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Symmetry in the Cuntz algebra on two generators

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ABSTRACT

We investigate the structure of the automorphism of $\mathcal{O}_2$ which exchanges the two canonical isometries. Our main observation is that the fixed point C*-subalgebra for this action is isomorphic to $\mathcal{O}_2$ and we detail the relationship between the crossed-product and fixed point subalgebra.

This paper studies the structure of the fixed point C*-algebra of the action of $\mathbb{Z}_2$ which switches the canonical generators of the Cuntz algebra $\mathcal{O}_2$. We show that both the C*-crossed-product and the fixed point C*-algebra for this action are *-isomorphic to $\mathcal{O}_2$.

This action is an example of an action of a finite group on a noncommutative C*-algebra, and in general the structures associated to such actions can be quite difficult to describe [7,5,6]. To any action α of a finite group G on a unital C*-algebra A, one can associate two new related C*-algebras: the fixed point C*-algebra A_1 and the C*-crossed-product $A \rtimes_\alpha G$ [9]. By construction, A_1 is a C*-subalgebra of A, while, if G is Abelian, then A is in fact the fixed point C*-subalgebra of $A \rtimes_\alpha G$ for the dual action of the Pontryagin dual of G – so A is itself a subalgebra of $A \rtimes_\alpha G$. In [8], Rosenberg shows that A_1 is *-isomorphic to a corner of $A \rtimes_\alpha G$, so that if $A \rtimes_\alpha G$ is simple, then it is Morita equivalent to A_1. In general, however, understanding the structure of A_1 or $A \rtimes_\alpha G$ can be quite complex, as demonstrated for instance in [2]. In this paper, when A is chosen to be $\mathcal{O}_2$ and the group is $\mathbb{Z}_2$, for the natural action swapping the generators of $\mathcal{O}_2$, we obtain a complete picture of the relative positions of these three C*-algebras, which we prove are all *-isomorphic to $\mathcal{O}_2$.

We shall say that two isometries S_1 and S_2 on some Hilbert space satisfy the Cuntz relation when

$$S_1S_1^* + S_2S_2^* = 1. \tag{0.1}$$

By [3], [4, Theorem V.4.6, p. 147], the Cuntz relation defines, up to *-isomorphism, a unique simple C*-algebra denoted by $\mathcal{O}_2$. Moreover, by universality, there is a unique *-automorphism σ of $\mathcal{O}_2$ which satisfies

$$\sigma(S_1) = S_2 \quad \text{and} \quad \sigma(S_2) = S_1.$$

Since σ^2 is the identity, we can define the C*-crossed-product $\mathcal{O}_2 \rtimes_\sigma \mathbb{Z}_2$ as the universal C*-algebra generated by two isometries S_1 and S_2 and a unitary w such that $w^2 = 1$ and $wS_1 = S_2 w$ [9]. We also can define the fixed point C*-subalgebra

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$[\mathcal{O}_2]_1$ of $\mathcal{O}_2$ as $\{a \in \mathcal{O}_2\colon \sigma(a) = a\}$. It should also be noted that Izumi [5, Example 5.7] studied the action of $\mathbb{Z}_2$ on $\mathcal{O}_2$ given by σ and proved that it has the Rohlin property. Thus, our examples fit in a larger family of "classifiable actions" in the sense of [5].

In the first section of this paper, we show that $[\mathcal{O}_2]_1$ is in fact *-isomorphic to $\mathcal{O}_2$. In the second section, we prove that $\mathcal{O}_2 \rtimes_\sigma \mathbb{Z}_2$ is also *-isomorphic to $\mathcal{O}_2$ and that σ is not inner. We also establish that in any representation of $\mathcal{O}_2$, the set of unitaries of order 2 exchanging the image of two generators is empty or a pair. In the third section, we study the symmetry between the relations of $\mathcal{O}_2 \rtimes_\sigma \mathbb{Z}_2$ with $\mathcal{O}_2$ on the one hand, and $\mathcal{O}_2$ and $[\mathcal{O}_2]_1$ on the other hand. Section four deals with a description of the C*-crossed-product $\mathcal{O}_2 \rtimes_\sigma \mathbb{Z}$, i.e. the universal C*-algebra generated by a copy of $\mathcal{O}_2$ and a unitary U such that $UaU^* = \sigma(a)$ for all $a \in \mathcal{O}_2$. We conclude this paper with concrete representations of $\mathcal{O}_2$ on function spaces.

1. Fixed point C*-subalgebra

This section investigates the structure of the C*-algebra $[\mathcal{O}_2]_1$ of fixed points of the automorphism σ. We start with a simple preliminary result, which introduces a useful unitary for our later purpose. We fix two isometries S_1 and S_2 satisfying Relation (0.1).

Proposition 1.1. *Let S_1 and S_2 be two isometries such that $S_1S_1^* + S_2S_2^* = 1$ and σ be the unique order 2 automorphism of $\mathcal{O}_2 = C^*(S_1, S_2)$ such that $\sigma(S_1) = S_2$. Let $U = S_1S_1^* - S_2S_2^*$. Then U is a unitary of $\mathcal{O}_2$ of order 2. Let $[\mathcal{O}_2]_1$ be the fixed point C*-subalgebra of $\mathcal{O}_2$ for σ. Then*

$$\mathcal{O}_2 = [\mathcal{O}_2]_1 \oplus [\mathcal{O}_2]_1 U$$

with U a unitary of order 2. Moreover, with this decomposition, if $a = a_1 + a_2U$ then $\sigma(a) = a_1 - a_2U$. Note that $\oplus$ is the direct sum for Banach spaces, not between algebras, since $[\mathcal{O}_2]_1U$ is not an algebra for the multiplication of $\mathcal{O}_2$.

Proof. For all $a \in \mathcal{O}_2$ we have $a = a_1 + a_{-1}$ with $a_\varepsilon = \frac{1}{2}(a + \varepsilon\sigma(a))$, so that $\sigma(a_\varepsilon) = \varepsilon a_\varepsilon$ for $\varepsilon \in \{-1, 1\}$. Let $[\mathcal{O}_2]_{-1}$ be the space of elements $a \in \mathcal{O}_2$ such that $\sigma(a) = -a$. It is then immediate that $\mathcal{O}_2 = [\mathcal{O}_2]_1 \oplus [\mathcal{O}_2]_{-1}$. Now, by construction, $U = U^*$ and

$$U^2 = \left(S_1S_1^* - S_2S_2^*\right)\left(S_1S_1^* - S_2S_2^*\right) = S_1S_1^* + S_2S_2^* = 1,$$

so U is an order 2 unitary. Moreover $\sigma(U) = -U$. Thus $a \in [\mathcal{O}_2]_{-1}$ if and only if $aU \in [\mathcal{O}_2]_1$. Hence our decomposition is proven. An immediate computation shows that σ is indeed implemented as shown. □

We now start the process to identify $[\mathcal{O}_2]_1$. Our proof will exhibit a specific and interesting choice of generators for $[\mathcal{O}_2]_1$, and for clarity of exposition it will be useful to keep track of the generators of the many *-isomorphic copies of $\mathcal{O}_2$ we will encounter in our proof. We start with the following lemmas:

Lemma 1.2. *Let S_1 and S_2 be two isometries such that $S_1S_1^* + S_2S_2^* = 1$, so that $\mathcal{O}_2 = C^*(S_1, S_2)$. We define the following elements in $M_2(C^*(S_1, S_2))$:*

$$T_1 = \frac{1}{\sqrt[2]{2}}\begin{bmatrix} S_1 & S_2 \\ S_1 & S_2 \end{bmatrix} \quad \text{and} \quad T_2 = \frac{1}{\sqrt[2]{2}}\begin{bmatrix} S_1 & -S_2 \\ -S_1 & S_2 \end{bmatrix}.$$

Then $T_1^*T_1 = T_2^*T_2 = T_1T_1^* + T_2T_2^* = 1$ in $M_2(C^*(S_1, S_2))$. Thus, by universality and simplicity of $\mathcal{O}_2$, the C-algebra $C^*(T_1, T_2)$ is *-isomorphic to $\mathcal{O}_2$. On the other hand*:

$$C^*(T_1, T_2) = \left\{\begin{bmatrix} A_1 & A_2 \\ \sigma(A_2) & \sigma(A_1) \end{bmatrix}\colon A_1, A_2 \in C^*(S_1, S_2)\right\}$$

where σ is the unique order 2 automorphism of $C^*(S_1, S_2)$ such that $\sigma(S_1) = S_2$.

Proof. Note that $\sigma(S_2) = \sigma(\sigma(S_1)) = S_1$ by assumption on σ. Now, we observe that

$$T_1 = \frac{1}{\sqrt[2]{2}}\left(\begin{bmatrix} S_1 & 0 \\ 0 & \sigma(S_1) \end{bmatrix} + \begin{bmatrix} 0 & S_2 \\ \sigma(S_2) & 0 \end{bmatrix}\right)$$

and

$$T_2 = \frac{1}{\sqrt[2]{2}}\left(\begin{bmatrix} S_1 & 0 \\ 0 & \sigma(S_1) \end{bmatrix} - \begin{bmatrix} 0 & S_2 \\ \sigma(S_2) & 0 \end{bmatrix}\right)$$

so $C^*(T_1, T_2) = C^*\left(\left[\begin{smallmatrix} S_1 & 0 \\ 0 & \sigma(S_1) \end{smallmatrix}\right], \left[\begin{smallmatrix} 0 & S_2 \\ \sigma(S_2) & 0 \end{smallmatrix}\right]\right)$.

On the other hand, we also have

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} S_1 & 0 \\ 0 & \sigma(S_1) \end{bmatrix} \begin{bmatrix} 0 & S_2 \\ \sigma(S_2) & 0 \end{bmatrix}^* + \begin{bmatrix} 0 & S_2 \\ \sigma(S_2) & 0 \end{bmatrix} \begin{bmatrix} S_1 & 0 \\ 0 & \sigma(S_1) \end{bmatrix}^*$$

so $\Xi = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right] \in C^*(T_1, T_2)$ and in fact

$$\begin{bmatrix} S_2 & 0 \\ 0 & \sigma(S_2) \end{bmatrix} = \begin{bmatrix} 0 & S_2 \\ \sigma(S_2) & 0 \end{bmatrix} \Xi.$$

Thus we conclude

$$C^*(T_1, T_2) = C^*\left(\begin{bmatrix} S_1 & 0 \\ 0 & \sigma(S_1) \end{bmatrix}, \begin{bmatrix} S_2 & 0 \\ 0 & \sigma(S_2) \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right). \tag{1.1}$$

We note that

$$C^*\left(\begin{bmatrix} S_1 & 0 \\ 0 & \sigma(S_1) \end{bmatrix}, \begin{bmatrix} S_2 & 0 \\ 0 & \sigma(S_2) \end{bmatrix} \right) = \left\{ \begin{bmatrix} A & 0 \\ 0 & \sigma(A) \end{bmatrix} : A \in C^*(S_1, S_2) \right\}.$$

Thus, if $A_1, A_2 \in C^*(S_1, S_2)$ then

$$\begin{bmatrix} A_1 & A_2 \\ \sigma(A_2) & \sigma(A_1) \end{bmatrix} = \begin{bmatrix} A_1 & 0 \\ 0 & \sigma(A_1) \end{bmatrix} + \begin{bmatrix} A_2 & 0 \\ 0 & \sigma(A_2) \end{bmatrix} \Xi$$

is in $C^*(T_1, T_2)$. Conversely, if we write $D_2(C^*(S_1, S_2))$ the algebra of diagonal matrices in $M_2(C^*(S_1, S_2))$ then

$$M_2\big(C^*(S_1, S_2)\big) = D_2\big(C^*(S_1, S_2)\big) \oplus D_2\big(C^*(S_1, S_2)\big)\Xi.$$

Hence any element of $C^*(T_1, T_2)$ must be of the desired form from Eq. (1.1), which concludes our lemma. □

Lemma 1.3. *We use the notations of Lemma* 1.2. *Let* $Z = \left[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right] \in M_2(C^*(S_1, S_2))$ *and let*

$$\tau : X \in C^*(T_1, T_2) \mapsto ZXZ.$$

Then τ *is an order* 2 *automorphism on* $C^*(T_1, T_2)$ *such that* $\tau(T_1) = T_2$ *and the fixed point C*-algebra of* τ *is given by*

$$\left\{ \begin{bmatrix} A & 0 \\ 0 & \sigma(A) \end{bmatrix} : A \in C^*(S_1, S_2) \right\}.$$

Proof. The fixed point algebra of $C^*(T_1, T_2)$ for τ is given by

$$\big\{a + \tau(a) : a \in C^*(T_1, T_2)\big\},$$

so this lemma follows from an immediate computation and Lemma 1.2. □

Theorem 1.4. *Let* S_1 *and* S_2 *be two isometries such that* $S_1S_1^* + S_2S_2^* = 1$ *and* σ *be the unique order* 2 *automorphism of* $\mathcal{O}_2 = C^*(S_1, S_2)$ *such that* $\sigma(S_1) = S_2$. *Let*

$$T = \frac{1}{\sqrt[2]{2}}(S_1 + S_2),$$
$$U = S_1S_1^* - S_2S_2^* \quad \text{and}$$
$$V = UTU = \frac{1}{\sqrt[2]{2}}(S_1 - S_2)\big(S_1S_1^* - S_2S_2^*\big).$$

Then the fixed point C-algebra* $[\mathcal{O}_2]_1$ *for* σ *is* $C^*(T, V)$ *and is *-isomorphic to* $\mathcal{O}_2$.

Proof. We shall use the notations of Lemma 1.2. First, let $\Phi : C^*(S_1, S_2) \to C^*(T_1, T_2)$ be the unique *-epimorphism defined by universality with $\Phi(S_j) = T_j$ $(j = 1, 2)$. Since $C^*(S_1, S_2)$ is simple, Φ is a *-isomorphism. Moreover, by construction, $\Phi \circ \sigma = \tau \circ \Phi$. Therefore, the fixed point C*-algebra for σ is *-isomorphic to the fixed point C*-algebra for τ.

Now, the fixed point C*-algebra for τ is given by Lemma 1.3 as

$$\left\{ \begin{bmatrix} A & 0 \\ 0 & \sigma(A) \end{bmatrix} : A \in C^*(S_1, S_2) \right\}$$

so it is the C*-algebra generated by $R_1 = \begin{bmatrix} S_1 & 0 \\ 0 & S_2 \end{bmatrix}$ and $R_2 = \begin{bmatrix} S_2 & 0 \\ 0 & S_1 \end{bmatrix}$, which are two isometries satisfying the Cuntz relation. So the fixed point C*-algebra for τ (hence for σ) is *-isomorphic to $\mathcal{O}_2$.

On the other hand, we have the relation:

$$R_1 = \frac{1}{\sqrt[2]{2}}(T_1 + T_2). \tag{1.2}$$

Moreover, if $Y = \Phi(U)$ then

$$\begin{aligned} Y &= T_1T_1^* - T_2T_2^* \\ &= \frac{1}{2}\left(\begin{bmatrix} S_1 & S_2 \\ S_1 & S_2 \end{bmatrix}\begin{bmatrix} S_1^* & S_1^* \\ S_2^* & S_2^* \end{bmatrix} - \begin{bmatrix} S_1 & -S_2 \\ -S_1 & S_2 \end{bmatrix}\begin{bmatrix} S_1^* & -S_1^* \\ -S_2^* & S_2^* \end{bmatrix}\right) \\ &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \end{aligned}$$

so we obtain the relation:

$$R_2 = YR_1Y. \tag{1.3}$$

This concludes our proof after application of Φ^{-1} to Relations (1.2) and (1.3). □

Thus, Proposition 1.1 can now be restated in the following manner: $\mathcal{O}_2$ is *-isomorphic to $\mathcal{O}_2 \oplus \mathcal{O}_2 U$, where $\sigma(a \oplus bU) = a - bU$ for any $a, b \in \mathcal{O}_2$. Moreover, we have a pair of natural generators for $[\mathcal{O}_2]_1$. It is natural to ask whether this decomposition, in fact, is a mean to recognize $\mathcal{O}_2$ as a crossed-product of an action on $\mathcal{O}_2$ implemented by $\operatorname{Ad} U$, and σ can then be seen as the dual action of $\mathbb{Z}_2$ on this crossed-product. We note that $\operatorname{Ad} U$ does swap the generators T and V of $[\mathcal{O}_2]_1$ with the notation of Theorem 1.4. The next two sections will make precise these informal observations. We start with a study of the structure of the C*-crossed-product $\mathcal{O}_2 \rtimes_\sigma \mathbb{Z}_2$.

2. C*-crossed-product

We first observe that the C*-crossed-product $\mathcal{O}_2 \rtimes_\sigma \mathbb{Z}_2$ is in fact *-isomorphic to $\mathcal{O}_2$:

Theorem 2.1. *Let S_1 and S_2 be two isometries such that $S_1S_1^* + S_2S_2^* = 1$ and σ be the unique order 2 automorphism of $\mathcal{O}_2 = C^*(S_1, S_2)$ such that $\sigma(S_1) = S_2$. Then*

$$\mathcal{O}_2 \rtimes_\sigma \mathbb{Z}_2 \text{ is *-isomorphic to } \mathcal{O}_2.$$

Proof. Let W be the canonical unitary in $\mathcal{O}_2 \rtimes_\sigma \mathbb{Z}_2$ such that $WS_1W = S_2$ and $W^2 = 1$. Then

$$\begin{aligned} S_1S_1^*W + \left(S_1S_1^*W\right)^* &= S_1S_1^*W + WS_1S_1^* \\ &= S_1S_1^*W + WS_1S_1^*W^2 \\ &= S_1S_1^*W + S_2S_2^*W \\ &= W. \end{aligned}$$

Hence, $W \in C^*(S_1, WS_1) \subseteq \mathcal{O}_2 \rtimes_\sigma \mathbb{Z}_2$. Since $\mathcal{O}_2 \rtimes_\sigma \mathbb{Z}_2$ is generated by S_1, S_2 and W and $S_2 = WS_1W \in C^*(S_1, WS_1)$ so $\mathcal{O}_2 \rtimes_\sigma \mathbb{Z}_2 = C^*(S_1, WS_1)$.

On the other hand:

$$(WS_1)^*WS_1 = S_1^*W^2S_1 = S_1^*S_1 = 1$$

and

$$S_1S_1^* + WS_1(WS_1)^* = S_1S_1^* + WS_1S_1^*W = S_1S_1^* + S_2S_2^* = 1.$$

Therefore, $C^*(S_1, WS_1)$ is *-isomorphic to $\mathcal{O}_2$. □

We can provide more details on the structure of the automorphism σ.

Proposition 2.2. *Let S_1 and S_2 be two isometries such that $S_1S_1^* + S_2S_2^* = 1$ and σ be the unique order 2 automorphism of $\mathcal{O}_2 = C^*(S_1, S_2)$ such that $\sigma(S_1) = S_2$. Then σ is not inner.*

Proof. Let $\mathcal{H} = l^2(\mathbb{N})$ whose canonical Hilbert basis is denoted by $(e_n)_{n\in\mathbb{N}}$ (namely, $(e_n)_m$ is 0 unless $n = m$, when it is 1). We define

$$T_1 e_n = e_{2n} \quad \text{and} \quad T_2 e_n = e_{2n+1}.$$

Then note that T_2 has no eigenvector while $T_1 e_0 = e_0$. Hence, T_1 and T_2 are not unitarily equivalent in $\mathcal{H}$. Yet, it is immediate that T_1 and T_2 are isometries which satisfy $T_1T_1^* + T_2T_2^* = 1$. Therefore, there exists a (unique) *-homomorphism φ from $\mathcal{O}_2$ onto $C^*(T_1, T_2)$ with $\varphi(S_j) = T_j$ for $j = 1, 2$, and since $\mathcal{O}_2$ is simple, φ is in fact a *-monomorphism. Now, if σ was inner, then there would exists some unitary $u \in \mathcal{O}_2$ such that $uS_1u^* = \sigma(S_1) = S_2$. This would imply that $\varphi(u)T_1\varphi(u)^* = T_2$ with $\varphi(u)$ a unitary. This is a contradiction. □

We can use Proposition 2.2 to see that, if we can find a covariant representation of $\mathcal{O}_2$, then the representation of $\mathbb{Z}_2$ is unique up to a sign.

Proposition 2.3. *Let S_1 and S_2 be two isometries such that $S_1S_1^* + S_2S_2^* = 1$. Let u and w be two unitaries such that $C^*(S_1, S_2, u) \subseteq C^*(S_1, S_2, w)$, and such that $u^2 = w^2 = 1$ with $uS_1 = S_2u$ and $wS_1 = S_2w$. Then $u = w$ or $u = -w$.*

Proof. By assumption, $C^*(S_1, S_2)$ is *-isomorphic to $\mathcal{O}_2$ since $\mathcal{O}_2$ is simple and universal for the given property. Moreover, by universality, there exists a (unique) *-morphism $\varphi : \mathcal{O}_2 \rtimes_\sigma \mathbb{Z}_2 \twoheadrightarrow C^*(S_1, S_2, w)$. Since $\mathcal{O}_2 \rtimes_\sigma \mathbb{Z}_2$ is $\mathcal{O}_2$, hence simple, φ is an isomorphism.

We can use φ to show that $C^*(S_1, S_2, w) = \mathcal{O}_2 \oplus \mathcal{O}_2 w$. Let us now write $u = a + bw$ with $a, b \in \mathcal{O}_2$. Then, for $j = 1, 2$ and by assumption, $uwS_j = S_j uw$, so

$$(b + aw)S_j = S_j(b + aw).$$

Since $\mathcal{O}_2$ and $\mathcal{O}_2 w$ are complementary spaces, we conclude that $bS_j = S_j b$ and $awS_j = S_j aw$ (note that $awS_j = aS_{j'}w$ with $aS_{j'} \in \mathcal{O}_2$). Thus b is in the center of $\mathcal{O}_2$ and thus is scalar. On the other hand, we have

$$\begin{cases} aS_2w = S_1aw, \\ aS_1w = S_2aw \end{cases} \quad \text{so} \quad \begin{cases} aS_2 = S_1a, \\ aS_1 = S_2a. \end{cases}$$

Consequently, a^2 commutes with S_1 and S_2 so it is central in $\mathcal{O}_2$, hence again a^2 is scalar, say $\lambda \in \mathbb{C}$. Now, since u is normal and w is normal, so are a and b (again, since $\mathcal{O}_2$ and $\mathcal{O}_2 w$ are complementary spaces). Assume $a \neq 0$. Then $v = \mu a$, where $\mu^2 = \lambda^{-1}$, is a unitary of order 2 in $C^*(S_1, S_2)$ which satisfies $vS_j = S_{j'}v$. By Proposition 2.2, this is not possible. Hence, $a^2 = 0$ and so $a = 0$ as a normal. Thus $u = bw$ with b scalar, and since $1 = u^2 = w^2$ we conclude that $b \in \{-1, 1\}$. □

Remark 2.4. We can recover the well-known fact that $\mathcal{O}_2 = M_2(\mathcal{O}_2)$ as proven in [1]. Indeed, let us use the notations of Lemma 1.2. Then

$$\mathcal{O}_2 = C^*(T_1, T_2) = \left\{ \begin{bmatrix} a & b \\ \sigma(b) & \sigma(a) \end{bmatrix} : a, b \in \mathcal{O}_2 \right\}$$

and, if $Z \in M_2(\mathcal{O}_2)$ with $Z = \left[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right]$ then

$$M_2(\mathcal{O}_2) = C^*(T_1, T_2, Z).$$

On the other hand, $C^*(T_1, T_2, Z) = C^*(T_1, T_2) \rtimes_\eta \mathbb{Z}_2$ where $\eta(T_1) = T_2$ is of order 2. Indeed, by universality, $C^*(T_1, T_2, Z)$ is a quotient of $C^*(T_1, T_2) \rtimes_\eta \mathbb{Z}_2$, yet the later is $\mathcal{O}_2$ by Theorem 2.1 so it is simple. Moreover, Theorem 2.1 provides us with a natural pair of generators for $M_2(\mathcal{O}_2)$.

Now, we wish to see that in some way, the embedding of $\mathcal{O}_2$ as the fixed point algebra for σ in $\mathcal{O}_2$ or the embedding of $\mathcal{O}_2$ into $\mathcal{O}_2 \rtimes_\sigma \mathbb{Z}_2$ are the same. The following section formalizes this statement.

3. A doubly infinite sequence of self-similar $\mathcal{O}_2$ embeddings

The C*-algebra $\mathcal{O}_2$ embeds into itself as a fixed point sub-C*-algebra for σ or as a subalgebra of its crossed-product. The second embedding can be seen as embedding a fixed point for the dual action to σ. In our case, these two embeddings are the same, as shown in the following proposition.

Proposition 3.1. *There exists a *-isomorphism $\tau : \mathcal{O}_2 \rtimes_\sigma \mathbb{Z}_2 \to \mathcal{O}_2$ such that $\tau(\mathcal{O}_2)$ is the fixed point C*-algebra of σ.*

Proof. In Theorem 2.1, we showed that $\mathcal{O}_2 \rtimes_\sigma \mathbb{Z}_2 = C^*(S_1, S_1 W)$. It follows that $\mathcal{O}_2 \rtimes_\sigma \mathbb{Z}_2 = C^*(\frac{S_1+S_1W}{\sqrt[2]{2}}, \frac{S_1-S_1W}{\sqrt[2]{2}})$ and a direct computation shows that $B_1 = \frac{S_1+S_1W}{\sqrt[2]{2}}$ and $B_2 = \frac{S_1-S_1W}{\sqrt[2]{2}}$ are isometries satisfying $B_1B_1^* + B_2B_2^* = 1$. By universality of $\mathcal{O}_2$ there exists a unique *-monomorphism $\tau : \mathcal{O}_2 \rtimes_\sigma \mathbb{Z}_2 \to \mathcal{O}_2$ defined by $\tau(B_i) = S_i$ for $i = 1, 2$. Now, $\tau(S_1) = \tau(\frac{\sqrt[2]{2}}{2}(B_1 + B_2)) = S_1 + S_2 = T$ and $\tau(S_2) = \tau(WS_1W) = UTU$ where $U = S_1S_1^* - S_2S_2^*$ and $T = S_1 + S_2$ following the notations of Theorem 1.4. Hence τ maps $\mathcal{O}_2 \subseteq \mathcal{O}_2 \rtimes_\sigma \mathbb{Z}_2$ onto the fixed point subalgebra of $\mathcal{O}_2$ for σ. □

Now, we can give a somewhat more detailed picture of the embeddings by seeing how we can construct a double infinite sequence of identical embeddings of $\mathcal{O}_2$ into itself using the crossed-product construction. More precisely, suppose we are given a copy of $\mathcal{O}_2$ generated by two isometries r_n and t_n with $r_nr_n^* + t_nt_n^* = 1$ where $n \in \mathbb{Z}$ arbitrary. Then we can define σ_n as before to be the automorphism such that $\sigma_n(r_n) = t_n$ and $\sigma_n^2 = 1$. Now, let w_n be the canonical unitary of $C^*(r_n, t_n) \rtimes_{\sigma_n} \mathbb{Z}_2$. Then we have the following relations:

$$\begin{cases} w_n^2 = 1, \\ w_nt_n = r_nw_n, \\ t_n^*t_n = r_n^*r_n = t_nt_n^* + r_nr_n^* = 1. \end{cases} \tag{3.1}$$

Now, the fixed point C*-subalgebra of $C^*(r_n, t_n)$ for σ_n is generated by

$$\begin{cases} r_{n-1} = \frac{1}{\sqrt[2]{2}}(r_n + t_n), \\ t_{n-1} = w_{n-1}(r_{n-1})w_{n-1}, \end{cases} \tag{3.2}$$

where

$$w_{n-1} = r_nr_n^* - t_nt_n^*. \tag{3.3}$$

By Proposition 1.1, we have $w_{n-1}^2 = 1$ and $w_{n-1}t_{n-1} = r_{n-1}w_{n-1}$. By Theorem 1.4 we have r_{n-1} and t_{n-1} are isometries, such that $r_{n-1}r_{n-1}^* + t_{n-1}t_{n-1}^* = 1$. Hence Relations (3.1) are satisfied for $n-1$. Therefore, $C^*(r_n, t_n)$ is the C*-crossed-product of $C^*(r_{n-1}, t_{n-1})$ for the action of $\mathbb{Z}_2$ generated by σ_{n-1} where $\sigma_{n-1}(r_{n-1}) = t_{n-1}$ and $\sigma_{n-1}^2 = 1$.

Now, it is natural to define

$$\begin{cases} r_{n+1} = \frac{1}{\sqrt[2]{2}}(r_n + r_nw_n), \\ t_{n+1} = \frac{1}{\sqrt[2]{2}}(r_n - r_nw_n). \end{cases}$$

Thus, by Theorem 2.1, we have that $C^*(r_n, t_n) \rtimes_{\sigma_n} \mathbb{Z}_2 = C^*(r_n, t_n, w_n)$ is $C^*(r_{n+1}, t_{n+1})$. Moreover, we note

$$w_n = r_{n+1}r_{n+1}^* - t_{n+1}t_{n+1}$$

which is of course Eq. (3.3). Moreover, one checks easily that Relation (3.2) is satisfied for n rather than $n-1$:

$$\frac{1}{\sqrt[2]{2}}(r_{n+1} + t_{n+1}) = \frac{1}{2}(2r_n) = r_n,$$

and

$$\begin{aligned} \frac{1}{\sqrt[2]{2}}w_n(r_{n+1} + t_{n+1})w_n &= \frac{1}{2}[w_nr_nw_n + w_nr_n + w_nr_nw_n - w_nr_n] \\ &= \frac{1}{2}[t_n + t_n] = t_n \quad \text{since } w_nr_nw_n = t_n. \end{aligned}$$

Thus, in particular, $C^*(t_n, r_n)$ is the fixed point C*-subalgebra of $C^*(t_{n+1}, r_{n+1})$ for the action of $\mathbb{Z}_2$ generated by σ_{n+1} which switches the two generators t_{n+1} and r_{n+1}. Thus, we have a pattern repeating for $n \in \mathbb{Z}$ where $\mathcal{O}_2$ embeds in $\mathcal{O}_2$ either as a fixed point C*-subalgebra for the action which exchanges a choice of generators of the target $\mathcal{O}_2$ or as the crossed-product of the source $\mathcal{O}_2$ by the action which exchanges a corresponding choice of generators of the source $\mathcal{O}_2$. Once a particular set of generators is chosen in our sequence, then all the other ones are determined uniquely. Note that by Proposition 2.3, the operators w_n ($n \in \mathbb{Z}$) are then unique up to a sign as well.

4. Crossed-product with $\mathbb{Z}$

Our study of the crossed-product $\mathcal{O}_2 \rtimes_\sigma \mathbb{Z}_2$ makes it easy to study the crossed-product $\mathcal{O}_2 \rtimes_\sigma \mathbb{Z}$. We now present a description of $\mathcal{O}_2 \rtimes_\sigma \mathbb{Z}$. We begin with a simple observation:

Proposition 4.1. *Let S_1, S_2 be two isometries with $S_1S_1^* + S_2S_2^* = 1$. Hence, $C^*(S_1, S_2) = \mathcal{O}_2$. Let σ be the automorphism defined by $\sigma(S_1) = S_2$ and $\sigma(S_2) = S_1$. Let π be an irreducible representation of $C^*(S_1, S_2) \rtimes_\sigma \mathbb{Z}$ and let V be the canonical unitary in $C^*(S_1, S_2) \rtimes_\sigma \mathbb{Z}$. Then there exists $t \in [-1, 1]$ such that*

$$\pi(V) = e^{i\frac{\pi}{2}t}W$$

with $W^2 = 1$ and $W\pi(S_1) = \pi(S_2)W$.

Proof. By construction, V^2 is in the center of $\mathcal{O}_2 \rtimes_\sigma \mathbb{Z}$. Since π is irreducible, $\pi(V^2)$ is a scalar of the form $e^{i\pi t}$ for some $t \in [-1, 1]$. Let $W = e^{-i\frac{\pi}{2}t}V$. Then by construction, W is a unitary such that $W^2 = 1$ and $W\pi(S_1)W = V\pi(S_1)V^* = \pi(S_2)$ as desired. □

We now can derive the following theorem:

Theorem 4.2. *Let S_1, S_2 be two isometries with $S_1S_1^* + S_2S_2^* = 1$. Hence, $C^*(S_1, S_2) = \mathcal{O}_2$. Let σ be the automorphism defined by $\sigma(S_1) = S_2$ and $\sigma(S_2) = S_1$. Then $\mathcal{O}_2 \rtimes_\sigma \mathbb{Z}$ is *-isomorphic to*

$$\left\{f \in C\left([-1, 1], \mathcal{O}_2 \rtimes_\sigma \mathbb{Z}_2\right):\ f(-1) = \widehat{\sigma}\left(f(1)\right)\right\}.$$

Proof. To fix notation, let us write

$$\mathcal{A} = \left\{f \in C\left([-1, 1], \mathcal{O}_2 \rtimes_\sigma \mathbb{Z}_2\right):\ f(-1) = \widehat{\sigma}\left(f(1)\right)\right\}$$

where we denote by w the canonical unitary in $\mathcal{O}_2 \rtimes_\sigma \mathbb{Z}_2$ and $\widehat{\sigma}$ is the automorphism defined uniquely by $\widehat{\sigma}(a) = a$ for $a \in \mathcal{O}_2$ and $\widehat{\sigma}(w) = -w$.

We introduce the following elements of $\mathcal{A}$:

$$\begin{cases} v : t \in [-1, 1] \mapsto e^{i\pi\frac{t}{2}}w, \\ s_1 : t \in [-1, 1] \mapsto S_1, \\ s_2 : t \in [-1, 1] \mapsto S_2. \end{cases}$$

Our proof consists of two steps: we show first that $\mathcal{A} = C^*(s_1, s_2, v)$. We then show that $C^*(s_1, s_2, v)$ is *-isomorphic to $\mathcal{O}_2 \rtimes_\sigma \mathbb{Z}$.

By construction, $C^*(s_1, s_2, v) \subseteq \mathcal{A}$. To show that $\mathcal{A} = C^*(s_1, s_2, v)$, we introduce the C*-subalgebra $\mathcal{B}$ of $\mathcal{A}$ defined by

$$\left\{f \in C\left([-1, 1], \mathcal{O}_2 \rtimes_\sigma \mathbb{Z}^2\right):\ f(-1) = f(1)\right\}.$$

Writing $f = \frac{1}{2}(f + \widehat{\sigma}(f)) + \frac{1}{2}(f - \widehat{\sigma}(f))v^2$ (since $v^2 = 1$), we easily see that

$$\mathcal{A} = \{f + gv:\ f, g \in \mathcal{B}\}.$$

Thus, to prove $\mathcal{A} \subseteq C^*(s_1, s_2, v)$ it is enough to show that $\mathcal{B} \subseteq C^*(s_1, s_2, v)$. Now, s_1, s_2 and $v^2 : t \in [-1, 1] \mapsto e^{i\pi t}$ are all in $\mathcal{B}$ by construction, and a standard argument shows that

$$\mathcal{B} = C^*\left(s_1, s_2, v^2\right) \cong \mathcal{O}_2 \otimes C\left([-1, 1]\right)$$

so $\mathcal{B} \subseteq C^*(s_1, s_2, v)$.

Now, it is sufficient to show that $C^*(s_1, s_2, v)$ is *-isomorphic to $\mathcal{O}_2 \rtimes_\sigma \mathbb{Z}$. Let V be the canonical unitary of $\mathcal{O}_2 \rtimes_\sigma \mathbb{Z}$. Since $vs_1v^* = s_2$ and $vs_2v^* = s_1$ by construction, there exists by universality of the crossed-product a unique *-epimorphism $\theta : \mathcal{O}_2 \rtimes_\sigma \mathbb{Z} \to C^*(s_1, s_2, v)$ with $\theta(S_1) = s_1$, $\theta(S_2) = s_2$ and $\theta(V) = v$. We wish to show that θ is in fact a *-isomorphism. Let $a \in \ker\theta$. Let π be an arbitrary irreducible representation of $\mathcal{O}_2 \rtimes_\sigma \mathbb{Z}$. By Proposition 4.1, there exists $t \in [-1, 1]$ such that $\pi(V) = e^{i\frac{\pi}{2}t}W$ with $W^2 = 1$ and $W\pi(S_1) = \pi(S_2)W$ with W unitary. By universality, there exists a representation ψ of $\mathcal{O}_2 \rtimes_\sigma \mathbb{Z}_2$ on the same Hilbert space on which π acts such that $\psi(S_1) = \pi(S_1)$, $\psi(S_2) = \pi(S_2)$ and $\psi(w) = W$. Let ε_t be the *-morphism from $\mathcal{A}$ onto $\mathcal{O}_2 \rtimes_\sigma \mathbb{Z}_2$ defined by $\varepsilon_t(f) = f(t)$ for all $f \in \mathcal{A}$. Then by construction, $\pi = \psi \circ \varepsilon_t \circ \theta$. Hence $\pi(a) = 0$. Since π was arbitrary irreducible, $a = 0$ and thus θ is injective. This completes our proof. □

Corollary 4.3. *Let S_1, S_2 be two isometries with $S_1S_1^* + S_2S_2^* = 1$. Hence, $C^*(S_1, S_2) = \mathcal{O}_2$. Let σ be the automorphism defined by $\sigma(S_1) = S_2$ and $\sigma(S_2) = S_1$. Then $\mathcal{O}_2 \rtimes_\sigma \mathbb{Z}$ is *-isomorphic to*

$$\big\{f \in C\big([-1,1], \mathcal{O}_2\big)\colon\ f(-1) = \sigma\big(f(1)\big)\big\}.$$

Proof. By Proposition 3.1, there exists a *-isomorphism:

$$\tau : \mathcal{O}_2 \rtimes \mathbb{Z}_2 \to \mathcal{O}_2$$

such that $\sigma \circ \tau = \tau \circ \widehat{\sigma}$, where we use the notations in the proof of Theorem 4.2.

The corollary follows from this observation and Theorem 4.2. □

We can rephrase the result above in a manner which may appear explicit. We call an element a of $\mathcal{O}_2$ *symmetric* if $a = \sigma(a)$ and *antisymmetric* if $a = -\sigma(a)$. Then we get immediately from Theorem 4.2:

Corollary 4.4. *We have*

$$\mathcal{O}_2 \rtimes_\sigma \mathbb{Z} = \left\{ f \in C\big([-1,1], \mathcal{O}_2\big) \;\middle|\; \begin{array}{l} f(1) + f(-1) \text{ is symmetric} \\ f(1) - f(-1) \text{ is antisymmetric} \end{array} \right\}.$$

Appendix A. Concrete irreducible representations

In this appendix, we present a concrete representation of $\mathcal{O}_2$ which fits the framework of this paper. Our representation is based upon the following group:

Definition A.1. Let $\mathcal{A}$ be the group of strictly increasing affine transformations of $\mathbb{R}$, i.e.

$$\mathcal{A} = \{\varphi_{a,b} : t \in \mathbb{R} \mapsto at + b\colon\ a > 0,\ b \in \mathbb{R}\}.$$

The group $\mathcal{A}$ is naturally isomorphic to $\{\left[\begin{smallmatrix}1 & b\\ 0 & a\end{smallmatrix}\right]\colon\ a > 0,\ b \in \mathbb{R}\}$ where $\left[\begin{smallmatrix}1 & b\\ 0 & a\end{smallmatrix}\right]$ is mapped to $\varphi_{a,b}$. We will use this isomorphism implicitly when convenient.

Proposition A.2. *For any $\varphi_{a,b} \in \mathcal{A}$ we define the bounded linear operator $\pi_{\varphi_{a,b}}$ of $L^2(\mathbb{R})$ by*

$$\pi_{\varphi_{a,b}} : f \in L^2(\mathbb{R}) \mapsto a^{\frac{1}{2}} f \circ \varphi_{a,b}.$$

Then π is a unitary representation of $\mathcal{A}$ on $L^2(\mathbb{R})$.

Proof. It is immediate that $f \mapsto f \circ \varphi_{a,b}$ is a linear operator on $L^2(\mathbb{R})$ and $\pi_{gg'} = \pi_g \pi_{g'}$ for all $g, g' \in \mathcal{A}$. Moreover $\pi_{\mathrm{id}} = \mathrm{id}$. Now for all $f, g \in L^2(\mathbb{R})$ we have

$$\int_{\mathbb{R}} f(at+b)g(t)\,dt = \int_{\mathbb{R}} f(t)\frac{1}{a} g\left(\frac{1}{a}(t-b)\right)dt$$

so $\langle \pi_{\varphi_{a,b}}(f), g\rangle = \langle f, \pi_{\varphi_{\frac{1}{a},-\frac{b}{a}}}(g)\rangle = \langle f, \pi_{\varphi_{a,b}^{-1}}(g)\rangle$ and thus $\pi_{\varphi_{a,b}}$ is bounded and unitary. □

Definition A.3. Let I be any closed subset in $\mathbb{R}$. The orthogonal projection from $L^2(\mathbb{R})$ onto $L^2(I)$ is denoted by P_I.

In other words, P_I is the multiplication operator by the indicator function χ_I of I.

Definition A.4. Let I, J be two compact intervals in $\mathbb{R}$. Let $\varphi \in \mathcal{A}$ be the unique increasing affine map such that $\varphi(I) = J$. Then we set

$$V(I, J) = P_I \pi_\varphi P_J.$$

Note that $P_I \pi = P_I \pi_\varphi P_J = \pi_\varphi P_J$ by construction in Definition A.4.

Theorem A.5. *The set*:

$$\Sigma = \{0\} \cup \big\{V(I,J)\colon\ I, J \text{ compact intervals in } \mathbb{R}\big\}$$

is a semigroup of partial isometries. Moreover:

(1) *For all compact interval I we have $V(I,I) = P_I$.*
(2) *For all compact intervals I, J we have $V(I,J) = V(J,I)^*$.*
(3) *For all four compact intervals I, J, K, L we have*

$$V(I,J)V(K,L) = V\big(\varphi_1^{-1}(J \cap K), \varphi_2(J \cap K)\big),$$

where φ_1 and φ_2 are the unique elements of $\mathcal{A}$ such that $\varphi_1(I) = J$ and $\varphi_2(K) = L$. Note that in particular

$$\varphi_{\varphi_1^{-1}(J\cap K),\varphi_2(J\cap K)} = \varphi_2\varphi_1.$$

In particular, the initial space of $V(I,J)$ is $L^2(J)$ and the final space is $L^2(I)$ for all compact intervals I, J of $\mathbb{R}$.

Proof. By uniqueness of the element in $\mathcal{A}$ which maps an interval to another, properties (1) and (2) are immediate. In general, given four compact intervals J_1, J_2, J_3 and J_4, and two affine maps φ_1 and φ_2 such that $\varphi(J_1) = J_2$ and $\varphi'(J_3) = J_4$, then we let $J = J_2 \cap J_3$.

$$\begin{aligned} V(J_1,J_2)V(J_3,J_4) &= P_{J_1}\pi_{\varphi_1}P_{J_2}P_{J_3}\pi_{\varphi_2}P_{J_4} \\ &= P_{J_1}\pi_{\varphi_1}P_{J_2\cap J_3}\pi_{\varphi_2}P_{J_4}. \end{aligned}$$

Now, for $\pi_{\varphi_2}(f)$ to be supported on $J_2 \cap J_3$ we must have that f is supported on $\varphi_2(J_2 \cap J_3)$. Hence

$$P_{J_2\cap J_3}\pi_{\varphi_2}(f)P_{J_4} = P_{J_2\cap J_3}\pi_{\varphi_2}(f)P_{\varphi_2(J_2\cap J_3)}.$$

Similarly, if f is supported on $J_2 \cap J_3$ then $\pi_{\varphi_1}(f)$ is supported on $\varphi_1^{-1}(J_2 \cap J_3)$ and

$$P_{J_1}\pi_{\varphi_1}P_{J_2\cap J_3} = P_{\varphi_1^{-1}(J_2\cap J_3)}\pi_{\varphi_1}P_{J_2\cap J_3}.$$

Hence the third property above. □

We will use two simple lemmas to prove that the representation of $\mathcal{O}_2$ introduced in Theorem A.8 is irreducible. Let λ be the usual Lebesgue measure on $[-1,1]$ (so that $\lambda([-1,1]) = 2$).

Lemma A.6. *Let $\mathfrak{a}$ be an arbitrary Borel subset of $(-1,1)$ of strictly positive Lebesgue measure in $(0,2)$. Then there exist a natural number m and two integers k_1 and k_2 such that*

$$\lambda\left(\mathfrak{a} \cap \left[\frac{k_1}{2^m}, \frac{k_1+1}{2^m}\right]\right) > \lambda\left(\mathfrak{a} \cap \left[\frac{k_2}{2^m}, \frac{k_2+1}{2^m}\right]\right)$$

with $-2^m \leqslant k_1, k_2 < 2^m$.

Proof. Since $\lambda(\mathfrak{a}) < 2$, there exists an open set $\mathfrak{g}$ in $(-1,1)$ such that $\mathfrak{a} \subseteq \mathfrak{g}$ and $\lambda(\mathfrak{g}) < 2$. Now, $\mathfrak{g}$ is the disjoint union countably many open intervals $\mathfrak{g}_i$ ($i \in \mathbb{N}$) in $[-1,1]$. In fact we may choose each $\mathfrak{g}_i$ of the form $(\frac{k_i}{2^{m_i}}, \frac{k_i+1}{2^{m_i}})$ for some $k_i \in \mathbb{Z}$ and $m_i \in \mathbb{N}$ for all $i \in \mathbb{N}$ – in which case the symmetric difference between $\mathfrak{g}$ and $\bigcup_{i\in\mathbb{N}} \mathfrak{g}_i$ has measure 0. Now

$$\sum_{i\in\mathbb{N}} \lambda(\mathfrak{g}_i \cap \mathfrak{a}) = \lambda(\mathfrak{a}) < \lambda(\mathfrak{a})\frac{1}{2}\lambda(\mathfrak{g}) = \frac{\lambda(\mathfrak{a})}{2}\sum_{i\in\mathbb{N}} \lambda(\mathfrak{g}_i)$$

so there exists $i \in \mathbb{N}$ such that $\lambda(\mathfrak{g}_i \cap \mathfrak{a}) > \frac{\lambda(\mathfrak{a})}{2}\lambda(\mathfrak{g}_i)$. To fix notations, let us write $\mathfrak{g}_i = [\frac{k_1}{2^m}, \frac{k_1+1}{2^m}]$, so that

$$\lambda\left(\mathfrak{a} \cap \left[\frac{k_1}{2^m}, \frac{k_1+1}{2^m}\right]\right) > \frac{1}{2^{m+1}}\lambda(\mathfrak{a}).$$

On the other hand:

$$\sum_{k=-2^m}^{2^m-1} \lambda\left(\mathfrak{a} \cap \left[\frac{k}{2^m}, \frac{k+1}{2^m}\right]\right) = \lambda(\mathfrak{a})$$

so there exists an integer k_2 such that

$$\lambda\left(\mathfrak{a} \cap \left[\frac{k_2}{2^m}, \frac{k_2+1}{2^m}\right]\right) \leqslant \frac{1}{2^{m+1}}\lambda(\mathfrak{a}).$$

Consequently

$$\lambda\left(\mathfrak{a}\cap\left[\frac{k_2}{2^m},\frac{k_2+1}{2^m}\right]\right)<\lambda\left(\mathfrak{a}\cap\left[\frac{k_1}{2^m},\frac{k_1+1}{2^m}\right]\right)$$

as desired. □

Lemma A.7. *Let J_1 and J_2 be two closed intervals in $[-1,1]$. Let $\mathfrak{a}$ be a Borel subset of $[-1,1]$ such that the projection $P_\mathfrak{a}$ commutes with $V(J_2,J_1)$. Then $\lambda(J_1\cap\mathfrak{a})=\lambda(J_2\cap\mathfrak{a})$.*

Proof. Write $J=J_1$ and define $c\in[-1,1]$ by $J_2=J_1+c$. Write $P=P_\mathfrak{a}$ and $V=V(J+c,J)$. Now for $f\in L^2([-1,1])$ and $t\in[-1,1]$ we have

$$Pf(t)=\chi_\mathfrak{a}(t)f(t)\quad\text{and}\quad Vf(t)=\chi_{J+c}(t)f(t-c).$$

Thus $PV=VP$ exactly when

$$\chi_\mathfrak{a}(t)\chi_{J+c}(t)f(t-c)=\chi_{J+c}(t)\chi_\mathfrak{a}(t-c)f(t-c)$$

for all $f\in L^2([-1,1])$ and $t\in[-1,1]$. Thus $\chi_{\mathfrak{a}\cap(J+c)}=\chi_{(\mathfrak{a}\cap J)+c}$, which implies the desired result. □

Theorem A.8. *Let*

$$S_1 \text{ be the restriction of } V\big([0,1],[-1,1]\big) \text{ to } L^2\big([-1,1]\big),$$
$$S_2 \text{ be the restriction of } V\big([-1,0],[-1,1]\big) \text{ to } L^2\big([-1,1]\big).$$

In other words, for $f\in L^2([-1,1])$ and $t\in[-1,1]$ we have

$$S_1(f)(t)=\sqrt[2]{2}f(2t-1)\quad\text{and}\quad S_2(f)(t)=\sqrt[2]{2}f(2t+1).$$

Then S_1 and S_2 are two isometries of $L^2([-1,1])$ such that $S_1S_1^*+S_2S_2^*=1$. Moreover $C^*(S_1,S_2)$ is irreducible and

$$C^*(S_1,S_2)=\overline{\operatorname{span}\big\{V(I,J)\colon I,J\in\mathcal{J}\big\}}$$

where

$$\mathcal{J}=\left\{\left[\frac{k}{2^m},\frac{k'}{2^m}\right]\colon m\in\mathbb{N},\ -2^m\leqslant k<k'\leqslant 2^m,\ k,k' \text{ integers}\right\}.$$

Proof. By construction, S_1 and S_2 are isometries. Moreover

$$S_1S_1^*=P_{[0,1]}\quad\text{and}\quad S_2S_2^*=P_{[-1,0]}$$

so $S_1S_1^*+S_2S_2^*=1$ in $L^2([-1,1])$.

Note that by definition, the semigroup generated by S_1 and S_2 is the semigroup generated by $V([0,1],[-1,1])$ and $V([-1,0],[-1,1])$ when regarded as operators acting on $L^2([-1,1])$ only. First, we observe that $\varphi_{S_1}: t\mapsto 2t+1$ and $\varphi_{S_2}: t\mapsto 2t-1$. Hence, $\varphi_{S_1}^{-1}: t\mapsto \frac{1}{2}t-\frac{1}{2}$ and $\varphi_{S_2}^{-1}: t\mapsto \frac{1}{2}t+\frac{1}{2}$ and thus, by Theorem A.5:

$$V\big([0,1],[-1,1]\big)V\left(\left[\frac{k}{2^m},\frac{k'}{2^m}\right],[-1,1]\right)=V\left(\left[\frac{k-1}{2^{m+1}},\frac{k'-1}{2^{m+1}}\right],[-1,1]\right),$$

and

$$V\big([-1,0],[-1,1]\big)V\left(\left[\frac{k}{2^m},\frac{k'}{2^m}\right],[-1,1]\right)=V\left(\left[\frac{k+1}{2^{m+1}},\frac{k'+1}{2^{m+1}}\right],[-1,1]\right).$$

Thus by induction, any finite product of S_1 and S_2 is of the form $V(I,[-1,1])$ where $I\in\mathcal{J}$ and moreover all such operators can be obtained as such finite products. So the semigroup generated by S_1 and S_2 is given by

$$\mathcal{S}=\big\{V\big(I,[-1,1]\big)\colon I\in\mathcal{J}\big\}.$$

Now, by Theorem A.5, we also have that the adjoint of the operators in $\mathcal{S}$ are of the form:

$$\mathcal{S}^*=\big\{V\big([-1,1],I\big)\colon I\in\mathcal{J}\big\}.$$

Hence by a direct computation and applying Theorem A.5, we get that arbitrary products of S_1, S_1^*, S_2, S_2^* are exactly given by 0 or

$$V\left(\left[\frac{k}{2^m},\frac{k'}{2^m}\right],\left[\frac{p}{2^m},\frac{q}{2^m}\right]\right)=V\left(\left[\frac{k}{2^m},\frac{k'}{2^m}\right],[-1,1]\right)V\left([-1,1],\left[\frac{p}{2^m},\frac{q}{2^m}\right]\right)$$

for all

$$-2^m \leqslant p,k<q,k' \leqslant 2^m,\quad p,q,k,k' \in \mathbb{Z} \text{ and } m \in \mathbb{N}.$$

Therefore

$$C^*(S_1,S_2)=\overline{\operatorname{span}\{V(I,J)\colon I,J\in\mathcal{J}\}}$$

as claimed.

Last, note that $C^*(S_1,S_2)$ contains P_I for all compact intervals I with dyadic end points. Hence, the commutant of $C^*(S_1,S_2)$ is contained in the von Neumann algebra of the multiplications operators by functions in $L^\infty([-1,1])$ on $L^2([-1,1])$. Consequently, if P is a projection commuting with $C^*(S_1,S_2)$ then there exists a measurable set $\mathfrak{a}\subseteq[-1,1]$ such that P is the multiplication operator with the indicator function $\chi_{\mathfrak{a}}$ of A. Let us assume that $\lambda(A)\in(0,2)$. Then by Lemma A.6 we can find a natural number m and two integers k_1 and k_2 such that

$$\lambda\left(\mathfrak{a}\cap\left[\frac{k_1}{2^m},\frac{k_1+1}{2^m}\right]\right)>\lambda\left(\mathfrak{a}\cap\left[\frac{k_2}{2^m},\frac{k_2+1}{2^m}\right]\right).$$

Yet, this contradicts Lemma A.7. So $\lambda(\mathfrak{a})\in\{0,2\}$ and thus our representation is irreducible. □

We now can construct a unitary which implements the action of $\mathbb{Z}_2$ which flips S_1 and S_2 and illustrate our work by applying Theorems 2.1 and 1.4 to describe the fixed point subalgebra and the crossed-product in term of concrete operators.

Remark A.9. Let $W : L^2([-1,1])\to L^2([-1,1])$ be defined by $Wf : t\in[-1,1]\mapsto f(-t)$ for all $f\in L^2([-1,1])$. Then W is an order 2 unitary such that $WS_2=S_1W$. Thus, as in Theorem 2.1, setting $\sigma(A)=WAW$ for $A\in C^*(S_1,S_2)$, we conclude that $C^*(S_1,S_2)\rtimes_\sigma\mathbb{Z}_2=C^*(S_1,S_1W)$ is *-isomorphic to $\mathcal{O}_2$. Note that for $f\in L^2([-1,1])$ and $t\in[-1,1]$, and our choice of representation in this section, we get $S_1(f)(t)=\sqrt[2]{2}f(2t-1)$ and $(WS_1)(f)(t)=\sqrt[2]{2}f(-2t-1)$ – thus both are isometries. One checks easily that $S_1S_1^*+(WS_1)(WS_1)^*=1$.

Remark A.10. We can describe the two generators of the fixed point C*-algebra of $C^*(S_1,S_2)$ given by Theorem 1.4 in our concrete representation. Keeping the notations of Theorem 1.4, we have, for $f\in L^2([-1,1])$ and $t\in[-1,1]$:

$$\begin{aligned}
&U=S_1S_1^*-S_2S_s^*=P_{[0,1]}-P_{[-1,0]},\\
&T=\sqrt[2]{2}(S_1+S_2)\quad\text{so}\quad T(f)(t)=f(2t-1)+f(2t+1),\\
&R=UTU\quad\text{so}\quad R(f)(t)=\begin{cases} f(2t-1) & \text{for } t\in(\frac{1}{2},1),\\ -f(2t-1) & \text{for } t\in(0,\frac{1}{2}),\\ -f(2t+1) & \text{for } t\in(-\frac{1}{2},0),\\ f(2t+1) & \text{for } t\in(-1,-\frac{1}{2}).\end{cases}
\end{aligned}$$

One then check that R and T are isometries such that $TT^*+RR^*=1$.

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