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One-loop divergences in massive gravity theory

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ABSTRACT

The one-loop divergences are calculated for the recently proposed ghost-free massive gravity model, where the action depends on both metric and external tensor field f. The non-polynomial structure of the massive term is reduced to a more standard form by means of auxiliary tensor field, which is settled on-shell after quantum calculations are performed. As one should expect, the counter-terms do not reproduce the form of the classical action. Moreover, the result has the form of the power series in f.

1. Introduction

It is well known that general relativity is a non-linear dynamical theory of symmetric second rank tensor field in curved space–time. In the linear approximation this theory describes a propagation of massless spin-2 irreducible representation of the Poincare group. Dynamical theory of massive spin-2 representation of the Poincare group has been constructed by Fierz and Pauli [1] in terms of symmetric second rank tensor field in Minkowski space. It is natural to think that there should exist a non-linear dynamical theory in term of symmetric second rank tensor field in curved space–time, whose linear limit will be the Fierz–Pauli theory. However, during a long time such theory has not been constructed.

An evident way to find non-linear generalization of Fierz–Pauli theory is to add some kind of massive term into Lagrangian of general relativity. It could be a cosmological constant, but in the linear limit the cosmological term does not provide a true mass term in Fierz–Pauli theory. As a result the only way to insert a mass term to Lagrangian of general relativity is to use, additionally to metric, an extra second rank tensor field (reference metric) and find a coupling of metric to this extra field in such a way that in the linear approximation for both metric and Minkowski reference metric, this coupling should generate a true mass term in Fierz–Pauli theory.

The extension of general relativity described above has been studied in details by Boulware and Deser [2]. They have shown that inserting the mass term with the help of additional second rank tensor field can yield, in general, the inconsistent theory with propagating ghost field (BD ghost). However recently there was a progress in constructing a family of the non-linear dynamical theories which are free of problem of BD ghost [3–6] (see also the review [7]). As a result, at present one can have the Lagrangians which satisfy the following set of conditions: (i) describe consistent ghost-free non-linear dynamics of symmetric second rank tensor field; (ii) have a number of propagating degrees of freedom exactly corresponding to massive spin-2 field, (iii) depend of reference metric and on some mass parameter m, while in the limit $m = 0$ reproduce the Lagrangian of general relativity without cosmological term; (iv) reproduce the Lagrangian of Fierz–Pauli theory in linear approximation for dynamical metric and for Minkowski reference metric.

In this Letter we study the quantum aspects of the massive gravity theory.[2] To be more precise, we compute the one-loop divergences of a minimal massive theory [4] and investigate their structure. Of course, the massive gravity theory is non-renormalizable as well as general relativity since massive gravity Lagrangian includes general relativity Lagrangian and hence should lead to the quadratic in curvature tensor counter-terms at one loop. In order to find the one-loop divergences we will use the background field method and Schwinger–DeWitt proper-time techniques [9].

Although the theory under consideration is non-renormalizable, one can expect it to have many interesting features in quantum domain. First, the massive term in the action possesses a complicated structure in tensor indices and its background-quantum

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[2] Alternative approach to quantum aspects of massive gravity is developed in the papers [8]. Unlike our approach, which starts with complete ghost-free non-linear theory, in [8] the massive gravity model is treated as deformation of Pauli–Fierz Lagrangian by some cubic interaction of the fields.

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splitting (decomposition of initial field into background and quantum field, which is the element of the background field method) is not trivial. Second, a functional determinant, defining the one-loop effective action, has different and in fact more complicated structure as compared to the massless case of general relativity. Therefore, evaluation of effective method may require some novelty in the method of calculation. Third, it is interesting to study whether the one-loop divergences in massive gravity theory vanish on-shell as it is the case for the general relativity without matter. Fourth, the one-loop divergences in massive gravity theory are expected to depend on the reference metric. As a result of computations we obtain the one-loop divergences in terms of the specific general covariant functionals depending on reference metric. They can be considered as the possible candidates for actions of reference metric if to treat this metric as a dynamical field. Problem of Lagrangian for reference metric is broadly discussed in the literature (see e.g. [4] and reference therein). Fifth, any new gravitational model deserves a study of quantum aspects in the hope to extend our understanding of a quantum gravity and to shead a light on a possible role of gravity in quantum domain.

The Letter is organized as follows. In Section 2 we consider an equivalent representation of the theory via auxiliary tensor field and the procedure of linearization. The derivation of bilinear form of the action and the details of background field method are treated in Section 3. In Section 4 we present the results of calculating the one-loop divergences. In Section 5 we present some discussions of the result and draw our conclusions.

2. Linearization of massive term

Consider the action of a minimal model of massive gravity [4]

$$S[g_{\mu\nu}] = S_0[g_{\mu\nu}] + S_m[g_{\mu\nu}] = \int d^4x\,\sqrt{-g}\,\big[R + 2\Lambda + m^2\,\mathrm{tr}\big(\sqrt{g^{-1}f}\big)\big]. \tag{1}$$

Here m is a mass parameter, f means an external tensor field (reference metric) $f_{\mu\nu}$, expression $g^{-1}f$ in action S_m means $g^{\mu\alpha}f_{\alpha\nu}$, all other notations are standard.[3] According to [4] we have to put $\Lambda = -3m^2$, however it is more convenient to perform all calculations for an arbitrary value of Λ and fix it only in the final result.

To calculate the one-loop divergences of the theory under consideration we should compute the second variational derivative of action with respect $g_{\mu\nu}$. Such computation for the term S_m is very non-trivial since the matrices $g^{\mu\alpha}$ and $f_{\alpha\nu}$ do not commute.[4] We will avoid this obstacle considering the classically equivalent theory formulated in terms of dynamical metric $g_{\mu\nu}$ and auxiliary field $\varphi^\mu{}_\nu$.

It is easy to see that the theory, described by the action (1) in terms of dynamical field $g_{\mu\nu}$, is classically equivalent to the theory in terms of the fields $g_{\mu\nu}$ and auxiliary field $\varphi^\mu{}_\nu$ with the following action

$$\tilde S\big[g_{\mu\nu}, \varphi^\mu{}_\nu\big] = S_0[g] + \tilde S_m\big[g_{\mu\nu}, \varphi^\mu{}_\nu\big], \tag{2}$$

where the action $\tilde S_m[g_{\mu\nu}, \varphi^\mu{}_\nu]$ is given by

$$\tilde S_m\big[g_{\mu\nu}, \varphi^\mu{}_\nu\big] = \frac{m^2}{2}\int d^4x\,\sqrt{-g}\,\big[g^{\mu\nu}f_{\nu\alpha}\varphi^\alpha{}_\mu + \big(\varphi^{-1}\big)^\mu{}_\mu\big]. \tag{3}$$

Indeed, the equation of motion that follows from the variation over $\varphi^\mu{}_\nu$ in (2) has the form

$$\frac{\delta\tilde S}{\delta\varphi^\mu{}_\nu} = \frac{\delta\tilde S_m}{\delta\varphi^\mu{}_\nu} = 0.$$

The solution to this equation is

$$\varphi^\mu{}_\nu = \big[\big(g^{-1}f\big)^{-1/2}\big]^\mu{}_\nu.$$

Replacing this solution back into (2), we obtain the action (1). It shows that the two actions are classically equivalent. Starting from this point we will use the action (2).[5]

3. Background field method and bilinear form to action

Our main purpose is to develop the background field method ([9], see also the details in [10]) to the theory (2) and use it to calculate the one-loop divergences of the theory. The first step is to obtain the bilinear form of the action (2).

According to the background field method, the fields $\varphi^\mu{}_\nu$ and $g_{\mu\nu}$ are replaced by sums of background and quantum fields as follows

$$\varphi^\mu{}_\nu \to \varphi^\mu{}_\nu + \psi^\mu{}_\nu, \qquad g_{\mu\nu} \to g_{\mu\nu} + h_{\mu\nu}. \tag{4}$$

Here $\varphi^\mu{}_\nu$ and $g_{\mu\nu}$ are background fields, while $\psi^\mu{}_\nu$ and $h_{\mu\nu}$ are quantum fields. By means of a simple algebra one can obtain the bilinear form for the actions S_0 and $\tilde S_m$ in the form

$$S_0^{(2)} + S_{gf} = \frac{1}{2}\int d^4x\,\sqrt{-g}\,h^{\mu\nu}\hat J_{\mu\nu,\alpha\beta}h^{\alpha\beta}, \tag{5}$$

and

$$\tilde S_m{}^{(2)} = m^2\int d^4x\,\sqrt{-g}\,\bigg\{\frac{1}{2}h^{\alpha\beta}\hat G_{\alpha\beta,\mu\nu}h^{\mu\nu} - \frac{1}{2}\psi^\alpha{}_\beta\hat A_\alpha{}^\beta{},_\mu{}^\nu\psi^\mu{}_\nu + \hat B^\beta{}_\alpha\psi^\alpha{}_\beta\bigg\}. \tag{6}$$

In Eq. (5) we have introduced the so-called minimal gauge fixing term

$$S_{gf} = -\frac{1}{2}\int d^4x\,\sqrt{-g}\,\chi^\mu\chi_\mu, \tag{7}$$

where

$$\chi_\mu = \nabla_\lambda h^\lambda{}_\mu - \frac{1}{2}\nabla_\mu h. \tag{8}$$

The operators $\hat J_{\mu\nu,\alpha\beta}$, $\hat G_{\alpha\beta,\mu\nu}$, $\hat A_\alpha{}^\beta{},_\mu{}^\nu$ and $\hat B^\beta{}_\alpha$, which were used in (5) and (6), have the form

$$\begin{aligned}
\hat J_{\mu\nu,\alpha\beta} &= \frac{1}{2}K_{\mu\nu,\alpha\beta}\Box + R_{\mu\alpha\nu\beta} + g_{\nu\beta}R_{\mu\alpha} \\
&\quad - \frac{1}{2}(g_{\mu\nu}R_{\alpha\beta} + g_{\alpha\beta}R_{\mu\nu}) - \frac{1}{2}(R + 2\Lambda)K_{\mu\nu,\alpha\beta}, \\
\hat G_{\alpha\beta,\mu\nu} &= -\frac{1}{4}K_{\alpha\beta,\mu\nu}\big[g^{\sigma\tau}f_{\tau\lambda}\varphi^\lambda{}_\sigma + \big(\varphi^{-1}\big)^\sigma{}_\sigma\big] \\
&\quad - \frac{1}{2}g_{\alpha\beta}f_{\nu\sigma}\varphi^\sigma{}_\mu + g_{\mu\beta}f_{\nu\sigma}\varphi^\sigma{}_\alpha, \\
\hat A_\alpha{}^\beta{},_\mu{}^\nu &= -\frac{1}{2}\big[\big(\varphi^{-1}\big)^\sigma{}_\alpha\big(\varphi^{-1}\big)^\beta{}_\mu\big(\varphi^{-1}\big)^\nu{}_\sigma \\
&\quad + \big(\varphi^{-1}\big)^\sigma{}_\mu\big(\varphi^{-1}\big)^\nu{}_\alpha\big(\varphi^{-1}\big)^\beta{}_\sigma\big], \\
\hat B^\beta{}_\alpha &= -\frac{1}{2}h^{\beta\lambda}f_{\lambda\alpha},
\end{aligned} \tag{9}$$

[3] The gravitational constant κ is suppressed.

[4] The first variational derivative is computed simply enough [4], however the second derivative cannot be computed analogously to [4].

[5] We assume that this equivalence is fulfilled on the quantum level as well, since there is no any source for possible anomaly.

where we define

$$K_{\alpha\beta,\mu\nu} = \delta_{\alpha\beta,\mu\nu} - \frac{1}{2} g_{\alpha\beta} g_{\mu\nu} \tag{10}$$

and use notation

$$\delta_{\alpha\beta,\mu\nu} = \frac{1}{2}(g_{\alpha\mu} g_{\beta\nu} + g_{\alpha\nu} g_{\beta\mu})$$

for the DeWitt identity matrix in the space of symmetric matrices. In the expressions (9) one has to assume symmetrization in both couples of indices $\mu\nu$ and $\alpha\beta$. Let us note that the expression for $\hat{J}_{\mu\nu,\alpha\beta}$ in (9) is one for the usual Einstein quantum gravity with the cosmological constant and the other terms here are because of the massive terms in Eq. (1).

It is easy to see that the path integral over the quantum field $\psi^\mu{}_\nu$ can be taken at once. It is well known that the following identity holds for Hermitian matrices $A(y,x)$:

$$\begin{aligned} &\int \mathcal{D}\psi \exp\left\{-\frac{i}{2}\int dy \int dx\, \psi(y) A(y,x) \psi(x) \right. \\ &\quad \left. + i \int dx\, B(x)\psi(x)\right\} \\ &= (\operatorname{Det} A)^{-1/2} \times \exp\left\{\frac{i}{2}\int dy \int dx\, B(y) A^{-1}(y,x) B(x)\right\}. \end{aligned} \tag{11}$$

Let us note that the quantity $(\operatorname{Det} A)^{-1/2}$ corresponds to the determinant of a numerical matrix. Since we assume dimensional regularization here, this object is irrelevant to the analysis of quantum corrections to the effective action and therefore will not be omitted. Using Eq. (11) in the expression for the generating functional of Green functions, we present the bilinear form for the action (2) as follows

$$\tilde{S}^{(2)} = S_0^{(2)} + S_{gf} + \tilde{S}_m^{(2)} = \frac{1}{2}\int d^4x \sqrt{-g}\, h^{\alpha\beta} \hat{H}_{\alpha\beta,\mu\nu} h^{\mu\nu},$$

where the operator $\hat{H}_{\alpha\beta,\mu\nu}$ is given by

$$\hat{H}_{\alpha\beta,\mu\nu} = \hat{J}_{\mu\nu,\alpha\beta} + m^2 \hat{G}_{\alpha\beta,\mu\nu} + \frac{1}{4} m^2 f_{\beta\lambda} \big(\hat{A}^{-1}\big)_\alpha{}^\lambda{}_,{}_\mu{}^\sigma f_{\nu\sigma}. \tag{12}$$

In order to obtain the matrix A^{-1}, let us consider the following procedure. The result (11) is valid for the Hermitian matrix A. Therefore we need to take only symmetric part of the matrix A. Consider first the matrix $\bar{A}$ which is not symmetrized. It is an easy exercise to find its inverse, however one has to work a little bit more to do the same with the symmetric part of it. One can write $\bar{A}$ as

$$\bar{A}_\alpha{}^\beta{}_,{}_\mu{}^\nu = -\big(\varphi^{-1}\big)^\sigma{}_\alpha \big(\varphi^{-1}\big)^\beta{}_\mu \big(\varphi^{-1}\big)^\nu{}_\sigma. \tag{13}$$

The corresponding inverse matrix is given by

$$\big(\bar{A}^{-1}\big)_\beta{}^\alpha{}_,{}_\sigma{}^\rho = -\varphi^\rho{}_\tau \varphi^\tau{}_\beta \varphi^\alpha{}_\sigma. \tag{14}$$

Consider now the following symmetric structure

$$X_\beta{}^\alpha{}_,{}_\sigma{}^\rho = -\varphi^\rho{}_\tau \varphi^\tau{}_\beta \varphi^\alpha{}_\sigma - \varphi^\alpha{}_\tau \varphi^\tau{}_\sigma \varphi^\rho{}_\beta. \tag{15}$$

Then we arrive at the equation

$$\hat{A}_\mu{}^\nu{}_,{}_\alpha{}^\beta \times X_\beta{}^\alpha{}_,{}_\sigma{}^\rho = Z_\mu{}^\rho{}_,{}_\sigma{}^\nu = \delta_\mu^\rho \delta_\sigma^\nu + Y_\mu{}^\rho{}_,{}_\sigma{}^\nu, \tag{16}$$

where

$$Y_\mu{}^\rho{}_,{}_\sigma{}^\nu = \frac{1}{2}\big(\varphi^{-1}\big)^\rho{}_\mu \varphi^\nu{}_\sigma + \frac{1}{2}\big(\varphi^{-1}\big)^\nu{}_\sigma \varphi^\rho{}_\mu. \tag{17}$$

Finally, we have the inverse to the symmetrized matrix in the form of the series

$$\big(\hat{A}^{-1}\big)_\mu{}^\nu{}_,{}_\alpha{}^\beta = X_\mu{}^\nu{}_,{}_\lambda{}^\sigma \times \big(Z^{-1}\big)_\sigma{}^\lambda{}_,{}_\alpha{}^\beta, \tag{18}$$

where the matrix $(Z^{-1})_\sigma{}^\lambda{}_,{}_\alpha{}^\beta$ is given by

$$\begin{aligned} \big(Z^{-1}\big)_\sigma{}^\lambda{}_,{}_\alpha{}^\beta &= \delta_\sigma^\lambda \delta_\alpha^\beta - Y_\sigma{}^\lambda{}_,{}_\alpha{}^\beta + Y_\sigma{}^\lambda{}_,{}_\rho{}^\tau Y_\tau{}^\rho{}_,{}_\alpha{}^\beta \\ &\quad - Y_\sigma{}^\lambda{}_,{}_\rho{}^\tau Y_\tau{}^\rho{}_,{}_\chi{}^\delta Y_\delta{}^\chi{}_,{}_\alpha{}^\beta + \cdots. \end{aligned} \tag{19}$$

Multiplying the operator $\hat{H}_{\alpha\beta,\mu\nu}$ by the operator $2\hat{K}^{-1}{}_{\lambda\sigma,}{}^{\alpha\beta}$, where

$$\hat{K}^{-1}{}_{\lambda\sigma,}{}^{\alpha\beta} = \delta_{\lambda\sigma,}{}^{\alpha\beta} - \frac{1}{2} g_{\lambda\sigma} g^{\alpha\beta},$$

we arrive at

$$2\hat{K}^{-1}{}_{\alpha\beta,}{}^{\lambda\sigma} \hat{H}_{\lambda\sigma,\mu\nu} \equiv \hat{O}_{\alpha\beta,\mu\nu} = \delta_{\alpha\beta,\mu\nu} \Box + \hat{\Pi}_{\alpha\beta,\mu\nu}, \tag{20}$$

where we have

$$\begin{aligned} \hat{\Pi}_{\alpha\beta,\mu\nu} &= 2R_{\alpha\mu\beta\nu} + 2g_{\beta\nu} R_{\alpha\mu} - g_{\alpha\beta} R_{\mu\nu} \\ &\quad - g_{\mu\nu} R_{\alpha\beta} - R K_{\alpha\beta,\mu\nu} - 2\Lambda \delta_{\alpha\beta,\mu\nu} \\ &\quad + \frac{m^2}{2} f_{\rho(\alpha} \big(\hat{A}^{-1}\big)_{\beta)}{}^\rho{}_,{}_\mu{}^\tau f_{\nu\tau} \\ &\quad - \frac{m^2}{4} g_{\alpha\beta} f_{\sigma\rho} \big(\hat{A}^{-1}\big)^{\sigma\rho}{}_,{}_\mu{}^\tau f_{\nu\tau} + 2m^2 f_{\nu\sigma} \varphi^\sigma{}_{(\beta} g_{\alpha)\mu} \\ &\quad - \frac{m^2}{2} \delta_{\alpha\beta,\mu\nu} \big[g^{\sigma\tau} f_{\tau\lambda} \varphi^\lambda{}_\sigma + \big(\varphi^{-1}\big)^\sigma{}_\sigma\big]. \end{aligned} \tag{21}$$

4. Derivation of one-loop divergences

The one-loop quantum corrections to effective action are written by standard way (see e.g. [11])

$$\bar{\Gamma}^{(1)} = \frac{i}{2} \operatorname{Ln} \operatorname{Det}(\hat{H}) = \frac{i}{2} \operatorname{Tr} \operatorname{Ln}(\hat{H}), \tag{22}$$

where the operator $\hat{H}$ corresponds to the bilinear part of the action in quantum fields and Tr means the functional trace.

The divergent part of $\operatorname{Tr}\operatorname{Ln}\hat{H}$ can be obtained by calculating $\operatorname{Tr}\operatorname{Ln}(\hat{K}^{-1}\hat{H})$ and then subtracting the $\operatorname{Tr}\operatorname{Ln}\hat{K}^{-1}$

$$-\operatorname{Tr}\operatorname{Ln}\hat{H} = -\operatorname{Tr}\operatorname{Ln}\big(\hat{K}^{-1}\hat{H}\big) + \operatorname{Tr}\operatorname{Ln}\hat{K}^{-1}. \tag{23}$$

However, as far as we are interested in the logarithmic divergent part of the effective action, the contribution of the last term can be safely omitted.

The computation of (22) can be performed by the use of the Schwinger–DeWitt proper-time technique [9]. This technique provides the efficient method of evaluating the $\operatorname{Ln}\operatorname{Det}\hat{O}$, where the operator $\hat{O}$ has the form (see e.g. [12,13])

$$\hat{O} = \hat{1}\Box + 2\hat{h}^\mu \nabla_\mu + \hat{\Pi}. \tag{24}$$

Also we introduce the operators

$$\begin{aligned} \hat{P} &= \hat{\Pi} + \frac{1}{6} R\hat{1} - \nabla_\mu \hat{h}^\mu - \hat{h}^\mu \hat{h}_\mu, \\ \hat{S}_{\mu\nu} &= \hat{1}[\nabla_\mu, \nabla_\nu] + \nabla_\nu \hat{h}_\mu - \nabla_\mu \hat{h}_\nu + [\hat{h}_\nu, \hat{h}_\mu]. \end{aligned} \tag{25}$$

In our case, exactly as for the Einstein quantum gravity, the $\hat{h}_\mu = 0$ and this essentially simplifies the calculations.

In the framework of dimensional regularization, the quantity $\overline{\Gamma}^{(1)}$ is written as follows

$$\overline{\Gamma}^{(1)}_{div} = -\frac{\mu^{n-4}}{(4\pi)^2(n-4)} \operatorname{Tr}\left\{ \frac{\hat{1}}{180}\left(R^2_{\mu\nu\alpha\beta} - R^2_{\mu\nu}\right) + \frac{1}{2}\hat{P}^2 + \frac{1}{12}\hat{S}^2_{\mu\nu} \right\}, \tag{26}$$

where μ is the parameter of dimensional regularization, the operators $\hat{P}$, $\hat{S}_{\mu\nu}$ are defined above and the surface terms are ignored.

In our case the operators $\hat{P}$ and $\hat{S}_{\mu\nu}$ have the form

$$\begin{aligned}
\hat{P}_{\alpha\beta,\mu\nu} &= 2R_{\alpha\mu\beta\nu} + 2g_{\beta\nu}R_{\alpha\mu} - g_{\alpha\beta}R_{\mu\nu} - g_{\mu\nu}R_{\alpha\beta} \\
&\quad - \frac{5}{6}R\delta_{\alpha\beta,\mu\nu} + \frac{1}{2}Rg_{\alpha\beta}g_{\mu\nu} - 2\Lambda\delta_{\alpha\beta,\mu\nu} \\
&\quad - \frac{m^2}{2}\delta_{\alpha\beta,\mu\nu}\left[g^{\sigma\tau} f_{\tau\lambda}\varphi^{\lambda}{}_{\sigma} + \left(\varphi^{-1}\right)^{\sigma}{}_{\sigma}\right] \\
&\quad + 2m^2 f_{\nu\sigma}\varphi^{\sigma}{}_{(\alpha}g_{\beta)\mu} + \frac{m^2}{2} f_{\rho(\alpha}\left(\hat{A}^{-1}\right)_{\beta)}{}^{\rho}{}_{,\mu}{}^{\tau} f_{\nu\tau} \\
&\quad - \frac{m^2}{4}g_{\alpha\beta}f_{\sigma\rho}\left(\hat{A}^{-1}\right)^{\sigma\rho}{}_{,\mu}{}^{\tau} f_{\nu\tau}, \\
\hat{S}_{\lambda\tau} &= [\hat{S}_{\lambda\tau}]_{\mu\nu,\alpha\beta} = -2R_{\mu\alpha\lambda\tau}g_{\nu\beta}.
\end{aligned} \tag{27}$$

Replacing these operators in the expression (26), after some algebra we obtain the expression for the divergent part of the one-loop effective action,

$$\begin{aligned}
\overline{\Gamma}^{(1)}_{div}|_{\Lambda=-3m^2} &= -\frac{2\mu^{n-4}}{(4\pi)^2(n-4)} \int d^n x \sqrt{-g} \left\{ \frac{53}{90}E + \frac{7}{20}R^2_{\mu\nu} \right. \\
&\quad + \frac{1}{120}R^2 + \frac{m^2}{2}R^{(\lambda|\alpha|\sigma)\beta} f_{\sigma\tau}\left(\hat{A}^{-1}\right)_{\lambda}{}^{\tau}{}_{(\alpha}{}^{|\rho|} f_{\beta)\rho} \\
&\quad - \frac{m^2}{48}R\left[948 + 236\left(\sqrt{g^{-1}f}\right)^{\mu}{}_{\mu} \right. \\
&\quad + 5\left(g^{-1}f\right)^{\lambda}{}_{\beta}\left(\hat{A}^{-1}\right)_{\alpha\lambda,}{}^{\alpha\sigma}\left(g^{-1}f\right)^{\beta}{}_{\sigma} \\
&\quad \left. + 5f_{\alpha\lambda}\left(\hat{A}^{-1}\right)^{\beta\lambda,\alpha\sigma} f_{\beta\sigma} - 5f_{\alpha\beta}\left(\hat{A}^{-1}\right)^{\alpha\beta,\mu\nu} f_{\mu\nu}\right] \\
&\quad + \frac{m^2}{4}R^{\lambda\alpha}\left[12\left(\sqrt{g^{-1}f}\right)_{\alpha\lambda} \right. \\
&\quad + \left(g^{-1}f\right)^{\beta}{}_{\rho}\left(\hat{A}^{-1}\right)_{\lambda}{}^{\rho}{}_{(\alpha}{}^{|\tau|} f_{\beta)\tau} \\
&\quad + f_{\lambda\rho}\left(\hat{A}^{-1}\right)^{\beta\rho}{}_{(\alpha}{}^{|\tau|} f_{\beta)\tau} \\
&\quad \left. - 2f_{\sigma\tau}\left(\hat{A}^{-1}\right)^{\sigma\tau}{}_{\alpha}{}^{\rho} f_{\lambda\rho}\right] + \frac{m^4}{16}\left[1440 \right. \\
&\quad + 12\left(g^{-1}f\right)^{\lambda}{}_{\beta}\left(\hat{A}^{-1}\right)_{\alpha\lambda,}{}^{\alpha\sigma}\left(g^{-1}f\right)^{\beta}{}_{\sigma} \\
&\quad + 12f_{\alpha\lambda}\left(\hat{A}^{-1}\right)^{\beta\lambda,\alpha\sigma} f_{\beta\sigma} - 12f_{\alpha\beta}\left(\hat{A}^{-1}\right)^{\alpha\beta,\mu\nu} f_{\mu\nu} \\
&\quad - 240\left(\sqrt{g^{-1}f}\right)^{\mu}{}_{\mu} + 24\left(g^{-1}f\right)^{\mu}{}_{\mu} \\
&\quad + 4\left(\sqrt{g^{-1}f}\right)^{\mu}{}_{\mu}\left(\sqrt{g^{-1}f}\right)^{\nu}{}_{\nu} \\
&\quad + \left(g^{-1}f\right)^{\lambda}{}_{(\rho}\left(\hat{A}^{-1}\right)_{\tau)\lambda\mu}{}^{\sigma}\left(g^{-1}f\right)^{\nu}{}_{\sigma} \\
&\quad \times \left(g^{-1}f\right)^{\rho}{}_{\theta}\left(\hat{A}^{-1}\right)^{\tau\theta\mu}{}_{\alpha}\left(g^{-1}f\right)^{\alpha}{}_{\nu} \\
&\quad + \left(g^{-1}f\right)^{\lambda}{}_{(\rho}\left(\hat{A}^{-1}\right)_{\tau)\lambda}{}^{\mu}{}_{\sigma}\left(g^{-1}f\right)^{\sigma}{}_{\nu} \\
&\quad \times \left(g^{-1}f\right)^{\rho}{}_{\theta}\left(\hat{A}^{-1}\right)^{\tau\theta\nu}{}_{\alpha}\left(g^{-1}f\right)^{\alpha}{}_{\mu} \\
&\quad - 4\left(\sqrt{g^{-1}f}\right)^{\lambda}{}_{\lambda}\left(g^{-1}f\right)^{\beta}{}_{\varphi} f_{\theta(\alpha}\left(\hat{A}^{-1}\right)_{\beta)}{}^{\theta\alpha\varphi} \\
&\quad + 2\left(\sqrt{g^{-1}f}\right)^{\lambda}{}_{\lambda} f_{\alpha\beta}\left(\hat{A}^{-1}\right)^{\alpha\beta,\mu\nu} f_{\mu\nu} \\
&\quad - 4f_{\rho\theta}\left(\sqrt{g^{-1}f}\right)^{\lambda}{}_{(\alpha} f_{\varphi)\lambda}\left(\hat{A}^{-1}\right)^{\rho\theta\alpha\varphi} \\
&\quad + 2f_{\alpha\theta}f_{\lambda\varphi}\left(\sqrt{g^{-1}f}\right)^{\lambda}{}_{\rho}\left(\hat{A}^{-1}\right)^{\rho\theta\alpha\varphi} \\
&\quad + 2f_{\beta\lambda}\left(g^{-1}f\right)^{\beta}{}_{\varphi}\left(\sqrt{g^{-1}f}\right)^{\lambda}{}_{\theta}\left(\hat{A}^{-1}\right)_{\alpha}{}^{\theta\alpha\varphi} \\
&\quad + 2\left(g^{-1}f\right)^{\theta}{}_{\lambda}\left(g^{-1}f\right)^{\lambda}{}_{\varphi}\left(\sqrt{g^{-1}f}\right)^{\alpha}{}_{\rho}\left(\hat{A}^{-1}\right)^{\rho}{}_{\theta\alpha}{}^{\varphi} \\
&\quad \left.\left. + 2\left(g^{-1}f\right)^{\tau}{}_{\theta} f_{\lambda\varphi}\left(\sqrt{g^{-1}f}\right)^{\alpha}{}_{\tau}\left(\hat{A}^{-1}\right)^{\lambda\theta}{}_{\alpha}{}^{\varphi}\right]\right\},
\end{aligned} \tag{28}$$

where we included the mass-independent ghost contribution and used the special value $\Lambda = -3m^2$. In Eq. (28) $E = R^2_{\mu\nu\alpha\beta} - 4R^2_{\mu\nu} + R^2$ is the integrand of the Gauss–Bonnet topological term and the expression $(\hat{A}^{-1})^{\alpha\beta\mu\nu}$ has been defined in Eq. (16). One has to note that the matrix $(\hat{A})^{-1}$ is an infinite power series on the external field f and hence the divergences (28) have essentially non-polynomial structure in this field too.

Let us note that before the use of the condition $\Lambda = -3m^2$ the divergences represent the corresponding expression for Einstein quantum gravity [12] with the contribution of the cosmological term and the rest of the expression is due to additional mass dependent term in the action. The reason for such a result is that we performed calculations is the situation when the diffeomorphism symmetry is unbroken. This means we treat $f_{\mu\nu}$ as external tensor field which does not violate general covariance of the theory, hence the number of physical degrees of freedom does not change due to the extra massive term in (1). Indeed, our approach follows the standard practice when, e.g., the divergences in the Yang–Mills theory with the spontaneous symmetry breaking are calculated in the unbroken phase. It would be, definitely, interesting to perform the calculation in the broken phase, however it is not immediately clear how this can be done in the softly broken non-Abelian theory such as quantum gravity.

An interesting observation concerning Eq. (28) is that there is an explicit simple hierarchy of the terms, for example the ones with higher derivatives do not depend on mass and/or on the field f. At the same time, if we consider the classical action with the algebraic structures presented in (28), we note that there are no derivatives acting on f there. However, despite there are no such derivatives of f, this field will be dynamical in action (28) because of the mixture with Ricci tensor and scalar curvature which emerge in the third line of the expression.

The next problem is to see what happens with the result (28) on-shell. For this end we have to derive the classical equations of motion and replace them into (28). The equation of motion for the theory (1) with $\Lambda = -3m^2$ has the form

$$\begin{aligned}
R^{\mu\nu} - \frac{1}{2}Rg^{\mu\nu} &= -3m^2 g^{\mu\nu} + \frac{1}{2}m^2 g^{\mu\nu}\left(\sqrt{g^{-1}f}\right)^{\alpha}{}_{\alpha} \\
&\quad - \frac{1}{2}m^2\left(\sqrt{g^{-1}f}\right)^{\nu\mu}.
\end{aligned} \tag{29}$$

After using this relation in Eq. (28), we arrive at the following on-shell result

$$\begin{aligned}
\overline{\Gamma}^{(1)}_{div}|_{on\ shell} &= -\frac{\mu^{n-4}}{(4\pi)^2(n-4)} \int d^n x \sqrt{-g}\left[\frac{53}{45}E \right. \\
&\quad \left. + m^2 R^{(\lambda|\alpha|\sigma)\beta} f_{\sigma\tau}\left(\hat{A}^{-1}\right)_{\lambda}{}^{\tau}{}_{(\alpha}{}^{|\rho|} f_{\beta)\rho}\right] \\
&\quad - \frac{m^4\mu^{n-4}}{8(4\pi)^2(n-4)} \int d^n x \sqrt{-g}\left[1.5 \cdot 32 \cdot 111 \right. \\
&\quad - 8f_{\alpha\lambda}\left(\hat{A}^{-1}\right)^{\beta\lambda,\alpha\sigma} f_{\beta\sigma} + 0.4 \cdot 77\left(\sqrt{g^{-1}f}\right)^{\mu}{}_{\mu}
\end{aligned}$$

$$
\begin{aligned}
&+\frac{7}{5}\left(g^{-1}f\right)^{\mu}{}_{\mu}\\
&+1.5\cdot 13\cdot 29\left(\sqrt{g^{-1}f}\right)^{\mu}{}_{\mu}\left(\sqrt{g^{-1}f}\right)^{\nu}{}_{\nu}\\
&+\left(g^{-1}f\right)^{\lambda}{}_{(\rho}\left(\hat{A}^{-1}\right)_{\tau)\lambda\mu}{}^{\sigma}\left(g^{-1}f\right)^{\nu}{}_{\sigma}\\
&\times\left(g^{-1}f\right)^{\rho}{}_{\theta}\left(\hat{A}^{-1}\right)^{\tau\theta\mu}{}_{\alpha}\left(g^{-1}f\right)^{\alpha}{}_{\nu}\\
&+\left(g^{-1}f\right)^{\lambda}{}_{(\rho}\left(\hat{A}^{-1}\right)_{\tau)\lambda}{}^{\mu}{}_{\sigma}\left(g^{-1}f\right)^{\sigma}{}_{\nu}\\
&\times\left(g^{-1}f\right)^{\rho}{}_{\theta}\left(\hat{A}^{-1}\right)^{\tau\theta\nu}{}_{\alpha}\left(g^{-1}f\right)^{\alpha}{}_{\mu}\\
&-4\left(\sqrt{g^{-1}f}\right)^{\lambda}{}_{\lambda}\left(g^{-1}f\right)^{\beta}{}_{\varphi}f_{\theta(\alpha}\left(\hat{A}^{-1}\right)_{\beta)}{}^{\theta\alpha\varphi}\\
&+\left[\frac{3}{2}\left(\sqrt{g^{-1}f}\right)^{\lambda}{}_{\lambda}+16\right]f_{\alpha\beta}\left(\hat{A}^{-1}\right)^{\alpha\beta,\mu\nu}f_{\mu\nu}\\
&-4f_{\rho\theta}\left(\sqrt{g^{-1}f}\right)^{\lambda}{}_{(\alpha}f_{\varphi)\lambda}\left(\hat{A}^{-1}\right)^{\rho\theta\alpha\varphi}\\
&+2f_{\alpha\theta}f_{\lambda\varphi}\left(\sqrt{g^{-1}f}\right)^{\lambda}{}_{\rho}\left(\hat{A}^{-1}\right)^{\rho\theta\alpha\varphi}\\
&+2f_{\beta\lambda}\left(g^{-1}f\right)^{\beta}{}_{\varphi}\left(\sqrt{g^{-1}f}\right)^{\lambda}{}_{\theta}\left(\hat{A}^{-1}\right)_{\alpha}{}^{\theta\alpha\varphi}\\
&+2\left(g^{-1}f\right)^{\theta}{}_{\lambda}\left(g^{-1}f\right)^{\lambda}{}_{\varphi}\left(\sqrt{g^{-1}f}\right)^{\alpha}{}_{\rho}\left(\hat{A}^{-1}\right)^{\rho}{}_{\theta\alpha}{}^{\varphi}\\
&-2\left(g^{-1}f\right)^{\tau}{}_{\theta}f_{\lambda\varphi}\left(\sqrt{g^{-1}f}\right)^{\alpha}{}_{\tau}\left(\hat{A}^{-1}\right)^{\lambda\theta}{}_{\alpha}{}^{\varphi}\\
&-\left[\frac{1}{2}\left(\sqrt{g^{-1}f}\right)^{\mu}{}_{\mu}+6\right]\\
&\times\left(g^{-1}f\right)^{\alpha}{}_{\rho}\left(\hat{A}^{-1}\right)^{\beta\rho}{}_{(\alpha}{}^{|\tau|}f_{\beta)\tau}\\
&+2\left(\sqrt{g^{-1}f}\right)^{\alpha}{}_{\lambda}\left(g^{-1}f\right)^{\lambda}{}_{\rho}\left(\hat{A}^{-1}\right)^{\beta\rho}{}_{(\alpha}{}^{|\tau|}f_{\beta)\tau}\\
&+2\left(\sqrt{g^{-1}f}\right)^{\alpha}{}_{\lambda}\left(g^{-1}f\right)^{\beta}{}_{\rho}\left(\hat{A}^{-1}\right)^{\lambda\rho}{}_{(\alpha}{}^{|\tau|}f_{\beta)\tau}\\
&+\left[32-\frac{5}{2}\left(\sqrt{g^{-1}f}\right)^{\mu}{}_{\mu}\right]\\
&\times\left(g^{-1}f\right)^{\lambda}{}_{\beta}\left(\hat{A}^{-1}\right)_{\alpha\lambda}{}^{\alpha\sigma}\left(g^{-1}f\right)^{\beta}{}_{\sigma}\Bigg].
\end{aligned}
\tag{30}
$$

It is easy to see that the on-shell result does not vanish as it was for the massless theory [12]. Moreover, in the first and second lines one can see the term which explicitly depends on the Riemann tensor which is mixed with the components of $f_{\sigma\tau}$. It is easy to see that the specific tuning of the mass term which results (after symmetry breaking) in the ghost-free massive theory, does not hold at the quantum level.

5. Conclusion

We have developed the background field method and calculated the one-loop divergences for minimal massive gravity models suggested in [4]. The divergences are formulated in terms of geometrical invariants constructed from metric and reference metric and contain the inverse matrix (16) which is an infinite power series in the reference metric $f_{\mu\nu}$. There are no doubts that the divergences for the non-minimal, more complicated actions of [4] will have qualitatively the same structure. The final expression (28) shows that the UV completion of the massive gravity theory would be essentially more complicated than the one of Einstein quantum gravity. Along with the usual fourth-derivative metric-dependent terms such completion should include also dependence on the reference metric $f_{\mu\nu}$. Furthermore, this field gains dynamics due to the mixture with curvature tensor components. Therefore the counter-term (28) can be considered as the action functional defining dynamics of reference metric.

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