similarly for filtrations). Graded and filtered algebras are defined dually. The coradical filtration is always a coalgebra filtration. We will always take C as so filtered, and will use, in addition, the (vector-space) filtration $F_0C = \{0\}$, $F_iC = C_i$ for $i \in P$. For X any graded vector space, coalgebra, or algebra, we have $GFX \simeq X$, but $FGX \simeq X$ only for vector spaces in general.

Let $K = \bigoplus_{i=1}^{\infty} K_i$ (with $K_0 = \{0\}$) be a graded subspace of C with $K \subset \ker \varepsilon$, so that, for each $i \in P$, $K_i \oplus C_{i-1} = C_i$, and let $L = FK$ be the associated filtered subspace. K is a *graded complementary subspace* (g.c.s.) of C. $C_0 \oplus K$ is a (vector space) grading of C whose associated filtration, $C_0 \oplus L$, is the coradical filtration.

Let $L_{g,h}$ (resp., $K_{g,h}$) $= \{p \in K : p$ is a (g,h)-pseudo-primitive$\} \cup \{0\}$ with the induced filtration (resp., grading). Then $K_{g,h}$ is a graded subspace of K. Taft and Wilson [13] have shown:

Proposition 1.3. (1) *$K_i = \bigoplus \{K_{g,h,i} : g, h \in G\}$ for each $i \in P$.*

(2) *There is a choice of K so that, for $P \in K_{g,h,i}$,*

$$\Delta p - g\otimes p - p\otimes h \in (FC\otimes FC)_i.$$

Thus K can be chosen with K_1 spanned by nearly primitives.

Define projections π_i and $\pi_{g,h}$ to be the natural projections of C onto K_i and $K_{g,h}$, respectively, and π_0 to be the projection of C onto C_0. Further, let $\pi_{g,h,i} = \pi_i \circ \pi_{g,h}$ for $i\in P$, and $\pi_{g,h,0}$ be the composition of π_0 with the projection of C_0 onto kg if $g=h$, and 0 if $g\neq h$.

The set of pseudo-primitives is related to those in K by:

Proposition 1.4. *c is a (g,h)-pseudo-primitive of degree i iff $c=p\oplus d$, for $0\neq p\in K_{g,h,i}$, and $d\in C_{i-1}$.*

Proof. If c is (g,h)-pseudo-primitive of degree i, then $c=p\oplus d$ for $d\in C_{i-1}$ and $p\in K_i$ (by 1.2) Computation of $\Delta(c-d)$ shows $p\in K_{g,h,i}$.

Conversely, if $c=p\oplus d, c$ is seen to be pseudo-primitive.

2. Graded complementary subspaces and the diagonal

Let C be a pointed coalgebra over a field k.

Theorem 2.1. *There is a g.c.s. K for C so that, for all $p\in L_{g,h}$,*

$$\Delta p - g\otimes p - p\otimes h \in \sum_{v\in G} L_{g,v}\otimes L_{v,h}.$$

Remarks. This theorem, together with the next, can be seen from the examples of cocommutative and incidence coalgebras to be as specific a result as possible for the diagronal of an arbitrary pointed coalgebra.

The proof of the theorem can be outlined as follows: Say (C,K) has property $(*)$ if K is a g.c.s. of C for which the diagonalization formula of the theorem holds. By 1.3, there is a K_1 so that (C_1,K_1) has property $(*)$. We now proceed by induction: Assume $(C_{i-1}, K=\bigoplus_{j=1}^{i-1}K_j)$ has property $(*)$. We show

(1) there is an $R=R_i$ complementary to C_{i-1} in C_i with projection of ΔR on $C_0\otimes C_0$ being 0; and

(2) there is K_i with $K\oplus R\simeq K\oplus K_i$ as graded subspaces of C (and $K\oplus R=K\oplus K_i$ as subspaces) so that $(C_i, K\oplus K_i)$ has property $(*)$.

Lemma 2.2. (1) *For $p\in K_{g,h,i}$,*

$$\Delta p - g\otimes p - p\otimes h \in (L\otimes L)_i + ((C_0\oplus L)\otimes(C_0\oplus L))_{i-1}\oplus(C_0\otimes C_0).$$

(2) *If (C,K) has property $(*)$, then L_i is a coideal of C for each i.*

Proof. (1) By 1.3,

$$\Delta p - g\otimes p - p\otimes h \in (FC\otimes FC)_i \subset ((C_0\oplus L)\otimes((C_0\oplus L))_i,$$

and the terms $L_i\otimes C_0$ and $C_0\otimes L_i$ do not occur.

(2) follows from (*) and 1.2.

Lemma 2.3. *Let (C_{i-1}, K) have property (*). Then there is an $R=R_i$ complementary to C_{i-1} in C_i with, for each $c\in R_i$, $(\pi_0\otimes\pi_0)\Delta c=0$.*

Proof. Let ψ be the natural projection of C_i onto $C_i/L=D$, and let $N=N_1$ be a g.c.s. of D spanned by nearly-primitives. Choose $R\subset\psi^{-1}(N)$ to be complementary to C_{i-1} in C_i. $K\oplus R$ is a g.c.s. of C_i so $R=\bigoplus R_{g,h}$. Let $c\in R_{g,h}$. Then

$$\Delta\psi c - \psi g\otimes\psi c - \psi c\otimes\psi h = 0, \quad \text{or} \quad (\psi\otimes\psi)(\Delta c - g\otimes c - c\otimes h)=0.$$

whence

$$\Delta c - g\otimes c - c\otimes h \in \ker(\psi\otimes\psi), \quad \text{or} \quad (\pi_0\otimes\pi_0)\Delta c = 0.$$

Proof of Theorem 2.1. Assume (C_{i-1}, K) has property (*) and R satisfies the conclusions of Lemma 2.3, and let $c\in R_{g,h}$. Then

$$\Delta c\in g\otimes c + c\otimes h + (L\otimes L)_i \oplus (C_0\otimes L_{i-1})\oplus(L_{i-1}\otimes C_0)$$

(by Lemma 2.2).

We would like to eliminate the last two terms in the summation, first finding an element $c_1\in c+L_{i-1}$ whose diagonal has zero projection on $L_{i-1}\otimes C_0$. Let $\psi=\sum_{j=1}^{i-1}\pi_j$ be the projection of $C_i=C_0\oplus L_{i-1}\oplus R$ onto L_{i-1}, and let π_i be the projection onto R. Also, let $\zeta_g=\varepsilon\circ\pi_{g,g,0}$. Using the formula for Δc and comparing terms of $(\Delta\otimes I)\Delta c$ and $(I\otimes\Delta)\Delta c$ in $L_{i-1}\otimes C_0\otimes C_0$, we have

$$\begin{aligned}(\psi_0\otimes\pi_0\otimes\pi_0)\Delta^2 c &= (I\otimes\Delta)(\psi\otimes\pi_0)\Delta c\\ &= (I\otimes\pi_0\otimes I)(\Delta\otimes I)(\psi\otimes\pi_0)\Delta c + ((\psi\otimes\pi_0)\Delta c)\otimes h,\end{aligned}$$

which, for $(\psi\otimes\pi_0)\Delta c=\sum_{b\in G} l_b\otimes b$, implies

$$\sum_{b\in G} l_b\otimes b\otimes b = \sum_{b\in G}(I\otimes\pi_0)\Delta l_b\otimes b + \sum_{b\in G} l_b\otimes b\otimes h.$$

Applying $(I\otimes\zeta_u\otimes\zeta_v)$ gives, for $v\neq h$,

$$(I\otimes\zeta_u)\Delta l_v = \delta_{uv} l_v, \quad \text{so} \quad (I\otimes\pi_0)\Delta l_v = l_v\otimes v;$$

and for $v=h$, $l_h=-\sum_{b\neq h} l_b$. Thus

$$(\psi\otimes\pi_0)\Delta c = \sum_{b\neq h} l_b\otimes b - \left(\sum_{b\neq h} l_b\otimes h\right).$$

Letting $c_1=c-\sum_{b\neq h} l_b$, we have $(\psi\otimes\pi_0)\Delta c_1=0$.

We would like to use the same technique to find a $c_2\in c_1+L_{i-1}$ with

$(\pi_0 \otimes \psi)\Delta c_2 = 0$, but first must check that we do not reintroduce terms in $L_{i-1} \otimes C_0$. Let $(\pi_0 \otimes \psi)\Delta c_1 = \sum_{b \in G} b \otimes r_b$. Then

$$(\pi_0 \otimes \psi \otimes \pi_0)\Delta^2 c_1 = \sum_{b \in G} b \otimes r_b \otimes h = \sum_{b \in G} b \otimes (I \otimes \pi_0)\Delta r_b,$$

which, applying $(\zeta_u \otimes I \otimes \zeta_v)$, yields $r_b \in \sum_{q \in G} K_{q,h}$. We now consider

$$(\pi_0 \otimes \pi_0 \otimes \psi)\Delta^2 c = \sum_{b \in G} g \otimes b \otimes r_b + \sum b \otimes (\pi_0 \otimes I)\Delta r_b$$
$$= \sum_{b \in G} b \otimes b \otimes r_b.$$

Applying $(\zeta_u \otimes \zeta_v \otimes I)$, we find that, for $b \neq g$, $(\zeta_v \otimes I)\Delta r_b = \delta_{vb} r_b$, so $r_b \in \sum_{n \in G} K_{b,n}$, which implies that

$$r_b \in \left(\sum_{q \in G} K_{q,h}\right) \cap \left(\sum_{n \in G} K_{b,n}\right) = K_{b,h};$$

and that $r_g = -\sum_{v \neq g} r_v$. Thus

$$(\pi_0 \otimes \psi)\Delta c_1 = \sum_{v \neq g} v \otimes r_v - g \otimes \left(\sum_{v \neq g} r_v\right).$$

Letting $c_2 = c_1 - \sum_{v \neq g} r_v$, we have

$$(\pi_0 \otimes \psi)\Delta c_2 = 0 = (\psi \otimes \pi_0)\Delta c_2,$$

or,

$$\Delta c_2 - g \otimes c_2 - c_2 \otimes h \in (L \otimes L)_i.$$

Thus it remains to show that the terms in $(L \otimes L)_i$ are also in $\sum_{v \in G} L_{g,v} \otimes L_{v,h}$. For each pair (g,h), choose a basis $\{c_\alpha\}$ for $R_{g,h}$, and let $K_{g,h,i} = \mathrm{span}\{(c_\alpha)_2\}$. For $c \in K_{g,h,i}$, write

$$\Delta c - g \otimes c - c \otimes h \in \sum L_{u,v} \otimes L_{x,y},$$

where the summation is taken over only those $(u, v, x, y) \in G^4$ which make non-zero contributions to the sum.

Then consideration of $(\psi \otimes \pi_0 \otimes \psi)$, $(\pi_0 \otimes \psi \otimes \psi)$, and $(\psi \otimes \psi \otimes \pi_0)$ applied to $\Delta^2 c$ show, respectively, that $v = x$, $u = g$, and $y = h$, so

$$\Delta c - g \otimes c - c \otimes h \in \sum_{v \in G} L_{g,v} \otimes L_{v,h}.$$

This shows that $C_{g,h,i} = (C_{i-1} \oplus K_{g,h,i}, K \oplus K_{g,h,i})$ has property (*), from which so does $(C_i, K \oplus K_i)$ where $C_i = \sum_{g,h} C_{g,h,i}$, and $K_i = \bigoplus_{g,h} K_{g,h,i}$. This completes the induction and the proof.

Henceforward we assume that all g.c.s.'s K of C have been chosen so that (C, K) has property (*).

Note that if C is the sum of its pointed irreducible components, i.e. $C = C_0 \oplus \sum_{g \in G} K_{g,g}$, then the theorem states that, for $c \in K_{g,g,i}$,

$$\Delta c - g\otimes c - c\otimes g \in (L_{g,g}\otimes L_{g,g})_i.$$

This result has long been known for the cocommutative and the irreducible cases.

Corollary 2.4. *For any $c\in C$:*
(1) $(\pi_{g,h}\otimes\pi_{s,t})\Delta c=0$ *if* $h\neq s$.
(2) *If* $(\pi_{g,r}\otimes\pi_{r,h})\Delta c\neq 0$, *then* $\pi_{g,h}c\neq 0$.
(3) *If* $(\pi_{g,\upsilon,i}\otimes\pi_{\upsilon,h,j})\Delta c\neq 0$, *then* $\pi_{g,h,s}c\neq 0$ *for some* $s\geq i+j$.

The following theorem provides a partial converse for the corollary, and, in so doing, describes the highest-weight terms of ΔC_i in $C\otimes C$, thus completing the general picture of the diagonal.

Lemma 2.5. *If $c\in C_i$ and $c\notin C_{i-1}$, then either*

$$(\pi_1\otimes\pi_{i-1})\Delta c\neq 0 \quad\text{or}\quad (\pi_{i-1}\otimes\pi_1)\Delta c\neq 0.$$

Proof. For $i=1$, this is clear.
For $i\geq 2$, note

$$\Delta c\in C_0\otimes C_i + C_i\otimes C_0 + K_{i-1}\otimes K_1 + K_1\otimes K_{i-1} + C_{i-2}\otimes C_{i-2}.$$

So if $(\pi_1\otimes\pi_{i-1})\Delta c$ and $(\pi_{i-1}\otimes\pi_1)\Delta c$ were 0, then

$$\Delta c\in C_0\otimes C_i + C_i\otimes C_0 + C_{i-2}\otimes C_{i-2}\subset C_{i-2}\otimes C + C\otimes C_0,$$

which would imply $c\in C_{i-1}$.

Theorem 2.6. *For all $0\leq i\leq s$, and all $0\neq c\in K_{g,h,s}$, there is a $\upsilon(i)$ (depending on c) such that $(\pi_{g,\upsilon(i),i}\otimes\pi_{\upsilon(i),h,s-i})\Delta c\neq 0$.*

Proof. This is clear if $i=0$ or $i=s$, and so, a fortiori, for $C=C_1$. Suppose the conclusion of the theorem holds for C_{s-1}, and $c\in K_{g,h,s}$. Then, by 2.4 and 2.5, there is either a $\upsilon(1)$ with $(\pi_{g,\upsilon(1),1}\otimes\pi_{\upsilon(1),h,s-1})\Delta c\neq 0$, or a $\upsilon(s-1)$; without loss of generality assume $\upsilon(1)$.

By induction, there is a $\upsilon(i)$ so that

$$(I\otimes\pi_{\upsilon(1),\upsilon(i),i-1}\otimes\pi_{\upsilon(i),h,s-i})(I\otimes\Delta)(\pi_{g,\upsilon(1),1}\otimes\pi_{\upsilon(1),h,s-1})\Delta c\neq 0.$$

But since $c\in C_s$, this is equal to

$$\begin{aligned}&(\pi_{g,\upsilon(1),1}\otimes\pi_{\upsilon(1),\upsilon(i),i-1}\otimes\pi_{\upsilon(i),h,s-i})\Delta^2 c\\ &\quad=(\pi_{g,\upsilon(1),1}\otimes\pi_{\upsilon(1),\upsilon(i),i-1}\otimes I)(\Delta\otimes I)(\pi_{g,\upsilon(i),i}\otimes\pi_{\upsilon(i),h,s-i})\Delta c,\end{aligned}$$

which shows $(\pi_{g,\upsilon(i),i}\otimes\pi_{\upsilon(i),h,s-i})\Delta c\neq 0$.

Corollary 2.7. *For any $c\in C_s$ and $0\leq i\leq s$, if $\pi_s c\neq 0$, then $(\pi_i\otimes\pi_{s-i})\Delta c\neq 0$.*

3. Incidence-type invariants of pointed coalgebras

An *invariant of (pointed) coalgebras* is a functor on the subcategory of (pointed) coalgebras with monomorphisms. For example, G is an invariant into Sets, whereas g.c.s. is not an invariant because there is no canonical choice of g.c.s. in a coalgebra. In this section we construct invariants into reflexive relations, indexed families of sets, and indexed families of subcoalgebras; and in the next section we will show that for an incidence coalgebra these are the reflexive relations, the family of intervals, and (in the special case of a partial order) the family of subcoalgebras generated by intervals, respectively. We also determine some some restrictions on choice of g.c.s. given by the invariants.

For a coalgebra C, define a relation r on the elements of G by $r(g,h)$ iff $g=h$ or $K_{g,h}\neq 0$, and let $\bar{r}$ be its transitive closure.

Let

$$N_{g,h}=\{v\in G : r(g,v) \text{ and } r(v,h)\}$$

and

$$M_{g,h}=\{v\in G : \bar{r}(g,v) \text{ and } \bar{r}(v,h)\}.$$

We then can obtain the following refinement of 2.6:

Proposition 3.1. *If $0\neq p\in K_{g,h,i}$ and $V(p)\subset N_{g,h}$ is minimal such that*

$$\Delta p-g\otimes p-p\otimes h\in \sum_{v\in V(p)} (L_{g,v}\otimes L_{v,h}),$$

then $V(p)$ is finite and contains elements $g=v(0), v(1),\ldots, v(i)=h$ (not necessarily distinct) with

$$(\pi_{v(0),v(1),1}\otimes\pi_{v(1),v(2),1}\otimes\cdots\otimes\pi_{v(i-1),v(i),1})\Delta^i p\neq 0).$$

Corollary 3.2. *A pointed coalgebra C is the sum of its irreducible components iff C_1 is cocommutative.*

Proof. If C is the sum of components, then C_1 is spanned by primitives and grouplikes. Conversely, if C_1 is cocommutative, $\pi_{g,h,1}=0$ if $g\neq h$. Thus, by 3.1, $K_{g,h}=\{0\}$ if $g\neq h$.

For g and h grouplikes, let $C(g,h)$ be the vector space

$$\left(\sum\{L_{u,v} : u,v\in M_{g,h}\}\right)+C(M_{g,h}\cup\{g,h\};k),$$

and let $C_s(g,h)=C(g,h)\cap C_s$.

Proposition 3.3. *$C(g,h)$ is a subcoalgebra of C, and $C_0(g,h)=C(M_{g,h}\cup\{g,h\};k)$.*

Proof. For $u,v\in M_{g,h}$, $C(u,v)\subset C(g,h)$.

Suppose (C,K) and (C,P) both have property $(*)$; let $L=FK$ and $C(g,h)$ be as above, with $S=FP$ and $D(g,h)$ the analogous objects for P. Note r and $M_{g,h}$ agree whether constructed via K or via P (by 1.4). Finally, let $L(g,h)=L\cap C(g,h)$, and $S(g,h)=S\cap D(g,h)$.

Lemma 3.4. *If $p\in C$ with*

$$\Delta p-g\otimes p-p\otimes h\in\sum_{v\in G}L_{g,v}\otimes L_{v,h},$$

then $p\in L_{g,h}+k(g-h)$.

Proof. It follows from 2.4 that $p\in L_{g,h}+C_0^+$, and thence (by 1.2 and 1.4), that $p\in L_{g,h}+k(g-h)$.

Lemma 3.5. *Let $B=\Delta^{(\infty)}C_0(g,h)$. Then $B_n=C_n(g,h)$.*

Proof. Clearly, $B_0=C_0(g,h)$ and $C_n(g,h)\subset\Delta^{(n)}C_0(g,h)=B_n$ for all n. But $B_1\subset C_1(g,h)$ since B_1 is spanned by nearly-primitives (by 3.3).

If $B_s=C_s(g,h)$ for all $s<n$, and $c\in B_n$, then

$$\Delta c\in B_0\otimes C+C\otimes B_{n-1}\quad\text{and}\quad\Delta c\in B_{n-1}\otimes C+C\otimes B_0.$$

Applying $(\pi_{x,y}\otimes\pi_{y,z})$ to each expression yields $(\pi_{x,y}\otimes\pi_{y,z})\Delta c=0$ unless $x,y,z\in B_0$. Thus $c=b+c_1$, for $b\in C_n(g,h)$ and $c_1\in C_1$. But $c_1\in C_1\cap B_n=B_1=C_1(g,h)$.

Proposition 3.6. *$C(g,h)=D(g,h)$. Further, $K_{g,h,i}\subset P_{g,h,i}+D^+{}_{i-1}(g,h)$.*

Proof. $C_0(g,h)=D_0(g,h)$ by 3.3, so $C_n(g,h)=\Delta^{(n)}C_0(g,h)=D_n(g,h)$. Also $K_{g,h,i}\subset D(g,h)$ and is spanned by pseudo-primitives of degree i, so

$$K_{g,h,i}\subset(P_{g,h,i}+D_{i-1}(g,h))\cap C^+.$$

Lemma 3.7. *Let D be a subcoalgebra of C, and let (D,K) have property $(*)$. Then there is a g.c.s. P of C with K a graded subspace of P.*

Proof. P_1 can be chosen to contain K_1. Examination of the proof of 2.1 then shows that, if R is chosen to contain K_i, P_i will also.

Lemma 3.8. *Let $\phi:C\to D$ be an isomorphism of coalgebras with K and P g.c.s. for C and D respectively. Then for each pair (g,h), $K_{g,h}\simeq P_{\phi g,\phi h}$ as graded vector spaces.*

Proof. ϕK is a g.c.s. for D.

Proposition 3.9. *Let $\tau: C \to D$ be a coalgebra monomorphism. Then*
(1) *$r_C(g,h)$ implies $r_D(\tau g, \tau h)$.*
(2) *$M_{g,h} \subset M_{\tau g, \tau h}$ and $N_{g,h} \subset N_{\tau g, \tau h}$.*
(3) *$C(g,h)$ is a subcoalgebra of $D(\tau g, \tau h)$.*

Proof. τ is the composition of an inclusion with an isomorphism.

Let $M = \{M_{g,h} : g, h \in G\}$, $N = \{N_{g,h} : g, h \in G\}$, and $C(G) = \{C(g,h) : g, h \in G\}$. We have shown:

Theorem 3.10. *r, M, N, and $C(G)$ are invariants of pointed coalgebras.*

Since the composition of functors is a functor, any functor on reflexive relations (or on indexed families of sets or of coalgebras) is an invariant of pointed coalgebras.

Let $A_*(r)$ be the free abelian chain complex with n-simplices all relation-preserving maps from $[n] = \{0, 1, \ldots, n\}$ into r, with the usual boundary, and let $A^*(r)$ be the cochain complex of group homomorphisms from $A_*(r)$ to $k^\#$, with coboundary δ. For a relation-preserving map $\phi: r \to r$, let $\bar{\phi}$ be the map induced on intervals of r, and $\phi^\#$ be the map induced on the cochains. Then the homology and cohomology groups of A are invariants of pointed coalgebras, (cf. Farmer [2]).

Finally, consider the category of indexed families of integers, $\{n_\alpha\}_{\alpha \in I}$, with morphisms all maps of index sets, $\phi: I \to J$, such that $n_{\phi(\alpha)} \geq n_\alpha$ for all $\alpha \in I$; and let $\dim_C = \{\dim K_{g,h,i} : g, h \in G, i \in P\}$. Then:

Proposition 3.11. *$\dim_C$ is an invariant of pointed coalgebras.*

Proof. Let (C, K) have property $(*)$, and let $D = C_i / L_{i-1}$. Let $X_{g,h} = \{d \in D : d$ is a (g,h)-nearly-primitive$\}$. $X_{g,h}$ is independent of choice of g.c.s. having property $(*)$ (by 3.6), and (by 3.3), $\dim K_{g,h,i} = \dim X_{g,h} - 1 + \delta_{g,h}$.

Direct computation shows $\dim_C$ is a functor.

Corollary 3.12. *$\dim_{C_1}$ is an invariant of pointed coalgebras.*

However $A = \dim_{C_1}$ can be viewed as a $G \times G$ matrix with cardinal number entries, and we can compute its powers A^i in the standard way.

Proposition 3.13. $\dim K_{g,h,i} \leq (A^i)_{g,h}$.

Proof. If $\{v_\alpha\}_{\alpha \in A}$ are linearly independent elements of $K_{g,h,i}$, then their projections $\pi_1^{i+1} \Delta^i v_\alpha$ must also be linearly independent. But these are sums of i-chains in $N_{g,h} \subset M_{g,h}$ (compare 3.1). But there are only $(A^i)_{g,h}$ i-chains in $M_{g,h}$.

4. The structure of incidence coalgebras

Let R be a locally-finite antisymmetric reflexive relation with underlying set S. R is *right locally-transitive* iff for each $x \in S$, R restricted to $\{y \in S : x R y\}$ is transitive, and *left locally-transitive* iff R restricted to $\{y \in S : y R x\}$ is. R is *locally-transitive* if it is both left and right locally-transitive. Call a locally-transitive locally-finite antisymmetric reflexive relation an admissible relation, and for such a relation R, let $C(R)$ be the free vector space on its non-empty intervals ($[g,h] = \{v \in S : g R v$ and $v R h\}$ if $g R h$, and is empty otherwise). For $\alpha \in A^2(R)$, define linear maps $\varepsilon : C(R) \to k$ and $\Delta_\alpha : C(R) \to C(R) \otimes C(R)$ by

$$\varepsilon[g,h] = \delta_{g,h} \quad \text{and} \quad \Delta_\alpha[g,h] = \sum_{g R v R h} \alpha(g,v,h)[g,v] \otimes [v,h].$$

Then $C(R,\alpha) = (C(R), \Delta_\alpha, \varepsilon)$ is a coalgebra iff α is a normalized cocycle. The coalgebras $C(R,\alpha)$ are *generalized incidence coalgebras* on R; $C(R,1)$ is the *standard incidence coalgebra*. Let $C_n(R,\alpha) = (C(R,\alpha))_n$. The length of a chain in R is the number of elements in the chain, and the dimension of $[g,h]$ is one less than the maximum length of chains from g to h. We identify the interval $[g,g]$ with the point g.

Proposition 4.1. (1) $C_0(R,\alpha) = C(R;k)$

(2) $C_i(R:\alpha) \supset \operatorname{span}\{[g,h] : \dim[g,h] \le i\}$.

(3) *If* $\dim[g,h] = t$, *then* $[g,h] \notin C_{t-1}(R,\alpha)$.

Proof. (1) $C(R;k) \subset C_0(R,\alpha)$ and $\Lambda^{(\infty)}C(R;k) = C(R,\alpha)$. (cf. 1.1).

(2) $C_i(R,\alpha) = \Lambda^{(i)}C(R;k)$.

(3) $(\bigoplus^t \pi_1)\Delta^{t-1}[g,h] \neq 0$, but $(\bigoplus^t \pi_1)\Delta^{t-1} \mid C_{t-1}(R,\alpha) = 0$.

Corollary 4.2. (1) $[g,h]$ *is a* (g,h)*-pseudo-primitive of degree* $\dim[g,h]$.

(2) *If* $\dim[g,h] = i$, *then* $K_{g,h,i} \neq 0$.

Proposition 4.3. *Let* $K_i = \operatorname{span}\{[g,h] \mid \dim[g,h] = i\}$. *Then* K *is a g.c.s. of* $C(R,\alpha)$ *and* $(C(R,\alpha), K)$ *has property* $(*)$.

Proof. $C(R)$ is spanned by intervals.

Corollary 4.4. (1) *The relation* r *defined by* $C(R,\alpha)$ *is* R.

(2) *If a coalgebra* C *is isomorphic to an incidence coalgebra* $C(R,\alpha)$, *then* $R \simeq (G(C), r_C)$.

Note that every incidence coalgebra satisfies, for every $g,h \in G$:

(I) $\dim(K_{g,h} + K_{h,g}) \le 1 - \delta_{g,h}$.

(II) If $0 \neq \pi_{g,v,i}$ and $0 \neq \pi_{v,h,j}$, then $\pi_{g,h} = \sum_{s \ge i+j} \pi_{g,h,s}$.

(III) If $0 \neq c \in K_{g,h}$ and $v \in N_{g,h} - \{g,h\}$, then $(\pi_{g,v} \otimes \pi_{v,h})\Delta c \neq 0$.

Let D be any pointed coalgebra satisfying properties (I), (II), and (III).

Lemma 4.5. (1) *r_D is an admissible relation.*

(2) *If $g \neq h$, then the dimension of the interval $[g,h]$ in r_D is $n > 0$ iff $K_{g,h,n} \neq 0$.*

Proof. (1) r_D is antisymmetric since D is pointed, and locally-finite since D is a coalgebra (and reflexive by definition). Property (II) implies local transitivity.

(2) $K_{g,h} \neq 0$ iff $\dim[g,h] > 0$. Assume $K_{g,h} = K_{g,h,m}$. Then (by antisymmetry and 3.1) $m \leq \dim[g,h]$. But if s is minimal such that $K_{g,h,s} \neq 0$ and $\dim[g,h] > s$, then property (II) gives a contradiction.

Theorem 4.6. *If D is a coalgebra satisfying properties* (I), (II), *and* (III), *then $D \cong C(R,\alpha)$ for some α.*

Proof. Choose, in each non-zero $K_{g,h}$, a (non-zero) element $\langle g,h\rangle$; and, for $\upsilon \in N_{g,h} - \{g,h\}$, let

$$(\pi_{g,\upsilon} \times \pi_{\upsilon,h})\Delta\langle g,h\rangle = \alpha(g,\upsilon,h)\langle g,\upsilon\rangle \otimes \langle \upsilon,h\rangle.$$

By property (III), $\alpha(g,\upsilon,h) \neq 0$, so, defining $\alpha(g,g,h) = \alpha(g,h,h) = 1$, $D \cong C(R,\alpha)$.

Corollary 4.7. *$D \cong C(P,\alpha)$ for P a partially ordered set iff D satisfies* (I), (III), *and*

(II′) *if $0 \neq \pi_{g,\upsilon,i}$ and $0 \neq \pi_{\upsilon,h,j}$, then $0 \neq \pi_{g,h} = \sum_{s \geq i+j} \pi_{g,h,s}$.*

The remainder of the section determines conditions under which two such coalgebras will be isomorphic (clearly the admissible relations must be isomorphic).

Lemma 4.8. (1) *Let ϕ be an automorphism of R, and $\varrho = \phi^{-1}$. Then $\bar{\phi} : C(R,\alpha) \to C(R, \varrho^{\#}\alpha)$ is an isomorphism.*

(2) *If α and γ are cohomologous, then $C(R,\alpha) \cong C(R,\gamma)$.*

Proof. (2) If $\alpha\gamma^{-1} = \delta\beta$, then β is normalized and $\bar{\beta}$ defined by $\bar{\beta}([g,h]) = \beta(g,h)[g,h]$ is a coalgebra isomorphism from $C(R,\gamma)$ to $C(R,\alpha)$.

Theorem 4.9. *$C(R,\alpha) \cong C(R,\gamma)$ iff there is an automorphism ϱ of R with α and $\varrho^{\#}\gamma$ cohomologous.*

Proof. ($\Leftarrow$) Let $\varrho^{-1} = \phi$, and $\alpha(\varrho^{\#}\gamma)^{-1} = \delta\beta$, and apply 4.8.

($\Rightarrow$) Let ψ be the isomorphism, and let $G\psi = \phi : R \to R$, with $\varrho = \phi^{-1}$. Then $\psi \circ \varrho : C(R, \varrho^{\#}\gamma) \to C(R,\alpha)$ is a coalgebra isomorphism with $G(\psi \circ \varrho) = \mathrm{id}_R$.

Thus it is sufficient to show: If $\psi : C(R,\alpha) \to C(R,\gamma)$ is an isomorphism with $G\psi = \mathrm{id}_R$, then α is cohomologous to γ.

Let $C_{s-1}(g,h)$ denote $(C_{s-1}(R,\alpha))(g,h)$, and define a cochain $\beta \in A^1(R)$ by $\psi[gh] \in \beta(g,h)[g,h] + C_{s-1}^{+}(g,h)$; $\beta(g,g) = 1$. Then, comparing $(\pi_{g,\upsilon} \otimes \pi_{\upsilon,h})\Delta\psi[g,h]$ and $(\pi_{g,\upsilon} \otimes \pi_{\upsilon,h})(\psi \otimes \psi)\Delta[g,h]$ yields

$$\alpha(g, \upsilon, h) = \gamma(g, \upsilon, h)\ \beta(\upsilon, h)\ \beta(g, h)^{-1}\ \beta(g, \upsilon), \quad \text{or} \quad \alpha\gamma^{-1} = \beta.$$

Corollary 4.10. (1) $C(R, \alpha) \cong C(R, 1)$ *iff* α *is a coboundary.*

(2) *If* $H^2(R, k^{\#}) \neq 1$, *then there is a non-standard incidence coalgebra on* R.

Thus if P is the partially-ordered set $\{e_1, e_2 f_1, f_2, g_1, g_2\}$ with the order $e_i < f_j < g_s$ for all i, j, s (i.e. the usual decomposition of S^2), then there is a distinct coalgebra for each $a \in k^{\#}$, where $\alpha(e_1, f_1, g_1) = a$, $\alpha = 1$ otherwise. It can be shown that the following locally-finite partially-ordered sets have $H^2(P, k^{\#}) = (0)$, and so support no non-standard incidence coalgebras: trees, directed sets and their duals, and certain finite (or initially or terminally finite) sets (cf. [1] and [8]).

That the generalization to reflexive relations is not trivial can be seen from the following two examples. First, consider the set $\{a_i\}_1^n$, with $a_i R a_j$ iff $i - j \leq m \bmod n$ (the m-transitive n-circle). Then, for $m < [n/2]$, this is an admissible relation whose transitive closure is not a partially-ordered set. Second, for the partially-ordered set P above, consider $Q = P \cup \{h\}$, where $h R e_i$ and $h R f_i$ (but not $h R g_i$). Then $\bar{R}$ is a partial order with trivial second cohomology, but Q can be viewed as the union on $S^1 = \{f_1, f_2, e_1, e_2\}$ of E^2 and S^2, and its second cohomology group seen to be free on two generators (by Mayer–Vietoris). Q has the cohomology of a wedge of two 2-spheres.

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