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Approximate dynamical systems on interval

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ABSTRACT

Let I be a an open real interval. We show that if a function $H : I \times \mathbb{R} \to I$ satisfies the inequality

$$|H(H(x_0, s), t) - H(x_0, s + t)| \le \delta \quad \text{for } s, t \in \mathbb{R}$$

with a $\delta \ge 0$ and an $x_0 \in I$ such that the function $H(x_0, \cdot)$ is a continuous surjection of $\mathbb{R}$ onto I, then there exists a dynamical system F on I such that

$$|H(x, t) - F(x, t)| \le 9\delta \quad \text{for } x \in I, t \in \mathbb{R}.$$

1. Introduction

Let I be a real interval. The translation equation, i.e. a functional equation of the form

$$F(F(x, s), t) = F(x, s + t) \tag{1}$$

plays a very important role in the theory of functional equations and iteration theory. It belongs to the class of composite type functional equations. Every continuous solution $F : I \times \mathbb{R} \to I$ of (1) satisfying $F(x, 0) = x$ for $x \in \mathbb{R}$ is called *a dynamical system on* I. For more details concerning (1) and its applications we refer to [1,2]. In the present paper we deal with the case where (1) is satisfied up to some possible error. More precisely, we consider the inequality

$$|H(H(x, s), t) - H(x, s + t)| \le \delta \quad \text{for } x \in I, s, t \in \mathbb{R}, \tag{2}$$

where δ is a fixed nonnegative real number. Our considerations are inspired by a paper [3], where the stability problem for the translation equation has been studied in a very general setting, but under relatively strong assumptions on H. In a recent paper [4] an analogous question has been investigated in the case of functions mapping $I \times (0, \infty)$ into I. Stability problem for (1) in the rings of formal power series has been considered in [5]. Several details concerning stability of functional equations and a number of references can be found e.g. in [6,7].

The main result of the paper reads as follows.

Theorem 1.1. *Let I be an open real interval and δ be a nonnegative real number. Assume that a function $H : I \times \mathbb{R} \to I$ satisfies inequality*

$$|H(H(x_0, s), t) - H(x_0, s + t)| \le \delta \quad \textit{for } s, t \in \mathbb{R} \tag{3}$$

with a $\delta > 0$ and an $x_0 \in I$ such that the function $H(x_0, \cdot)$ is a continuous surjection of $\mathbb{R}$ onto I. Then there exists a homeomorphism $g : \mathbb{R} \to I$ such that

$$|H(x, t) - g(t + g^{-1}(x))| \le 9\delta \quad \textit{for } x \in I, t \in \mathbb{R}. \tag{4}$$

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Since, for every homeomorphism $g:\mathbb{R}\to I$, a function $F:I\times\mathbb{R}\to I$ of the form $F(x,t)=g(t+g^{-1}(x))$ for $x\in\mathbb{R}$, $t\in I$, satisfies (1) and, for every $x\in\mathbb{R}$, $F(x,\cdot)$ is a continuous surjection of $\mathbb{R}$ onto I, from Theorem 1.1 we derive the following stability result for (1).

Theorem 1.2. *Let I be an open real interval. The translation equation is stable in the Hyers–Ulam sense in the class of functions*

$$\mathfrak{F}_{cs}:=\{F:I\times\mathbb{R}\to I|F(x_0,\cdot)\text{ is a continuous surjection for some }x_0\in I\},$$

that is, for every $\varepsilon>0$ there is a δ $(=\frac{1}{9}\varepsilon)$ such that, for every $H\in\mathfrak{F}_{cs}$ satisfying (2), there exists $F\in\mathfrak{F}_{cs}$ satisfying (1) such that

$$|H(x,t)-F(x,t)|\le\varepsilon\quad\text{for } x\in I,\ t\in\mathbb{R}.$$

2. Proof of Theorem 1.1

Let $h:=H(x_0,\cdot)$. If h is bijective, then (see [3, p.193]), we get the assertion with $g:=h$ and δ instead of 9δ (note that in [3] the continuity of h is not assumed). So, from now on we will assume that h is not injective. A remaining part of the proof will be divided into four steps.

Step 1. We show that for every $\alpha,\beta\in\mathbb{R}$ with

$$h(\alpha)=h(\beta)\tag{5}$$

it holds

$$|h(s)-h(s+(\alpha-\beta))|\le 2\delta\quad\text{for } s\in\mathbb{R}.\tag{6}$$

Fix $\alpha,\beta\in\mathbb{R}$ and suppose that (5) is valid. Then $H(x_0,\alpha)=H(x_0,\beta)$, so in view of (3), for every $s\in\mathbb{R}$, we obtain

$$\begin{aligned}|h(s)-h(s+(\alpha-\beta))| &= |H(x_0,s)-H(x_0,s+(\alpha-\beta))|\\ &\le |H(x_0,s)-H(H(x_0,\beta),s-\beta)|+|H(H(x_0,\alpha),s-\beta)-H(x_0,s+(\alpha-\beta))|\le 2\delta.\end{aligned}$$

Step 2. For every $c\in\mathbb{R}$, let $\mathrm{J}(c)$ be a family (possibly empty) of all non-degenerated closed intervals $[u,v]\subset\mathbb{R}$ satisfying the following two conditions:

(C_1) $h(u)=h(v)=c$;

(C_2) $h(t)\le c$ for $t\in[u,v]$; or $h(t)\ge c$ for $t\in[u,v]$.

Let $\mathrm{J}:=\bigcup_{c\in\mathbb{R}}\mathrm{J}(c)$. Note that as h is not injective, there exist $\alpha,\beta\in\mathbb{R}$ such that $\alpha<\beta$ and $h(\alpha)=h(\beta)=:c$. If $h(t)=c$ for $t\in[\alpha,\beta]$ then $[\alpha,\beta]\in\mathrm{J}(c)$. If $h(t_0)\ne c$ for some $t_0\in[\alpha,\beta]$ then the sets $A^-:=\{t\in[\alpha,t_0]:h(t)=c\}$ and $A^+:=\{t\in[t_0,\beta]:h(t)=c\}$ are nonempty and closed. Therefore, taking $\alpha^-:=\max A^-$ and $\alpha^+:=\min A^+$, we get $[\alpha^-,\alpha^+]\in\mathrm{J}(c)$. So we have proved that $\mathrm{J}\ne\emptyset$.

Let

$$S:=\sup\{v-u:[u,v]\in\mathrm{J}\}.\tag{7}$$

We show that for every $s_1,s_2\in\mathbb{R}$, the following implication holds

$$|s_1-s_2|<S\Rightarrow|h(s_1)-h(s_2)|\le 2\delta.\tag{8}$$

To this end, fix $s_1,s_2\in\mathbb{R}$ such that $|s_1-s_2|<S$. Assume for instance that $s_1<s_2$. Then there exist $u,v\in\mathbb{R}$ such that $[u,v]\in\mathrm{J}$ and $s_2-s_1<v-u$. Let $c\in\mathbb{R}$ be such that $[u,v]\in\mathrm{J}(c)$ and let a function $\phi:[0,v-u-(s_2-s_1)]\to\mathbb{R}$ be defined by:

$$\phi(t)=h(u+t)-h(u+t+s_2-s_1)\quad\text{for } t\in[0,v-u-(s_2-s_1)].\tag{9}$$

Clearly, $u+s_2-s_1\in[u,v]$ and $v-(s_2-s_1)\in[u,v]$. Since $[u,v]\in\mathrm{J}(c)$, by (C_1), we have

$$\phi(0)\phi(v-u-(s_2-s_1))=[c-h(u+s_2-s_1)][h(v-(s_2-s_1))-c]\le 0.$$

As ϕ is continuous, from the latter inequality it follows that there exists a $z\in[0,v-u-(s_2-s_1)]$ such that $\phi(z)=0$, that is

$$h(u+z)=h(u+z+s_2-s_1).$$

Thus, taking $\alpha:=u+z$ and $\beta:=u+z+s_2-s_1$, we get (5). Hence according to Step 1, we obtain that

$$|h(s_1)-h(s_2)|\le 2\delta.$$

In particular, if $S=\infty$ then for every $s_1,s_2\in\mathbb{R}$ there exists an interval $[u,v]\in\mathrm{J}$ such that $s_1-s_2<v-u$. Hence, we get that $|h(s_1)-h(s_2)|\le 2\delta$ for every $s_1,s_2\in\mathbb{R}$. Thus, as h is surjective, we get $b-a\le 2\delta$. So, taking an arbitrary homeomorphism $g:\mathbb{R}\to(a,b)$, we get

$$|H(x,t)-g(t+g^{-1}(x))|\le 2\delta\quad\text{for } x\in I,\ t\in\mathbb{R}.$$

Step 3. Assume that $S < \infty$. Let $a := \inf I$ and $b := \sup I$. As h is surjective, we have either

$$\limsup_{t\to\infty} h(t) = a$$

or

$$\limsup_{t\to\infty} h(t) = b. \tag{10}$$

Since the proof in both cases is similar, assume that (10) holds. Fix a $p \in (\frac{1}{2}S, S)$. Then there exists an interval $[u, v] \in \mathrm{J}$ such that $p < v - u$. Furthermore, according to Step 2, for every $s, t \in \mathbb{R}$ it holds

$$|s - t| \le p \Rightarrow |h(s) - h(t)| \le 2\delta. \tag{11}$$

For every $n \in \mathbb{Z}$, we define an interval I_n in a following way

$$I_n := [u + 2np, u + 2(n + 1)p]. \tag{12}$$

Let

$$M_n := \max\ h(I_n) \quad \text{for } n \in \mathbb{Z}$$

and

$$m_n := \min\ h(I_n) \quad \text{for } n \in \mathbb{Z}.$$

Since the length of every I_n is $2p$, making use of (11), we conclude that

$$M_n - m_n \le 4\delta \quad \text{for } n \in \mathbb{Z}. \tag{13}$$

Furthermore, the continuity of h yields that

$$m_n \le M_{n-1} \quad \text{for } n \in \mathbb{Z}. \tag{14}$$

Therefore

$$M_n - M_{n-1} \le M_n - m_n \le 4\delta \quad \text{for } n \in \mathbb{Z}. \tag{15}$$

Now, we show that a sequence $(M_n : n \in \mathbb{N})$ is strictly increasing. For the proof by contradiction suppose that $M_n \le M_{n-1}$ for some $n \in \mathbb{N}$. Since h is continuous and (10) holds, this means that there exists an $\alpha_0 \ge u + 2(n + 1)p$ such that

$$h(\alpha_0) = M_{n-1} \tag{16}$$

and

$$h(t) \le M_{n-1} \quad \text{for } t \in [u + 2(n + 1)p, \alpha_0]. \tag{17}$$

Let $\alpha_1 \in I_{n-1}$ be such that $h(\alpha_1) = M_{n-1}$. Then, in view of (16) and (17), we obtain that

$$\alpha_0 - \alpha_1 \ge u + 2(n + 1)p - (u + 2np) = 2p > S,$$

which contradicts the definition of S. So, we have proved that the sequence $(M_n : n \in \mathbb{N})$ is strictly increasing, which together with (10) gives

$$\lim_{n\to\infty} M_n = b. \tag{18}$$

Next, note that if $\liminf_{t\to\infty} h(t) = a$, then $\liminf_{n\to\infty} m_n = a$, so by (13) and (18), we obtain

$$b - a = \liminf_{n\to\infty}(M_n - m_n) \le 4\delta.$$

Thus we get (4) with an arbitrary homeomorphism $g : \mathbb{R} \to I$. In the case where $\liminf_{t\to\infty} h(t) > a$, using the fact that h is surjective, we have $\liminf_{t\to-\infty} h(t) = a$. Therefore, arguing as previously, we obtain that

$$m_{-n} - m_{-n-1} \le 4\delta \quad \text{for } n \in \mathbb{N}, \tag{19}$$

a sequence $(m_{-n} : n \in \mathbb{N})$ is strictly decreasing and so

$$\lim_{n\to\infty} m_{-n} = a. \tag{20}$$

Let $g : \mathbb{R} \to I$ be a piecewise linear mapping such that

$$g(u + 2np) = \begin{cases} M_{n-1} & \text{for } n = 1, 2, \ldots \\ m_n & \text{for } n = 0, -1, \ldots. \end{cases} \tag{21}$$

Since the sequence $(M_n : n \in \mathbb{N})$ is strictly increasing and the sequence $(m_{-n} : n \in \mathbb{N})$ is strictly decreasing, from (18) and (20) it follows that g is a homeomorphism of $\mathbb{R}$ onto I. Furthermore, for every $n \in \mathbb{Z}$ and $t \in I_n$, we get

$$|h(t) - g(t)| \le M_n - m_n.$$

Thus, in view of (13), we obtain

$$|h(t) - g(t)| \le 4\delta \quad \text{for } t \in \mathbb{R}. \tag{22}$$

Next, making use of (15) and (19), for every $n \in \mathbb{Z}$ and $s, t \in I_n$ with $s \neq t$, we get

$$|g(s) - g(t)| \le 2\frac{\delta}{p}|s - t| \quad \text{for } s, t \in \mathbb{R}. \tag{23}$$

Now, we show that for every $x \in I$ it holds that

$$\inf\{|g^{-1}(x) - s_x| : s_x \in \mathbb{R}, h(s_x) = x\} \le 2p. \tag{24}$$

The case where $x \in [m_0, M_0]$ is obvious. If $x > M_0$ then $x \in [M_{n-1}, M_n]$ for some $n \in \mathbb{N}$. Thus, by (14), $x \in [m_n, M_n]$, so

$$\{s_x \in \mathbb{R}, h(s_x) = x\} \cap I_n \neq \emptyset.$$

On the other hand, in view of (21), we get

$$g(u + 2np) = M_{n-1}$$

and

$$g(u + 2(n+1)p) = M_n,$$

which implies that $g^{-1}(x) \in I_n$. Since the length of I_n is $2p$, this yields (24). If $x < m_0$, the similar arguments work.

Step 4. We show that the estimation (4) holds. To this end fix an $x \in I$ and a $t \in \mathbb{R}$. Then, by (24), there exists a $s_x \in \mathbb{R}$ such that

$$x = h(s_x) = H(x_0, s_x)$$

and

$$|s_x - g^{-1}(x)| \le 2p.$$

Thus, making use of (3), (22) and (23), we obtain

$$\begin{aligned} |H(x,t) - g(t + g^{-1}(x))| &= |H(H(x_0, s_x), t) - g(t + g^{-1}(x))| \\ &\le |H(H(x_0, s_x), t) - H(x_0, s_x + t)| + |H(x_0, s_x + t) - g(t + g^{-1}(x))| \\ &\le \delta + |h(s_x + t) - g(t + g^{-1}(x))| \\ &\le \delta + |h(s_x + t) - g(s_x + t)| + |g(s_x + t) - g(t + g^{-1}(x))| \\ &\le 5\delta + 2\frac{\delta}{p}|s_x - g^{-1}(x)| \le 9\delta. \end{aligned}$$

3. Concluding remarks

Let us begin this section with the following simple example showing that in general a function $H \in \mathfrak{F}_{cs}$ satisfying (2) with some positive δ can be approximate by several dynamical systems belonging to $\mathfrak{F}_{cs}$.

Example 3.1. Let $d : \mathbb{R} \to \mathbb{R}$ be given by

$$d(t) = \min\{|t - n| : n \in \mathbb{Z}\} \quad \text{for } t \in \mathbb{R}.$$

Note that

$$|d(s + t) - d(s) - d(t)| \le 1 \quad \text{for } s, t \in \mathbb{R}. \tag{25}$$

Define a function $H : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by

$$H(x, t) = x + t + d(t) \quad \text{for } x, t \in \mathbb{R}.$$

Then $H \in \mathfrak{F}_{cs}$ (with $I = \mathbb{R}$). Moreover, making use of (25), we obtain that (2) holds with $\delta = 1$. Next, given an $\alpha \in [0, 1)$, define a function $g_\alpha : \mathbb{R} \to \mathbb{R}$ by

$$g_\alpha(t) = t + \alpha d(t) \quad \text{for } t \in \mathbb{R}.$$

Then, for every $\alpha \in [0, 1)$, g_α is a homeomorphism on $\mathbb{R}$ with $g_\alpha(0) = 0$. Furthermore, we have

$$|H(0, t) - g_\alpha(t)| = (1 - \alpha)d(t) \le \frac{1 - \alpha}{2} \quad \text{for } t \in \mathbb{R} \tag{26}$$

and

$$|g_\alpha(s) - g_\alpha(t)| \le (1+\alpha)|s-t| \quad \text{for } t \in \mathbb{R}. \tag{27}$$

Now, for every $\alpha \in [0, 1)$, put

$$F_\alpha(x,t) = g_\alpha(g_\alpha^{-1}(x) + t) \quad \text{for } x, t \in \mathbb{R}.$$

Since, for every $\alpha \in [0, 1)$, it holds that $F_\alpha(0, t) = g_\alpha(t)$ for $t \in \mathbb{R}$, we conclude that $F_\alpha \in \mathfrak{F}_{cs}$ and $F_\alpha \ne F_\beta$ whenever $\alpha \ne \beta$. Note also that similarly as in the proof of Theorem 1.1, we obtain that for every $x \in \mathbb{R}$ there is a $s_x \in \mathbb{R}$ such that $H(0, s_x) = x$ and $|s_x - g^{-1}(x)| \le 1$. Therefore, taking into account (25)–(27), for every $x, t \in \mathbb{R}$, we get

$$\begin{aligned}
|H(x,t) - F_\alpha(x,t)| &= |H(H(0,s_x),t) - g_\alpha(g_\alpha^{-1}(x)+t)| \\
&\le |H(H(0,s_x),t) - H(0,s_x+t)| + |H(0,s_x+t) - g_\alpha(s_x+t)| \\
&\quad + |g_\alpha(s_x+t) - g_\alpha(g_\alpha^{-1}(x)+t)| \\
&\le |d(s_x+t) - d(s_x) - d(t)| + \frac{1-\alpha}{2} + (1+\alpha)|s_x - g_\alpha^{-1}(x)| \\
&\le 1 + \frac{1-\alpha}{2} + 1 + \alpha = \frac{5+\alpha}{2} < 3 = 3\delta.
\end{aligned}$$

The next remarks concern the case where the interval I is not open.

Remark 3.1. The assertions of Steps 1 and 2 are true (with the same argumentation) also in the case where I is not open.

Remark 3.2. Assume that I is a not open real interval, say $a := \inf I \in I$. Suppose that $H \in \mathfrak{F}_{cs}$ satisfies (3) with a positive δ and an $x_0 \in I$ such that a function $h := H(x_0, \cdot)$ is a continuous surjection of $\mathbb{R}$ onto I. We claim that the length of I is at most 4δ. First note that as h is a continuous surjection of $\mathbb{R}$ onto I, there is a $t_a \in \mathbb{R}$ such that $h(t_a) = a$ and either $h((-\infty, t_a)) \subset h((t_a, \infty)) = I$ or $h((t_a, \infty)) \subset h((-\infty, t_a)) = I$. Assume for instance that the first possibility is valid. Suppose that the length of I is greater than 4δ. Then $h(t_0) > a + 4\delta$ for some $t_0 \in (t_a, \infty)$. Fix a $t_1 \in (-\infty, 2t_a - t_0)$. Then $t_1 < t_a + (t_a - t_0) < t_a$, so there exists a $t_2 \in (t_a, \infty)$ with $h(t_1) = h(t_2)$. Thus, applying Remark 3.1, from (6) we deduce that

$$|a - h(t_a + (t_2 - t_1))| = |h(t_a) - h(t_a + (t_2 - t_1))| \le 2\delta,$$

that is

$$h(t_a + (t_2 - t_1)) \le a + 2\delta.$$

Furthermore, we have

$$t_a < t_0 < 2t_a - t_1 < t_a + (t_2 - t_1).$$

Hence, as h is continuous and $h(t_0) > a+4\delta$, we conclude that there exist a $u \in (t_a, t_0)$ and a $v \in (t_0, t_a + (t_2 - t_1)]$ such that $h(u) = h(v) = a + 2\delta$ and $h(t) \ge a + 2\delta$ for $t \in [u, v]$. Therefore $[u, v] \in \mathrm{J}(a + 2\delta)$ and so $t_0 - u < v - u < S$, where S is given by (7). Hence, applying again Remark 3.1, from (8) we derive that $|h(t_0) - h(u)| \le 2\delta$. Thus $h(t_0) \le h(u) + 2\delta = a + 4\delta$, which yields a contradiction.

Remark 3.3. From Remark 3.2 it follows that if the interval I is non-degenerated and not open then for sufficiently small δ (namely, for δ smaller than $\frac{1}{4}$ of the length of I) there is no $H \in \mathfrak{F}_{cs}$ satisfying (2). In particular, taking $\delta = 0$, we get that if the interval I is non-degenerated and not open then Eq. (1) has no solutions in the class $\mathfrak{F}_{cs}$.

Remark 3.4. Theorem 1.2 and Remark 3.3 imply that for every real interval I, Eq. (1) is stable in the Hyers–Ulam sense in the class $\mathfrak{F}_{cs}$. Note however that if I is a not open bounded interval with a positive length $|I|$ then taking an arbitrary $H \in \mathfrak{F}_{cs}$, we get (2) with $\delta = |I|$, but according to Remark 3.3, Eq. (1) has no solutions in the class $\mathfrak{F}_{cs}$. Hence, H can not be approximated by such a solution.

We conclude the paper with the following problem.

Problem 3.1. Let I be an open real interval and $\delta : \mathbb{R}^2 \to [0, \infty)$. Under what reasonable assumptions on δ, every function $H : I \times \mathbb{R} \to I$ satisfying the inequality

$$|H(H(x_0, s), t) - H(x_0, s + t)| \le \delta(s, t) \quad \text{for } s, t \in \mathbb{R}$$

with an $x_0 \in I$ such that the function $H(x_0, \cdot)$ is a continuous surjection of $\mathbb{R}$ onto I, can be approximated (in some sense) by a dynamical system belonging to $\mathfrak{F}_{cs}$.

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