JOURNAL OF ALGEBRA 172, 764–786 (1995)

On Generalized Biderivations and Related Maps*

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Communicated by Susan Montgomery

Received January 18, 1994

1. INTRODUCTION

Throughout, R will be a ring with center Z. Let S be a subset of R. A map $f: S \to R$ is said to be *centralizing* on S if $[f(x), x] \in Z$ for every $x \in S$. In the special case where $[f(x), x] = 0$ for every $x \in S$, the map f is called *commuting* on S. The study of centralizing maps was initiated by a well-known theorem of Posner [19] which states that the existence of a nonzero centralizing derivation on a prime ring R implies that R is commutative. A number of authors have extended Posner's theorem in several ways. They have showed that nonzero derivations cannot be centralizing on various subsets of noncommutative prime rings (see [14] for probably the most general results of that kind), and that similar conclusions hold for some other maps, for example, automorphisms (see [5] for references).

In [5] the present author studied maps that are centralizing and additive, and no further assumption was required. First, it was shown that under rather mild assumptions this study can be reduced to the study of commuting maps (more precisely, it was proved that every centralizing additive map on a Jordan subring of a 2-torsion free semiprime ring is commuting; see also [4] and [9] for generalizations). The main result of [5] characterizes commuting additive maps on prime rings R: every such map is of the form $x \to \lambda x + \zeta(x)$ where $\lambda \in C$, the extended centroid of R, and ζ is an additive map of R into C. Somewhat later, similar results for semiprime rings [1, 7], von Neumann [3] and C^*-algebras [1], and for skew elements of a prime ring with involution t[9] were obtained. We also mention the papers [6] and [10] where commuting traces of biadditive maps of certain rings were characterized.

*The work was supported in part by a grant from the Ministry of Science and Technology of the Republic of Slovenia.

0021-8693/95 $6.00

The notion of additive commuting maps is closely connected with the notion of biderivations. Let S be a subring of R. A biadditive map D: $S \times S \to R$ is called a *biderivation* of S if it is a derivation in each argument; that is, for every $x \in S$, the maps $y \to D(x, y)$ and $y \to D(y, x)$ are derivations of S into R. Every commuting additive map f: $S \to R$ gives rise to a biderivation of S. Namely, linearizing $[f(x), x] = 0$, $x \in S$, we get $[f(x), y] = [x, f(y)]$, $x, y \in S$, and hence we note that the map $(x,y) \to [f(x), y]$ is a biderivation (moreover, all derivations appearing are inner). In [9] it was shown that every biderivation D of a noncommutative prime ring R is of the form $D(x, y) = \lambda[x, y]$ for some $\lambda \in C$. As an immediate application of this result we obtain a characterization of commuting additive maps of prime rings mentioned above.

Let σ be an automorphism of R. Recall that an additive map d of a subring S of R into R is called a *σ-derivation* if $d(xy) = \sigma(x)d(y) + d(x)y$, $x, y \in S$. A biadditive map Δ: $S \times S \to R$ will be called a *σ-biderivation* of S if for every $x \in S$, the maps $y \to \Delta(x, y)$ and $y \to \Delta(y, x)$ are σ-derivations of S into R. In Section 3, we determine the structure of an arbitrary σ-biderivation of a prime ring R. As an application, additive maps f of R satisfying $f(x)x = \sigma(x)f(x)$, $x \in R$ are characterized.

Recently, Lanski [15, 16] dealt with the situation where a nonzero derivation d of a prime ring R satisfies $c_1xd(y) + c_2d(x)y + c_3yd(x) + c_4d(y)x \in C$ for some $c_i \in C$ and all $x, y \in S$, where S is a certain subset of R (the case when S is a two-sided or a Lie ideal was considered in [15], and the case when R has an involution and S is the set of skew or symmetric elements in R was considered in [16]). Neglecting rings of characteristic 2 or 3, in any case the conclusion was the same: either all $c_i = 0$ or R satisfies S_4, the standard identity of degree 4 (however, the exact statements are much more precise). The condition considered by Lanski clearly covers the case of centralizing derivations; namely, a linearization of $[d(x), x] \in Z$ gives $xd(y) - d(x)y + yd(x) - d(y)x \in Z$. The same is true for the case of skew-centralizing derivations—a map f of a subset S of R into R is said to be *skew-centralizing* on S if $f(x)x + xf(x) \in Z$ for al $x \in S$. In the special case where $f(x)x + xf(x) = 0$ for all $x \in S$, the map f is called *skew-commuting* on S. In [8] the present author proved that there is no nonzero additive maps that are skew-commuting on ideals of prime rings of characteristic not 2.

In Section 4, motivated by the work of Lanski and our results on centralizing, commuting and skew-commuting maps, we consider the following situation: R is a prime ring, I is an ideal of R, and f_1, f_2, f_3, f_4: $I \to R$ are additive maps such that $\pi(x, y) = f_1(x)y + xf_2(y) + f_3(y)x + yf_4(x)$ lies in Z for all $x, y \in I$. It is shown that if the characteristic of R is not 2 or 3, then either $\pi(x, y) = 0$, $x, y \in I$ or R satisfies S_4. For the case where $\pi(x, y) = 0$, $x, y \in I$ we obtain a complete description of the maps

f_1, f_2, f_3, f_4. For instance, it turns out that f_1 must be of the form $f_1(x) = -xa + \mu(x)$ for some additive map μ of R into C and some $a \in Q_s$, the symmetric Martindale ring of quotients of R; similar conclusions hold for f_2, f_3, and f_4.

A map g of a subset S of R into R is called a *generalized (inner) derivation* if $g(x) = ax + xb$, $x \in S$, for some $a, b \in R$. Such maps have been extensively studied in the theory of operator algebras. Writing the relation $\pi(x, y) = 0$, $x, y \in I$, in the form $f_1(x)y + yf_4(x) = -xf_2(y) - f_2(y)x$, we see that the map $(x, y) \to f_1(x)y + yf_4(x)$ is a generalized derivation in each argument. Such maps will be called generalized biderivations; more precisely, a biadditive map $G: S \times S \to R$ is a *generalized biderivation* of S if for every $x \in S$ the maps $y \to G(x, y)$ and $y \to G(y, x)$ are generalized derivations. Our main result states that every generalized biderivation G of an ideal I of a prime ring R is of the form $G(x, y) = xay + ybx$ for some $a, b \in Q_s$.

In Section 5 we consider biderivations, commuting maps, and skew-commuting maps of right ideals T of prime rings R. The conclusions are the same as for the two-sided ideals, unless TC is a minimal right ideal of the central closure of R.

2. Preliminaries

In this section we recall basic definitions and gather together a few results of general interest that will be needed in the subsequent sections. We remark that all main theorems in this paper depend upon entirely elementary computations in the proofs of Lemmas 2.3 and 2.5.

Throughout the paper, *R will denote a prime ring* with center Z. By Q_r we will denote the *right Martindale ring of quotients* of R. It is known that this ring, introduced by Martindale in [17], can be characterized by the following four properties (compare [18, Proposition 10.2]):

(i) $R \subseteq Q_r$,

(ii) for every $q \in Q_r$ there exists a nonzero ideal I of R such that $qI \subseteq R$,

(iii) if $q \in Q_r$ and I is a nonzero ideal of R such that $qI = 0$, then $q = 0$.

(iv) if I is an ideal of R and $k: I \to R$ is a right R-module map, then there exists $q \in Q_r$ such that $k(u) = qu$ for all $u \in U$.

The center of Q_r, which will be denoted by C, is called the *extended centroid of* R. It is well-known that C is a field. Also, it is easily seen that C is the centralizer of R in Q_r. In particular, $Z \subseteq C$. The subring of Q_r

generated by R and C is called the *central closure* of R and will be denoted by R_c. Another important subring of Q_r is $Q_s = \{q \in Q_r | Iq \subseteq R$ for some nonzero ideal I of $R\}$. It is called the *symmetric Martindale ring of quotients*. We point out that $R \subseteq R_c \subseteq Q_s \subseteq Q_r$. Note that $q_1Rq_2 = 0$ with $q_1, q_2 \in Q_r$ implies that $q_1 = 0$ or $q_2 = 0$. Whence we see that all R_c, Q_s, and Q_r are prime algebras (over the field C). The following lemma indicates one of the reasons why the extended centroid is important.

LEMMA 2.1. *Suppose that nonzero elements $a_i, b_i \in Q_r$, $i = 1, \ldots, m$, satisfy $\sum_{i=1}^m a_i x b_i = 0$ for all x in some nonzero ideal of R. Then the a_i's are linearly dependent over C, and the b_i's are linearly dependent over C.*

The special case of Lemma 2.1 where a_i and b_i are elements of R is a well-known result. However, essentially the same argument shows that the result is still true in this more general situation (see [13, Theorem 1]).

The following lemma is a slight generalization of [2, Lemma] and [9, Lemma 3.2]. Although the same proofs works, we include it for it is rather short.

LEMMA 2.2. *Let M be any set. Suppose that maps $F, G \colon M \to Q_r$ satisfy $F(s)xG(t) = G(s)xF(t)$ for all $s, t \in M$ and all x in some nonzero ideal I of R. If $F \neq 0$, then there exists $\lambda \in C$ such that $G(s) = \lambda F(s)$ for all $s \in M$.*

Proof. Pick $t_0 \in M$ such that $F(t_0) \neq 0$. Of course, we may assume that $G \neq 0$. By Lemma 2.1 then follows that $G(t_0)$ and $F(t_0)$ are C-dependent. Thus, $G(t_0) = \lambda F(t_0)$ for some $\lambda \in C$. Therefore, the relation $F(s)xG(t_0) = G(s)xF(t_0)$, $x \in I$, $s \in M$, implies that $(\lambda F(s) - G(s))IF(t_0) = 0$ for every $s \in S$. But then $G(s) = \lambda F(s)$.

In the following lemmas we derive basic relations.

LEMMA 2.3. *Let B be a ring, A be a subring, and σ be an automorphism of B. If $\Delta \colon A \times A \to B$ is a σ-biderivation, then*

$$\Delta(x, y)z[u, v] = [\sigma(x), \sigma(y)]\sigma(z)\Delta(u, v)$$

for all $x, y, z, u, v \in A$.

Proof. We compute $\Delta(xu, yv)$ in two different ways. Since Δ is a σ-derivation in the first argument, we have

$$\Delta(xu, yv) = \Delta(x, yv)u + \sigma(x)\Delta(u, yv).$$

Using the fact that Δ is a σ-derivation in the second argument, it follows that

$$\Delta(xu, yv) = \Delta(x, y)vu + \sigma(y)\Delta(x, v)u + \sigma(x)\Delta(u, y)v + \sigma(x)\sigma(y)\Delta(u, v).$$

On the other hand, we have

$$\begin{aligned}\Delta(xu, yv) &= \Delta(xu, y)v + \sigma(y)\Delta(xu, v) \\ &= \Delta(x, y)uv + \sigma(x)\Delta(u, y)v \\ &\quad + \sigma(y)\Delta(x, v)u + \sigma(y)\sigma(x)\Delta(u, v).\end{aligned}$$

Comparing the relations so obtained for $\Delta(xu, yv)$, we get

$$\Delta(x, y)[u, v] = [\sigma(x), \sigma(y)]\Delta(u, v).$$

Substituting zu for u, and using

$$[zu, v] = [z, v]u + z[u, v], \qquad \Delta(zu, v) = \Delta(z, v)u + \sigma(z)\Delta(u, v),$$

we obtain the assertion of the lemma.

In the special case where σ is the identity, we obtain the known result [9, Lemma 3.1]:

COROLLARY 2.4. *Let B be a ring and A be a subring. If D: $A \times A \to B$ is a biderivation, then*

$$D(x, y)z[u, v] = [xy]zD(u, v)$$

for all $x, y, z, u, v \in A$.

LEMMA 2.5. *Let B be ring, A be a subring, and T be a right ideal of A. Let f_1, f_2, f_3, f_4: $T \to B$ be any maps, and set*

$$\pi(x, y) = f_1(x)y + xf_2(y) + f_3(y)x + yf_4(x).$$

Then

$$\begin{aligned}x[r, f_2(yr) - f_2(y)r] + y[r, f_4(xr) - f_4(x)r] \\ = \pi(x, y)r^2 - (\pi(xr, y) + \pi(x, yr))r + \pi(xr, yr)\end{aligned}$$

for all $x, y \in T$, $r \in A$.

Proof. Verify.

COROLLARY 2.6. *Let B be a ring, A be a subring, and T be a right ideal of A. Suppose that maps $f_1, f_2, f_3, f_4\colon T \to B$ satisfy*

$$f_1(x)y + xf_2(y) + f_3(y)x + yf_4(x) = 0$$

for all $x, y \in T$. Then

$$x[r, f_2(yr) - f_2(y)r] + y[r, f_4(xr) - f_4(x)r] = 0$$

for all $x, y \in T$, $r \in A$.

3. σ-BIDERIVATIONS

We say that an automorphism σ of the prime ring R is *X-inner* if there exists an invertible element $a \in Q_s$ such that $\sigma(r) = ara^{-1}$ for all $r \in R$.

LEMMA 3.1 [18, LEMMA 12.1]. *Let σ be an automorphism of R. If there exist nonzero elements $a_1, a_2, a_3, a_4 \in Q_r$ such that $a_1ra_2 = a_3\sigma(r)a_4$ for all $r \in R$, then σ is X-inner.*

We are now in a position to prove

THEOREM 3.2. *Let R be a noncommutative prime ring, σ be an automorphism of R, and $\Delta\colon R \times R \to R$ be a nonzero σ-biderivation of R. Then σ is X-inner and there exists an invertible element $b \in Q_s$ such that*

$$\sigma(x) = bxb^{-1}, \qquad \Delta(x, y) = b[x, y]$$

for all $x, y \in R$.

Proof. By Lemma 2.3,

$$\Delta(x, y)r[u, v] = [\sigma(x), \sigma(y)]\sigma(r)\Delta(u, v)$$

for all $x, y, u, v, r \in R$. Since R is noncommutative and $\Delta \neq 0$, we can find $x_0, y_0, u_0, v_0 \in R$ such that $a_1 = \Delta(x_0, y_0) \neq 0$, $a_2 = [u_0, v_0] \neq 0$, $a_3 = [\sigma(x_0), \sigma(y_0)] \neq 0$ and $a_4 = \Delta(u_0, v_0) \neq 0$ (this can be, for instance, proved by observing that $[u, v] \neq 0$ if and only if $\Delta(u, v) \neq 0$). We have $a_1ra_2 = a_3\sigma(r)a_4$, $r \in R$, and so σ is X-inner by Lemma 3.1. That is, $\sigma(x) = axa^{-1}$ for some $a \in Q_s$. Therefore,

$$\Delta(x, y)r[u, v] = a[x, y]ra^{-1}\Delta(u, v)$$

for all $x, y, u, v, r \in R$. Multiplying from the left by a^{-1} we obtain

$$a^{-1}\Delta(x, y)r[u, v] = [x, y]ra^{-1}\Delta(u, v).$$

Let $M = R \times R$ and note that maps $F, G\colon M \to Q_r$ defined by $F(x, y) = [x, y]$, $G(x, y) = a^{-1}\Delta(x, y)$ satisfy all the requirements of Lemma 2.2. Thus, there exists $\lambda \in C$ such that $G(x, y) = \lambda F(x, y)$, that is, $\Delta(x, y) = b[x, y]$ where $b = \lambda a$. Note that $b \neq 0$ for $\Delta \neq 0$, whence b is invertible and $\sigma(x) = bxb^{-1}$, $x \in R$. The proof of the theorem is complete.

We remark that even for biderivations, the assumption that R is noncommutative cannot be removed. Namely, note that any derivations d_1, d_2 of a commutative ring generate a biderivation $(x, y) \to d_1(x)d_2(y)$.

COROLLARY 3.3. *Suppose that an automorphism σ of R and a nonzero additive map $f\colon r \to R$ satisfy*

$$f(x)x = \sigma(x)f(x) \qquad \text{for all } x \in R.$$

Then there exists an invertible element $b \in Q_s$ and an additive map ζ: $R \to C$ such that $\sigma(x) = bxb^{-1}$, $x \in R$ (thus σ is X-inner), and f is either of the form $f(x) = bx + \zeta(x)b$, $x \in R$, or of the form $f(x) = \zeta(x)b$, $x \in R$.

Proof. Assume first that R is commutative. In this case it suffices to show that σ is the identity on R. We have $f(x)(x - \sigma(x)) = 0$, $x \in R$. Therefore, for $x \in R$, either $f(x) = 0$ of $x = \sigma(x)$. In other words, R is the union of its additive subgroups $\{x \in R | f(x) = 0\}$ and $\{x \in R | x = \sigma(x)\}$. Since a group cannot be the union of two proper subgroups, and since $f \neq 0$, it follows that $\sigma(x) = x$ for every $x \in R$.

We may therefore assume that R is noncommutative. Linearizing $f(x)x = \sigma(x)f(x)$ we obtain $f(x)y - \sigma(y)f(x) = \sigma(x)f(y) - f(y)x$, and hence we see that the map $\Delta(x, y) = f(x)y - \sigma(y)f(x)$ is a σ-biderivation of R. Suppose that $\Delta \neq 0$. By Theorem 3.2, we then have $\Delta(x, y) = b[x, y]$ and $\sigma(x) = bxb^{-1}$ for some invertible $b \in Q_s$. Thus, $b^{-1}\Delta(x, y) = [x, y]$. This relation can be written in the form

$$(b^{-1}f(x) - x)y = y(b^{-1}f(x) - x).$$

This means that $\zeta(x) = b^{-1}f(x) - x$ lies in C. Thus, in the case when $\Delta \neq 0$, f has the form $f(x) = bx + \zeta(x)b$. It remains to consider the case when $\Delta = 0$, that is, $f(x)y = \sigma(y)f(x)$ for all $x, y \in R$. By Lemma 3.1 it follows that $\sigma(x) = bxb^{-1}$, $x \in R$, for some invertible $b \in Q_s$. Whence $f(x)y = byb^{-1}f(x)$; that is, $b^{-1}f(x)y = yb^{-1}f(x)$. Therefore $\zeta(x) = b^{-1}f(x)$ lies in C. Thus, in the case when $\Delta = 0$, f has the form $f(x) = \zeta(x)b$. The prof is complete.

4. Results on Two-Sided Ideals

Throughout this section, I *will be an ideal of a prime ring* R. Our goal is to characterize generalized biderivations of I. In order to do this we need some preliminary results. We first prove a somewhat more general version of [9, Theorem 3.3].

Proposition 4.1. *Let* $D: I \times I \to Q_r$ *be a biderivation. If* R *is noncommutative, then there exists* $\lambda \in C$ *such that* $D(x, y) = \lambda[x, y]$ *for all* $x, y \in I$.

Proof. There is nothing to prove if $I = 0$, so assume $I \neq 0$. It is easily seen that I is then noncommutative. That is, a map $F: I \times I \to Q_r$ defined by $F(x, y) = [x, y]$ is not zero. Set $M = I \times I$ and note that in view of Corollary 2.4, the maps $F, D: M \to Q_r$ satisfy the requirements of Lemma 2.2. Thus there exists $\lambda \in C$ such that $D(x, y) = \lambda F(x, y) = \lambda[x, y]$.

We shall say that a map f is *additive modulo* C if

$$f(x + y) - f(x) - f(y) \in C$$

for all x, y in a domain of f. This includes additive maps, maps with range in C, and sums of such maps. As a consequence of Proposition 4.1 we obtain a slight generalization of the main result of [5].

Corollary 4.2. *Let* $f: I \to Q_r$ *be additive modulo* C. *If* f *is commuting then there exists* $\lambda \in C$ *and a map* $\zeta: I \to C$ *such that* $f(x) = \lambda x + \xi(x)$ *for all* $x \in I$.

Proof. Clearly, we may assume that R is noncommutative and $I \neq 0$. Replacing x by $x + y$ in $[f(x), x] = 0$ we arrive at $[f(x), y] = [x, f(y)]$. Therefore, the map $(x, y) \to [f(x), y]$ is a biderivation of I. By Proposition 4.1 there exists $\lambda \in C$ such that $[f(x), y] = \lambda[x, y]$, $x, y \in I$. That is, $[f(x) - \lambda x, y] = 0$, and hence it follows at once that $\zeta(x) = f(x) - \lambda x$ lies in C.

Lemma 4.3. *Let* $f: I \to R$ *be additive modulo* C. *Suppose that*

$$[r, f(yr) - f(y)r] = 0$$

for all $y \in I$, $r \in R$. *Then there exists* $a \in Q_r$ *and a map* $\lambda: I \to C$ *such that* $f(y) = ay - \lambda(y)$ *for all* $y \in I$.

Proof. Again, we may assume that R is noncommutative, and that $I \neq 0$.

Pick $y \in I$. By assumption, the map $r \to f(yr) - f(y)r$ is additive modulo C and commuting on R. Therefore, by Corollary 4.2 there exist

$\lambda(y) \in C$ and a map ζ_y: $R \to C$ such that

$$f(yr) - f(y)r = \lambda(y)r + \zeta_y(r)$$

for all $r \in R$, $y \in I$. Define k: $I \to R + C$ by $k(y) = f(y) + \lambda(y)$. For $y \in I$, $r \in R$, we have

$$\begin{aligned} k(yr) &= f(yr) + \lambda(yr) \\ &= f(y)r + \lambda(y)r + \zeta_y(r) + \lambda(yr) \\ &= k(y)r + \zeta_y(r) + \lambda(yr). \end{aligned}$$

Thus, $k(yr) = k(y)r + \varepsilon(y, r)$ where $\varepsilon(y, r) \in C$. Take $y \in I$, $r, s \in R$, and consider $k(yrs)$. On the one hand we have

$$k(y(rs)) = k(y)rs + \varepsilon(y, rs).$$

On the other hand,

$$\begin{aligned} k((yr)s) &= k(yr)s + \varepsilon(yr, s) \\ &= k(y)rs + \varepsilon(y, r)s + \varepsilon(yr, s). \end{aligned}$$

Comparing both relations we see that $\varepsilon(y, r)s \in C$ for all $y \in I$, $r, s \in R$. Since we assumed that R is noncommutative, we are forced to conclude that $\varepsilon(y, r) = 0$ for all $y \in I$, $r \in R$. That is,

$$k(yr) = k(y)r \qquad \text{for all } y \in I, r \in R.$$

Our next goal is to show that k is additive. Obviously, k is additive modulo C. This means that given $x_1, x_2 \in I$,

$$k(x_1 + x_2) = k(x_1) + k(x_2) + \omega(x_1, x_2)$$

for some $\omega(x_1, x_2) \in C$. Thus, for $y_1, y_2 \in I$, $r \in R$, we have

$$\begin{aligned} k(y_1r + y_2r) &= k(y_1r) + k(y_2r) + \omega(y_1r, y_2r) \\ &= k(y_1)r + k(y_2)r + \omega(y_1r, y_2r). \end{aligned}$$

On the other hand,

$$\begin{aligned} k(y_1r + y_2r) &= k((y_1 + y_2)r) = k(y_1 + y_2)r \\ &= k(y_1)r + k(y_2)r + \omega(y_1, y_2)r. \end{aligned}$$

Comparing we get $\omega(y_1, y_2)r \in C$ for all $y_1, y_2, \in I$, $r \in R$. But then $\omega(y_1, y_2) = 0$ for R is noncommutative. Whence k is additive.

Set $J = I[R, R]$, and (using the identity $[r, s]t = [r, st] - s[r, t]$) note that J is a nonzero ideal of R contained in I. We claim that $k(J) \subseteq R$. Indeed, given $y \in I$, $r, s \in R$, we have

$$k(y[r,s]) = k(y)[r,s] = [(y)r,s] - [k(y),s]r$$
$$= [k(yr),s] - [k(y),s]r,$$

and since $k(I) \subseteq R + C$, it follows that $k(y[r, s]) \in R$. Whence we see that the restriction of k to J is a right R-module map of J into R. Therefore, there exists $a \in Q_r$ such that $k(w) = aw$ for all $w \in J$. For $y \in I$, $w \in J$, we then have $k(yw) = ayw$, and on the other hand, $k(yw) = k(y)w$. Thus, $(k(y) - ay)J = 0$, and hence $k(y) = ay$ for every $y \in I$. Now $f(y) = k(y) - \lambda(y) = ay - \lambda(y)$, $y \in I$, and the lemma is proved.

LEMMA 4.4. *Let g_1, g_2: $I \to Q_r$ be any maps. Suppose that one of the following conditions is fulfilled*:

(i) $xg_1(y) + yg_2(x) = 0$ *for all* $x, y \in I$,

(ii) $g_1(x)y + g_2(y)x = 0$ *for all* $x, y \in I$.

Then either $g_1 = g_2 = 0$ *or R is commutative.*

Proof. Suppose that (i) is fulfilled. Then $rxg_1(y) = -ryg_2(x)$ for $x, y \in I$, $r \in R$. But on the other hand, since $rx \in I$, we have $rxg_1(y) = -yg_2(rx)$. Thus, $ryg_2(x) = yg_2(rx)$ for all $r \in R$, $x, y \in I$. If $g_2(x) \neq 0$ for some $x \in I$, then Lemma 2.1 implies that r and 1 are C-dependent for any $r \in R$, which means that R is commutative. On the other hand, if $g_2 = 0$, then we clearly have $g_1 = 0$ too. Similarly we consider the condition (ii).

LEMMA 4.5. *Suppose that maps f_1, f_2, f_3, f_4: $I \to R$ are additive modulo C and that*

$$f_1(x)y + xf_2(y) + f_3(y)x + yf_4(x) = 0$$

for all $x, y \in I$. Then there exist $a, b \in Q_s$ and maps λ, μ: $I \to C$ such that

$$f_1(x) = -xa + \mu(x), \qquad f_2(x) = ax - \lambda(x)$$
$$f_3(x) = -xb + \lambda(x), \qquad f_4(x) = bx - \mu(x).$$

for all $x \in I$.

Proof. We first consider the case when R is commutative. Then we have

$$(f_1(x) + f_4(x))y + (f_2(y) + f_3(y))x = 0,$$

$x, y \in I$. Of course, we may assume that $I \neq 0$. Pick a nonzero $y_0 \in I$.

Then y_0 is invertible in C, and hence $f_1(x) + f_4(x) = -cx$ where $c = y_0^{-1}(f_2(y_0) + f_3(y_0)) \in C$. Consequently, $(f_2(y) + f_3(y) - cy)x = 0$ for all $x, y \in I$, and therefore, $f_2(y) + f_3(y) = cy$, $y \in I$. Letting $a = c$, $b = 0$, $\lambda(x) = f_3(x)$, and $\mu(x) = -f_4(x)$ we obtain the desired result.

We may therefore assume that R is noncommutative. Fix $r \in R$. By Corollary 2.6 we have $xg_1(y) + yg_2(x) = 0$, $x, y \in I$, where

$$g_1(y) = [r, f_2(yr) - f_2(y)r], \qquad g_2(x) = [r, f_4(xr) - f_4(x)r].$$

Lemma 4.4 yields $g_1 = g_2 = 0$. Thus,

$$[r, f_2(yr) - f_2(y)r] = 0, \qquad [r, f_4(yr) - f_4(y)r] = 0$$

for all $y \in I$, $r \in R$. By Lemma 4.3 it follows that there exist $a, b \in Q_r$ and maps $\lambda, \mu \colon I \to C$ such that

$$f_2(y) = ay - \lambda(y), \qquad f_4(y) = by - \mu(y)$$

for all $y \in I$. Our hypothesis can now be written as

$$(f_1(x) + xa - \mu(x))y + (f_3(y) + yb - \lambda(y))x = 0$$

for all $x, y \in I$. Applying Lemma 4.4 again we arrive at

$$f_1(x) + xa - \mu(x) = 0, \qquad f_3(x) + xb - \lambda(x) = 0.$$

It remains to prove that a and b are elements of Q_s. Since $xa = -f_1(x) + \mu(x) \in R + C$ for every $x \in I$, we have $[r, xa] \in R$ for all $r \in R$, $x \in I$. Therefore, $[r, s]xa = [r, sxa] - s[r, xa] \in R$ for all $r, s \in R$, $x \in I$, which means that the ideal $K = [R, R]I \neq 0$ satisfies $Ka \subseteq R$. Thus $a \in Q_s$. Similarly we see that $b \in Q_s$. The proof is complete.

LEMMA 4.6. *Suppose that $I \neq 0$. If $q_1, q_2 \in Q_r$ are such that $q_1x + xq_2 \in C$ for all $x \in I$, then either $q_1 = -q_2 \in C$ or R is commutative. In particular, if $q \in Q_r$ is such that $qI \subseteq C$ or $Iq \subseteq C$, then either $q = 0$ or R is commutative.*

Proof. Set $\tau(x) = q_1x + xq_2$, $x \in I$. By assumption, $\tau(x) \in C$. For $x \in I$, $r \in R$, we have $\tau(xr) - \tau(x)r = x[r, q_2]$, and hence we see that $x[r, q_2]$ commutes with r. That is, $rx[r, q_2] = x[r, q_2]r$ for all $x \in I$, $r \in R$. Lemma 2.1 implies that given $r \in R$, we have either $[r, q_2] = 0$ or r is C-dependent with 1, that is, $r \in Z$. In any case, $[r, q_2] = 0$ and so $q_2 \in C$. Similarly one shows that $q_1 \in C$. Therefore, $(q_1 + q_2)I \subseteq C$. Thus, if $q_1 + q_2 \neq 0$, then $I \subseteq C$, and hence R is commutative.

We now have enough information to prove

THEOREM 4.7. *Let I be an ideal of a prime ring. If $G: I \times I \to R$ is a generalized biderivation, then there exist $a, b \in Q_s$ such that*

$$G(x, y) = xay + ybx$$

for all $x, y \in I$.

Proof. As G is a generalized derivation in the first argument, for any $y \in I$ there exist $g_1(y), g_2(y) \in R$ such that

$$G(x, y) = g_1(y)x + xg_2(y).$$

Thus,

$$G(x, y_1 + y_2) = g_1(y_1 + y_2)x + xg_2(y_1 + y_2).$$

On the other hand, since G is additive in the second argument, we have

$$\begin{aligned} G(x, y_1 + y_2) &= G(x, y_1) + G(x, y_2) \\ &= g_1(y_1)x + xg_2(y_1) + g_1(y_2)x + xg_2(y_2). \end{aligned}$$

Comparing we obtain

$$\begin{aligned} &\{g_1(y_1 + y_2) - g_1(y_1) - g_1(y_2)\}x \\ &\quad + x\{g_2(y_1 + y_2) - g_2(y_1) - g_2(y_2)\} = 0 \end{aligned}$$

for all $x, y_1, y_2 \in I$. By Lemma 4.6 we conclude that

$$\begin{aligned} g_1(y_1 + y_2) - g_1(y_1) - g_1(y_2) &= -\{g_2(y_1 + y_2) - g_2(y_1) - g_2(y_2)\} \\ &\in C. \end{aligned}$$

Thus, the maps $g_1, g_2: I \to R$ are additive modulo C.

Using the fact that G is a generalized derivation in the second argument, one shows in an analogous way that

$$G(x, y) = g_3(x)y + yg_4(x),$$

$x, y \in I$, where $g_3, g_4: I \to R$ are additive modulo C. We compare both expressions of $G(x, y)$ and obtain

$$g_3(x)y - xg_2(y) - g_1(y)x + yg_4(x) = 0$$

for all $x, y \in I$. Now we can apply Lemma 4.5. Thus, there exist $a, b \in Q_s$

and maps $\lambda, \mu\colon I \to C$ such that

$$g_3(x) = xa + \mu(x), \qquad g_2(x) = ax + \lambda(x),$$
$$g_1(x) = xb - \lambda(x), \qquad g_4(x) = bx - \mu(x).$$

But then $G(x, y) = g_3(x)y + yg_4(x) = xay + ybx$. The theorem is proved.

As Theorem 4.7 follows quite easily from Lemma 4.5, one might wonder whether the lemma is actually a deeper result. This is not true. Namely, Lemma 4.5 can be easily deduced from Theorem 4.7. Essentially they are just different forms of the same result.

COROLLARY 4.8. *Let R be noncommutative and let $G\colon I \times I \to R$ be a generalized biderivation.*

(i) *If G is symmetric (that is, $G(x, y) = G(y, x)$, $x, y \in I$) then there exists $a \in Q_s$ such that $G(x, y) = xay + yax$, $x, y \in I$.*

(ii) *If G is anti-symmetric (that is, $G(x, y) = -G(y, x)$, $x, y \in I$) then there exists $a \in Q_s$ such that $G(x, y) = xay - yax$, $x, y \in I$.*

Proof. By Theorem 4.7 we have $G(x, y) = xay + ybx$ for some $a, b \in Q_s$. Let G be symmetric. Then $xay + ybx = yax + xby$, that is, $x(a - b)y + y(b - a)x = 0$, $x, y \in I$. By Lemma 4.4 it follows that $(a - b)y = 0$, $y \in I$, and so $a = b$. This proves (i). Similarly one proves (ii).

We are now in a position to generalize the results on commuting and skew-commuting maps mentioned in the introduction. The first such generalization is

COROLLARY 4.9. *Suppose that additive maps $f, g\colon I \to R$ satisfy*

$$f(x)x = xg(x)$$

for all $x \in I$. Then there exist $a \in Q_s$ and an additive map $\zeta\colon I \to C$ such that

$$f(x) = xa + \zeta(x), \; g(x) = ax + \zeta(x)$$

for all $x \in I$.

Proof. In the case when R is commutative, the result is trivial. Namely, we have $(f(x) - g(x))x = 0$, $x \in I$, and hence $f(x) = g(x)$ for every $x \in I$. Therefore, set $a = 0$, $\zeta = f = g$ and the result is proved.

Now let R be noncommutative. A linearization of $f(x)x = xg(x)$ shows that $(x, y) \to xg(y) - f(y)x$ is an anti-symmetric generalized biderivation of I. Therefore, Corollary 4.8 tells us that there exists $a \in Q_s$ such that

$xg(y) - f(y)x = xay - yax$, $x, y \in I$, that is, $x(g(y) - ay) + (ya - f(y))x = 0$, $x, y \in I$. By Lemma 4.6 it follows that $g(y) - ay = f(y) - ya \in C$, $y \in I$. This proves the result.

COROLLARY 4.10. *Let R be noncommutative. Suppose that an additive map $f\colon I \to R$ satisfies*

$$c_1 f(x)y + c_2 xf(y) + c_3 f(y)x + c_4 yf(x) = 0$$

for all $x, y \in I$ and some $c_1, c_2, c_3, c_4 \in Z$, not all zero. Then f is of the form $f(x) = \lambda x + \zeta(x)$ where $\lambda \in C$ and ζ is an additive map of I into C. Moreover, if $\lambda \neq 0$ then $c_1 = -c_2$ and $c_3 = -c_4$, and if $\zeta \neq 0$ then $c_1 = -c_4$ and $c_2 = -c_3$.

Proof. By Lemma 4.5 there exist $a, b \in Q_s$ and additive maps $\lambda, \mu\colon I \to C$ such that

$$c_1 f(x) = -xa + \mu(x), \qquad c_2 f(x) = ax - \lambda(x),$$
$$c_3 f(x) = -xb + \lambda(x), \qquad c_4 f(x) = bx - \mu(x).$$

First, assume that $c_1 \neq 0$ and $c_2 \neq 0$. Set $\alpha_i = c_i^{-1} \in C$, $i = 1, 2$. We have $f(x) = -x(\alpha_1 a) + \alpha_1 \mu(x) = (\alpha_2 a)x - \alpha_2 \lambda(x)$. Whence $(\alpha_2 a)x + x\alpha_1 a) \in C$, $x \in I$. Clearly, we may assume that $I \neq 0$, and therefore $\alpha_2 a = -\alpha_1 a \in C$ by Lemma 4.6. Thus, $f(x) = \lambda x + \zeta(x)$ where $\lambda = \alpha_2 a$, $\zeta(x) = -\alpha_2 \lambda(x)$.

Now assume that $c_1 \neq 0$ and $c_2 = 0$. Then $ax = \lambda(x)$, $x \in I$, and hence $a = 0$ by Lemma 4.6. But then $c_1 f(x) = \mu(x)$, $x \in I$, which means that f maps I into C. The same conclusion holds if $c_1 = 0$ and $c_2 \neq 0$.

The remaining case, namely, $c_1 = c_2 = 0$ and one of c_3, c_4 is not 0, can be considered analogously. Thus, we have $f(x) = \lambda x + \zeta(x)$. Using this in our initial hypothesis we obtain

$$\{\lambda(c_1 + c_2)x + (c_1 + c_4)\zeta(x)\}y + \{\lambda(c_3 + c_4)y + (c_2 + c_3)\zeta(y)\}x = 0$$

for all $x, y \in I$. By Lemma 4.4 it follows that

$$\lambda(c_1 + c_2)x + (c_1 + c_4)\zeta(x) = 0, \qquad x \in I,$$
$$\lambda(c_3 + c_4)y + (c_2 + c_3)\zeta(y) = 0, \qquad y \in I.$$

Since R is noncommutative, I cannot be contained in Z. Whence it follows that $\lambda(c_1 + c_2) = \lambda(c_3 + c_4) = 0$, $(c_1 + c_4)\zeta(I) = (c_2 + c_3)\zeta(I) = 0$; this clearly yields the desired result.

Remark. As we have mentioned in the introduction, Lanski [15, 16] recently considered conditions similar to the one in Corollary 4.10, for the case where f is a derivation. One difference is that in Lanski's work the elements c_i were supposed to belong to C (and not necessarily to Z). Anyway, Corollary 4.10 remains true if we assume that $c_i \in C$. The reason why we cannot prove this directly is that the ranges of the maps $x \to c_i f(x)$ then lie in R_c (and not necessarily in R), so that Lemma 4.5 cannot be applied. However, throughout this section (including Lemma 4.5) we could consider maps with ranges in R_c. It turns out that the only difference would then be that instead of dealing with Q_s we would have to deal with the symmetric Martindale ring of quotients of R_c (these two rings are not always equal—see, for example, [11, pp. 268–269]).

COROLLARY 4.11. *Suppose that an additive map f: $I \to R$ satisfies $c_1 f(x)x = c_2 x f(x)$ for all $x \in I$ and some $c_1, c_2 \in Z$, not both zero. Then f is of the form $f(x) = \lambda x + \zeta(x)$ where $\lambda \in C$ and ζ is an additive map of I into C. Moreover, if $f \neq 0$ then $c_1 = c_2$.*

Proof. Linearizing $c_1 f(x)x = c_2 x f(x)$ we get $c_1 f(x)y - c_2 x f(y) + c_1 f(y)x - c_2 y f(x) = 0$. Now apply Corollary 4.10.

As an immediate consequence we obtain the following result which was recently obtained by quite different means in [8].

COROLLARY 4.12. *Let f: $I \to R$ be an additive skew-commuting map. If the characteristic of R is not 2, the $f = 0$.*

As an illustration how our results can be applied to the study of derivations, we now establish

COROLLARY 4.13. *Let f_1, f_2, f_3, f_4: $I \to R$ be additive maps such that*

$$f_1(x)y + xf_2(y) + f_3(y)x + yf_4(x) = 0$$

for all $x, y \in I$. If for some $i \in \{1,2,3,4\}$, f_i is a nonzero derivation, then R is commutative.

Proof. By Lemma 4.5 there are $a \in Q_s$ and a map ζ: $I \to C$ such that either $f_i(x) = ax + \zeta(x)$ or $f_i(x) = xa + \zeta(x)$. For the sake of symmetry it suffices to consider just one possibility, say, the second one. Denote f_i by d. For $x, y \in I$, we have

$$xya + \zeta(xy) = d(xy) = d(x)y + xd(y) = d(x)y + xya + \zeta(y)x.$$

Thus, $d(x)y = \zeta(xy) - \zeta(y)x$, and hence $d(x)y$ commutes with x. That is, $d(x)yx = xd(x)y$, $x, y \in I$. By Lemma 2.1 we conclude that given $x \in I$ either $x \in Z$ or $d(x) = 0$. Since the sets of $x \in I$ for which these two alternatives hold form two additive subgroups of I with union equal to

I, we conclude that either $I \subseteq Z$ or $d(I) = 0$. However, we assumed that $d \neq 0$ and so $I \subseteq Z$ which implies that R is commutative.

Our next aim is to consider the situation where additive maps $f_i\colon I \to R$ satisfy $f_1(x)y + xf_2(y) + f_3(y)x + yf_4(x) \in Z$ for all $x, y \in I$. First we need some preliminary results.

For an element $r \in R$, we denote by $C(r)$ the centralizer of r in Q_r; that is, $C(r) = \{x \in Q_r | [x, r] = 0\}$.

LEMMA 4.14. *Let $g_1, g_2\colon I \to Q_r$ be any maps and let $r \in R$. Suppose that one of the following two conditions is fulfilled*:

(i) $xg_1(y) + yg_2(x) \in C(r)$ *for all* $x, y \in I$,

(ii) $g_1(x)y + g_2(y)x \in C(r)$ *for all* $x, y \in I$.

Then either $g_1 = g_2 = 0$ *or* $r^2 = \lambda r + \mu$ *for some* $\lambda, \mu \in C$.

Proof. For symmetry it suffices to consider the condition (i) and to show that $g_1 \neq 0$ implies that $r^2 = \lambda r + \mu$.

Replacing in (i) y by ry we obtain $xg_1(ry) + ryg_2(x) \in C(r)$. That is,

$$xg_1(ry) - rxg_1(y) + r(xg_1(y) + yg_2(x)) \in C(r).$$

The second summand itself is contained in $C(r)$, so that $xg_1(ry) - rxg_1(y) \in C(r)$. Thus, $[xg_1(ry) - rxg_1(y), r] = 0$ which can be written in the form

$$r^2xg_1(y) - rx(g_1(ry) + g_1(y)r) + xg_1(ry)r = 0$$

for all $x, y \in I$. Since $g_1(y) \neq 0$ for some $y \in I$, Lemma 2.1 implies that r^2, r and 1 are C-dependent. This proves the lemma.

The following lemma is known by standard PI theory.

LEMMA 4.15. *The following statements are equivalent*:

(i) *R satisfies S_4*,

(ii) *R is commutative or R embeds in $M_2(F)$ for some field F*,

(iii) *R is algebraic of bounded degree 2 over C (that is, for any $r \in R$ there exist $\lambda, \mu \in C$ such that $r^2 = \lambda r + \mu$)*,

(iv) $[[r^2, s], [r, s]] = 0$ *for all* $r, s \in R$.

LEMMA 4.16. *Let $g_1, g_2\colon I \to Q_r$ be any maps. Suppose that one of the following two conditions is fulfilled*:

(i) $xg_1(y) + yg_2(x) \in C$ *for all* $x, y \in I$,

(ii) $g_1(x)y + g_2(y)x \in C$ *for all* $x, y \in I$.

Then either $g_1 = g_2 = 0$ *or R satisfies S_4*.

Proof. Note that $C = \bigcap_{r \in R} C(r)$. Now apply Lemmas 4.14 and 4.15.

We continue with a technical lemma.

LEMMA 4.17. *Let G and H be additive groups, and let maps $\Gamma: G \times G \times G \to H$ and $\Omega: G \times G \to H$ be additive in each argument. Suppose that for any $x \in G$ either $\Gamma(x, x, x) = 0$ or $\Omega(x, x) = 0$. If H is 2-torsion-free and 3-torsion-free, then either $\Gamma(x, x, x) = 0$ for all $x \in G$ or $\Omega(x, x) = 0$ for all $x \in G$.*

Proof. Let $A = \{x \in G | \Gamma(x, x, x) = 0\}$, $B = \{x \in G | \Omega(x, x) = 0\}$. By our assumption, $G = A \cup B$. We must show that $A = G$ or $G = B$. Suppose this is not true. Thus, there are $x, y \in G$ such that $x \notin A$ and $y \notin B$. Therefore, $x \in B$ and $y \in A$.

Suppose that $x + y \in B$, that is, $\Omega(x + y, x + y) = 0$. Since $\Omega(x, x) = 0$ it follows that $h + k = 0$ where $h = \Omega(x, y) + \Omega(y, x)$, $k = \Omega(y, y)$. If also $x - y \in B$, then we have $-h + k = 0$. However, since H is 2-torsion-free, this implies $k = 0$ which contradicts the assumption $y \notin B$. Whence $x - y \in A$, and similarly one proves that $x + 2y$ and $x - 2y \in A$. Thus, $\Gamma(y, y, y) = 0$, $\Gamma(x - y, x - y, x - y) = 0$, $\Gamma(x + 2y, x + 2y, x + 2y) = 0$, $\Gamma(x - 2y, x - 2y, x - 2y) = 0$, and note that these relations imply

$$\begin{aligned} a - b + c &= 0, \\ a + 2b + 4c &= 0, \\ a - 2b + 4c &= 0, \end{aligned}$$

where $a = \Gamma(x, x, x)$, $b = \Gamma(y, x, x) + \Gamma(x, y, x) + \Gamma(x, x, y)$, $c = \Gamma(y, y, x) + \Gamma(y, x, y) + \Gamma(x, y, y)$. Using the assumption that H is 2-torsion-free and 3-torsion-free it follows that $a = 0$, contrary to the assumption.

This contradiction shows that $x + y \notin B$, and hence $x + y \in A$. Similarly we see that $x - y \in A$ and $x + 2y \in A$. However, then it follows easily that x belongs to A, contrary to the assumption. The lemma is thereby proved.

We are now in a position to prove

THEOREM 4.18. *Let I be an ideal of a prime ring R. Let $f_1, f_2, f_3, f_4: I \to R$ be additive maps and set*

$$\pi(x, y) = f_1(x)y + xf_2(y) + f_3(y)x + yf_4(x).$$

(i) *If $\pi(x, y) \in Z$ for all $x, y \in I$ and the characteristic of R is not 2 or 3, then either R satisfies S_4 or $\pi(x, y) = 0$ for all $x, y \in I$.*

(ii) *If* $\pi(x,y)=0$ *for all* $x,y\in I$ *then there exist* $a,b\in Q_s$ *and additive maps* $\lambda,\mu\colon I\to C$ *such that for every* $x\in I$,

$$f_1(x)=-xa+\mu(x),\qquad f_2(x)=ax-\lambda(x)$$
$$f_3(x)=-xb+\lambda(x),\qquad f_4(x)=bx-\mu(x).$$

Proof. By Lemma 2.5, for all $x,y\in I$, $r\in R$ we have

$$x[r,f_2(yr)-f_2(y)r]+y[r,f_4(xr)-f_4(x)r]$$
$$=\pi(x,y)r^2-(\pi(xr,y)+\pi(x,yr))r+\pi(xr,yr).$$

Therefore, given $r\in R$, we have $xg_1(y)+yg_2(x)\in C(r)$, $x,y\in I$, where $g_1(y)=[r,f_2(yr)-f_2(y)r]$, $g_2(x)=[r,f_4(xr)-f_4(x)r]$. In view of Lemma 4.14, either $g_1=g_2=0$, that is,

$$[r,f_2(yr)-f_2(y)r]=0,\ [r,f_4(yr)-f_4(y)r]=0\qquad\text{for all } y\in I,$$

or $r^2=\lambda r+\mu$ for some $\lambda,\mu\in C$, and hence

$$[[r^2,s],[r,s]]=0\qquad\text{for all } s\in R.$$

We assume henceforth that R does not satisfy S_4. Therefore, there exist $r_0,s_0\in R$ such that $[[r_0^2,s_0],[r_0,s_0]]\neq 0$ (Lemma 4.15). Define $\Gamma\colon R\times R\times R\to R$ by

$$\Gamma(r,s,t)=[[rs,s_0],[t,s_0]],$$

and for any $y\in I$ define $\Omega_y\colon R\times R\to R$ by

$$\Omega_y(r,s)=[r,f_2(ys)-f_2(y)s].$$

We proved that for any $r\in R$, either $\Gamma(r,r,r)=0$ or $\Omega_y(r,r)=0$ for any $y\in I$. By assumption, $\Gamma(r_0,r_0,r_0)\neq 0$. Therefore, Lemma 4.17 implies that, provided the characteristic of R is not 2 or 3, $\Omega_y(r,r)=0$- for all $r\in R$ and $y\in I$. That is, $[r,f_2(yr)-f_2(y)r]=0$ for all $r\in R$, $y\in I$. Similarly we see that $[r,f_4(yr)-f_4(y)r]=0$, $r\in R$, $y\in I$. According to Lemma 4.3 there exist $a,b\in Q_r$ and additive maps $\lambda,\mu\colon I\to C$ such that $f_2(y)=ay-\lambda(y)$, $f_4(y)=by-\mu(y)$, $y\in I$. Therefore,

$$\pi(x,y)=(f_1(x)+xa-\mu(x))y+(f_3(y)+yb-\lambda(y))x.$$

Since $\pi(x,y)\in Z$, Lemma 4.16 now yields $\pi(x,y)=0$, $x,y\in I$. This proves (i). The second statement follows from Lemma 4.5; on the other hand, it can be derived easily from the present proof using 4.4, 4.3.

5. Results on Right Ideals

Throughout this section, *T will be a right ideal of a prime ring R*. We shall see that for additive maps of T into R similar results hold as for maps of I into R obtained in the previous section, unless T is too "small." The following lemma indicates what we mean by that.

Lemma 5.1. *Suppose $T \neq 0$. Then $[T, T]T = 0$ if and only if TC is a minimal right ideal of R_c and there exists an idempotent $e \in R_c$ such that $TC = eR_c$ and $eR_ce = Ce$.*

Proof. Note that $U = TC$ is a nonzero right ideal of R_c, and that $[T, T]T = 0$ implies $[U, U]U = 0$. Thus, for $u, v, w \in U$ we have $[u, v]w = 0$; replacing v by vr where $r \in R_c$ it follows that $uvrw = vruw$. Therefore, by Lemma 2.1, uv and v are C-dependent for any $u, v \in U$; thus, there exists $\lambda(u, v) \in C$ such that $uv = \lambda(u, v)v$. From $uvrw = vruw$ we then get $(\lambda(u, v) - \lambda(u, w))vR_cw = 0$, and hence, if $v \neq 0$ and $w \neq 0$, $\lambda(u, v) = \lambda(u, w)$. This shows that $\lambda(u, v)$ does not depend on v. Thus, for any $u \in U$ there is $\lambda(u) \in C$ such that $uv = \lambda(u)v$ for every $v \in U$. That is, $(u - \lambda(u))U = 0$.

In order to show that U is a minimal right ideal of R_c, pick a nonzero right ideal V of R_c contained in U. We must show that $U = V$. Take a nonzero element $u \in U$. For any $v \in V \subseteq U$ we have $vu = \lambda(v)u$. Since R is prime and $V \neq 0$, we may choose $v_0 \in V$ such that $v_0u \neq 0$, and hence $\lambda(v_0) \neq 0$. Consequently, $u = v_0(\lambda(v_0)^{-1}u)$ lies in V for V is a right ideal of R_c. Whence $U = V$.

Thus, U is indeed a minimal right ideal of R_c. Therefore, as it is well-known , there exists an idempotent $e \in R_c$ such that $U = eR_c$ [12, Proposition 1, p. 57]. Pick $r \in R_c$. Then $er \in U$ and hence $(er - \lambda(er))U = 0$. In particular, $ere = \lambda(er)e$. This shows that $eR_ce = Ce$.

The converse implication can be verified by a direct computation.

The main purpose of this section is to prove

Theorem 5.2. *Let T be a right ideal of a prime ring R. Then the following four assertions hold*:

(i) *every biderivation $D: T \times T \to R$ is of the form $D(x, y) = \lambda[x, y]$, $x, y \in T$, for some $\lambda \in C$,*

(ii) *every commuting additive map $f: T \to R$ is of the form $f(x) = \lambda x + \zeta(x)$, $x \in T$, for some $\lambda \in C$ and an additive map ζ of T into C,*

(iii) *if an additive map $f: T \to R$ satisfies $xf(x) = 0$ for all $x \in T$, then $f = 0$,*

(iv) *if the characteristic of R is not 2 and $f\colon T \to R$ is an additive skew-commuting map, then $f = 0$,*

unless TC is a minimal right ideal of R_c and there exists an idempotent $e \in R_c$ such that $TC = eR_c$ and $eR_ce = Ce$.

Proof. Obviously, we may assume that $T \neq 0$.

(i) Since T is a right ideal of R, it follows from Corollary 2.4 that

$$D(x, y)zr[u, v] = [x, y]zrD(u, v)$$

for all $x, y, z, u, v \in T$, $r \in r$. In view of Lemma 5.1 we may assume that $a = [x, y]z \neq 0$ for some $x, y, z \in T$. Set $b = D(x, y)z$, and so we have $br[u, v] = arD(u, v)$ for all $u, v \in T$, $r \in R$. There is nothing to prove if $D = 0$; therefore, by Lemma 2.1, we may assume that a and b are C-dependent. That is, $b = \lambda a$ for some $\lambda \in C$. But then $ar(\lambda[u, v] - D(u, v)) = 0$ for all $u, v \in T$, $r \in R$. Whence $D(u, v) = \lambda[u, v]$.

(ii) As in the proof of Corollary 4.2, we observe that the map $(x, y) \to [f(x), y]$ is a biderivation of T. Applying (i) and noting that elements in C are the only elements commuting with every element in T, we obtain the desired result.

(iii) Linearizing $xf(x) = 0$, $x \in T$, we get $xf(y) + yf(x) = 0$, $x, y \in T$. Replacing x by xz we arrive at $xzf(y) + yf(xz) = 0$, $x, y, z \in T$. On the other hand, $xzf(y) = -xyf(z)$, and hence $xyf(z) = yf(xz)$, $x, y, z \in T$. But then, for $x, y, z, w \in T$, we have $x(yw)f(z) = ywf(xz) = y(wf(xz)) = y(xwf(z))$. Thus, $[T, T]Tf(T) = 0$. But then either $[T, T]T = 0$ or $f = 0$.

(iv) We have $f(x)x + xf(x) = 0$, $x \in T$, and hence $f(x)y + xf(y) + f(y)x + yf(x) = 0$, $x, y \in T$. By Corollary 2.6 it follows that

$$x[r, f(yr) - f(y)r] + y[r, f(xr) - f(x)r] = 0$$

for all $x, y \in T$, $r \in R$. Letting $x = y$, since the characteristic of R is not 2, we obtain $x[r, f(xr) - f(x)r] = 0$, $x \in T$, $r \in R$. That is, for any $r \in R$ we have $xf_r(x) = 0$ for all $x \in T$, where $f_r\colon T \to R$ is an additive map defined by $f_r(x) = [r, f(xr) - f(x)r]$. In view of (iii), we may assume that $f_r = 0$ for every $r \in R$.

Now fix $x \in I$. The fact that $f_r(x) = 0$ for every $r \in R$ means that $r \to f(xr) - f(x)r$ is a commuting additive map of R into itself. By Corollary 4.2 we therefore conclude that there exist $\lambda \in C$ and an additive map $\zeta\colon R \to C$ (both, of course, depending on x) such that

$$f(xr) - f(x)r = \lambda r + \zeta(r)$$

for every $r \in R$. Since f is skew-commuting, we have $f(xr)xr + xrf(xr) = 0$, $r \in R$, which can be written in the form

$$f(x)rxr + \lambda rxr + 2\zeta(r)xr + xrf(x)r + \lambda xr^2 = 0.$$

That is, $h(r)r = 0$ for every $r \in R$, where $h(r) = f(x)rx + \lambda rx + \lambda xr + xrf(x) + 2\zeta(r)x$. Of course, h is additive, and therefore, $h(r)s + h(s)r = 0$ for all $s, r \in R$. But then it follows from Lemma 4.4 that $h = 0$ or R is commutative; however, in the latter case, $h = 0$ follows at once from $h(r)r = 0$, $r \in R$. Thus, we have

$$yrx + xry = -2\zeta(r)x$$

for all $r \in R$, where $y = f(x) + \lambda$. Our immediate goal is to show that $[x, y] = 0$. Replacing r by rx in the relation above we get $yrx^2 + xrxy = -2\xi(rx)x$. On the other hand, $yrx^2 = (yrx)x = -2\zeta(r)x^2 - xryx$. Comparing the last two relations we arrive at $xr[x, y] = 2\zeta(r)x^2 - 2\zeta(rx)x$, $r \in R$. In particular, it follows that $xr[x, y]$ commutes with x. That is, $xr[x, y]x = x^2r[x, y]$ for every $r \in R$. By Lemma 2.1 we see that $[x, y] = 0$ unless x^2 and x are C-dependent. Thus, assume that $x^2 = \mu x$ for some $\mu \in C$. Multiplying $yrx + xry = -2\zeta(r)x$ from the left by x, we then get $xyrx + \alpha xry = \alpha(-2\zeta(r)x) = \alpha yrx + \alpha xry$. Whence $(xy - \alpha y)Rx = 0$ which gives $xy = \alpha y$. Similarly, multiplying $yrx + xry = -2\zeta(r)x$ from the right by x one derives that $yx = \alpha y$. Thus, x and y commute in this case as well. Since $y = f(x) + \lambda$ it follows that $[f(x), x] = 0$. Comparing this relation with $f(x)x + xf(x) = 0$, since the characteristic of R is not 2, we are forced to conclude that $f(x)x = xf(x) = 0$ where x is an arbitrary element in T. Applying (iii) we then see that (iv) holds.

The proof of the theorem is complete.

In the following lines we illustrate why we have to exclude the case when TC is a minimal right ideal.

EXAMPLES. 1. It is very easy to construct nonzero additive maps f of minimal right ideals such that $f(x)x = 0 = xf(x)$ for every x. Assume that $Z = C$ and let e be an idempotent in R such that $eRe = Ze$. Set $T = eR$ and note that for any $x \in T$ there is $\lambda(x) \in Z$ such that $(x - \lambda(x))T = 0$. Replacing x by $x + y$ it follows easily that $x \to \lambda(x)$ is an additive map. Therefore, setting $f(x) = x - \lambda(x)$ we see that f is additive and that $xf(x) = 0 = f(x)x$ for every $x \in T$. In particular, f is skew-commuting.

2. Let $R = M_n(F)$, the algebra of all $n \times n$ matrices over a field F, where $n \geq 3$. Then $R = R_c$ and $C = Z = FI$. Let e_{ij} be a matrix whose only nonzero entry is 1 in a position (i, j). Note that the minimal right

ideal $T = e_{11}R$ is equal to a linear span of $e_{11}, e_{12}, \ldots, e_{1n}$. Define f: $T \to R$ by $f(\lambda_1 e_{11} + \lambda_2 e_{12} + \cdots + \lambda_n e_{1n}) = \lambda_3 e_{2n} - \lambda_2 e_{3n}$. Then $f(x)x = 0 = xf(x)$ for every $x \in T$. In particular, f is commuting, but there is no $\lambda \in C$ and a map ζ: $T \to C$ such that $f(x) = \lambda x + \zeta(x)$. Of course, $(x, y) \to [f(x), y]$ is then a biderivation which is not of the form $(x, y) \to \lambda[x, y]$ for some $\lambda \in C$.

COROLLARY 5.3. *Let f be an additive map of T into itself. Then*:

(i) *If f if commuting then there exist $\lambda \in C$ and an additive map ζ: $T \to C$ such that $f(x) = \lambda x + \zeta(x)$ for every $x \in T$.*

(ii) *If $xf(x) = 0$ for all $x \in T$, then $f = 0$.*

(iii) *If f is skew-commuting and the characteristic of R is not 2, then $f = 0$.*

Proof. In view of Theorem 5.2 it suffices to consider the case where $TC = eR_c$ for some idempotent $e \in R_c$ such that $eR_c e = Ce$. For every $u \in T$ then there exists $\lambda(u) \in C$ such that $(u - \lambda(u))T = 0$. Note that $u \to \lambda(u)$ is an additive map of T into C. If $\lambda(u) = 0$ for every $u \in T$, then $T^2 = 0$ and hence $T = 0$. Therefore we may assume that $\lambda(T) \neq 0$. We remark that given $u, v \in T$, we have $f(u)v = \lambda(f(u))v$, $uf(v) = \lambda(u)f(v)$ for f maps T into T.

(i) Pick $a \in T$ such that $\lambda(a) \neq 0$. The relation $f(a)a = af(a)$ can be written in the form $\lambda(f(a))a = \lambda(a)f(a)$. We set $\alpha = \lambda(f(a))$, $\beta = \lambda(a)^{-1}$. Thus, $f(a) = \alpha\beta a$.

Replacing u by $u + a$ in $f(u)u = uf(u)$ we arrive at

$$f(u)a + f(a)u = uf(a) + af(u).$$

That is,

$$\lambda(f(u))a + \lambda(f(a))u = \lambda(u)f(a) + \lambda(a)f(u),$$

and hence

$$f(u) = \beta\lambda(f(u))a + \alpha\beta u - \alpha\beta^2\lambda(u)a.$$

Set $\nu(u) = \beta\lambda(f(u)) - \alpha\beta^2\lambda(u) \in C$. Thus, $f(u) = \alpha\beta u + \nu(u)a$ for all $u \in T$. Clearly, ν is an additive map of T into C. The result will be proved by showing that $\nu(u)a$ lies in C for every $u \in T$. From $[f(u), u] = 0$ it follows $\nu(u)[a, u] = 0$. Thus, for every $u \in T$ we have either $\nu(u) = 0$ or $[a, u] = 0$. Using the fact that a group cannot be the union of two proper subgroups it follows at once that either $\nu(T) = 0$ or $[a, T] = 0$. If the latter case occurs, then it is immediate that $a \in Z$. Thus, in any case $\nu(T)a$ lies in C. This proves (i).

(ii) For $x \in T$, we have $0 = xf(x) = \lambda(x)f(x)$. Whence either $\lambda(x) = 0$ or $f(x) = 0$. Using same group theory argument as in the proof of (i) we conclude that $f(T) = 0$ for we assumed that $\lambda(T) \neq 0$.

(iii) Pick $x \in T$ such that $\lambda(x) \neq 0$. The relation $f(x)x + xf(x) = 0$ yields $\lambda(f(x))x + \lambda(x)f(x) = 0$. Whence $f(x) = \gamma x$ where $\gamma = -\lambda(x)^{-1}\lambda(f(x))$. Consequently, $0 = f(x)x + xf(x) = 2\gamma x^2 = 2\gamma\lambda(x)x$. But then $\gamma = 0$, that is, $f(x) = 0$.

Thus we proved that given $x \in T$, either $\lambda(x) = 0$ or $f(x) = 0$. But then, as in the proof of (ii), it follows at once that $f = 0$.

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