Euclidean $N = 2$ supergravity

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1. Introduction

Special geometry was first discovered in the study of the coupling of $N = 2$ supergravity to vector multiplets [1]. In recent years, this geometry has provided an important ingredient in the understanding of non-perturbative structure in field theory, supergravity, string compactifications [see for example: [2]], as well as in the study and analysis of black hole physics [3]. More recently, the Euclidean version of special geometry has been investigated in the context of Euclidean supersymmetric theories [4–6]. The Euclidean versions of the special geometries can be obtained from their standard counterparts by replacing $i$ by the object $e$ with the properties $e^2 = 1$ and $e = -e$. In the context of finding instanton solutions, this replacement was first done in [7] in the study of D-instantons in type IIB supergravity. Geometrically, this change of $i$ into $e$, is effectively the replacement of the complex structure by a para-complex structure. Details on para-complex geometry, para-holomorphic bundles, para-Kähler manifolds and affine special para-Kähler manifolds can be found in [4]. In the rigid case, the scalar manifold is described by affine special para-Kähler geometry. Starting from the general five dimensional vector multiplet action, the dimensional reduction over a time-like circle was considered in [4]. The Euclidean action, together with the supersymmetry transformations when expressed in terms of para-holomorphic coordinates, are of the same form as their Minkowskian counterparts.

In [6] the results of the rigid case were generalised by considering the dimensional reduction of the five dimensional supergravity theory of [8]. The dimensional reduction with respect to a time-like and space-like direction, gives respectively the Euclidean and Lorentzian theories in four dimensions. The bosonic action for both types of reductions was obtained in [6]. The target geometries of Euclidean vector multiplets coupled to supergravity are given by projective special para-Kähler manifolds. This Letter, we derive the Killing spinor equations of Euclidean $N = 2$ supergravity theories coupled to vector multiplets. These equations provide the starting point for finding general supersymmetric instanton solutions.

2. Bosonic reduction and special $e$-Kähler geometry

In this section we review the bosonic reduction of the five dimensional supergravity theory [6]. The Lagrangian of the five dimensional theory is given by [8]

$$
\hat{L}_5 = \frac{1}{2} \hat{R} - \frac{1}{2} G_i^{\hat{h}_i h_1 h_2 h_3} - \frac{1}{4} G_{ij}(\mathcal{F}^i)^{\hat{h}_i h_1 h_2 h_3} (\mathcal{F}^j)^{\hat{h}_i h_1 h_2 h_3} + \frac{1}{48} C_{ijk} \epsilon^{\hat{a}_i \hat{a}_j \hat{a}_k} (\mathcal{F}^i)^{\hat{a}_i} (\mathcal{F}^j)^{\hat{a}_j} (\mathcal{F}^k)^{\hat{a}_k}.
$$

(2.1)

This is related to the original Lagrangian via the following identifications:

$$
\mathcal{F}^i \rightarrow \frac{6^{1/6}}{2} \hat{F}^i, \quad h_i \rightarrow 6^{-1/3} \hat{h}_i, \quad a_{ij} \rightarrow 4.6^{-1/3} G_{ij}.
$$

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Here $\hat{e}$ is the determinant of the fünfbein and $\hat{R}$ the space–time Ricci scalar, $C_{ijk}$ are real constants, symmetric in $i, j, k$. All the physical quantities of the theory are determined in terms of a homogeneous cubic polynomial $\mathcal{V}$ which defines very special geometry.

\[ G_{ij} = \left. -\frac{1}{2} \frac{\partial}{\partial h_f} \frac{\partial}{\partial h^i} (\ln \mathcal{V}) \right|_{\mathcal{V}=1} = -\frac{1}{2} \epsilon_{ijk} h^j - \frac{1}{2} C_{ijk} h^k \tag{2.2} \]

where

\[ \mathcal{V} = \frac{1}{6} C_{ijk} h^j h^k = h^i h_i = 1, \quad h_i = -\frac{1}{6} C_{ijk} h^k. \tag{2.3} \]

In particular we have the relation

\[ G_{ij} h^i = \frac{3}{2} h_i. \]

The reduction ansatz is given by [6]:

\[ \tilde{e}^0 = e^{-\phi/2} e^\theta, \quad \tilde{e}^0 = e^\theta (dt - A^0). \tag{2.5} \]

Here $\hat{e}$ are the fünfbeins, $e^\theta$ are the vielbeins, $A^0$ and $\phi$ are, respectively, a gauge field and a scalar field. All fields are independent of the coordinate $t$, and $\tilde{e}^0 = 0, A^0 = 0$. The five dimensional flat metric is denoted by $\eta_{\hat{a}\hat{b}} = (-\epsilon, +, +, +, +, \epsilon)$ while the four dimensional one is denoted by $\eta_{ab} = (+, +, +, +, +, \epsilon)$: Roman indices $m, n$ denote $D = 5$ frame indices, whereas $a, b, \ldots$ are $D = 4$ frame indices. Here $\epsilon = -1$ for reduction on a space-like direction and $\epsilon = 1$ for reduction on a time-like direction.

Note that the non-vanishing components of the $D = 5$ spin connection $\omega$, written in the frame basis, are given by

\[ \omega_{0,\hat{a}} = -\epsilon e^{\theta/2} \partial_\theta \phi, \]
\[ \omega_{\hat{a},\hat{b}} = -\frac{\epsilon}{2} e^{2\theta} (F_0)_{ab}, \]
\[ \omega_{0,\hat{b}} = -\frac{\epsilon}{2} e^{2\theta} (F_0)_{0b}, \]
\[ \omega_{\hat{a},\hat{b}} = e^{\theta} (\omega_{ab} + \frac{1}{2} \eta_{ab} \partial_\theta \phi - \frac{1}{2} \eta_{ab} \partial_\phi \phi) \tag{2.6} \]

where indices on the LHS are $D = 5$ frame indices, taken with respect to the basis $\hat{e}$, whereas the indices on the RHS are $e^\theta$ frame indices, and $F_0 = dA^0$. The spin connection associated with the $D = 4$ basis $e^\theta$ has components $\omega_{ab}$. The $D = 5$ gauge potentials $A^i (\mathcal{F}^i = dA^i)$ are decomposed as

\[ A^i = x^i (dt - A^0) + A^i, \quad A^i = 0 \tag{2.7} \]

where $A^i$ are the $D = 4$ gauge potentials; the scalar fields $x^i$ and gauge potentials $A^i$ are also independent of $t$. So the components of the $D = 5$ gauge field strengths $\mathcal{F}$ in the frame basis are given by

\[ \mathcal{F}^{\theta}_{0a} = -e^{-\theta/2} \partial_a x^\theta, \]
\[ \mathcal{F}^{\theta}_{ab} = e^{\theta} (F^i - x^i F_0)_{ab} \tag{2.8} \]

where $F^i = dA^i$, and on the LHS, the indices are frame indices defined with respect to (2.5), and on the RHS $e^\theta$ frame indices are used.

Then, after performing the redefinitions:

\[ h^i = e^{-\phi} y^i, \quad G_{ij} = -2 \epsilon g_{ij} e^{2\theta}, \tag{2.9} \]

and rescaling the $D = 4$ gauge fields $F^0$ and $F^i$ by a factor of $\sqrt{\mathcal{V}}$, we obtain from (2.1)

\[ e^{-1} \mathcal{L} = \frac{1}{2} R - g_{ij} (\partial_a x^i \partial^a x^j - \epsilon \partial_a y^i \partial^a y^j) \]
\[ + C_{xy} \left[ \frac{\epsilon}{24} \cdot F^0 \right. \]
\[ + \epsilon \left. \frac{1}{6} (g_{xx} F^0 + g_{ij} F^i F^j - 2 (g_{xx}) F^i F^j) \right] \]
\[ + \frac{1}{12} \left[ 3 (C_{xj}) F^j F^i - 3 (C_{xx}) F^i F^0 \right. \]
\[ + (C_{xxx}) F^0 F^0 \]
\[ F_I = \frac{\partial F}{\partial x^I} \] and \[ F_{IJ} = \frac{\partial^2 F}{\partial x^I \partial x^J}. \] The constraint (2.17) can be solved by setting \[ X^I = e^{K(z)/2}X^I(z) \] (2.19)

where \( K(z) \) is the Kähler potential. Then we have \[ e^{-K(z)} = -\bar{N}_{IJ}X^I(z)\bar{X}^J(z). \] (2.20)

The resulting geometry of the physical scalar fields \( z^i \) of the vector multiplets is then given by a special Kähler manifold with Kähler metric

\[ \bar{g}_{ij} = \frac{\partial^2 K(z, \bar{z})}{\partial z^i \partial \bar{z}^j}. \] (2.21)

A convenient choice of inhomogeneous coordinates \( z^i \) is the special coordinates defined by

\[ X^0(z) = 1, \quad X^i(z) = z^i. \]

The gauge field coupling matrix is

\[ \bar{N}_{IJ} = F_{IJ}(X) + i \epsilon (N X) J \] (2.22)

For theories with cubic prepotentials in (2.15), we obtain

\[ g_{ij} = \epsilon \left( \frac{3}{2} Cy_{ij} - \frac{9}{4} (Cyy)(Cyy)^2 \right) \] (2.23)

and

\[ N_{00} = \frac{1}{3} Cxxx + \epsilon e Cy_{ij} \left( \frac{2}{3} gxx + \frac{1}{6} \right), \]

\[ N_{0i} = -\frac{1}{2} (Cxx) - \frac{2}{3} e Cy_{ij} (gxx), \]

\[ N_{ij} = (Cxx) + \frac{2}{3} e g_{ij} Cy_{yy}. \] (2.24)

Therefore the kinetic term of the scalar fields agrees with the reduced theory where

\[ z^I = x^i - \epsilon e y^i. \] (2.25)

Using (2.24) the gauge part of the action (2.14) gives

\[ \frac{1}{6} e Cy_{ij} \left( \frac{3}{4} F^0 + gxx F^0 - 2gxx F^i - F^0 + g_{ij} F^i + F^j \right) \]

\[ + \frac{1}{12} e Cy_{ij} \left( gxx F^0 + \tilde{F}^0 - 3(Cxx) F^i + \tilde{F}^0 + 3(Cxx) F^i + \tilde{F}^j \right) \] (2.26)

which agrees with the reduced Lagrangian.

### 3. Reduced Killing spinor equations

In this section we start with the supersymmetry variation of the gravitini and gaugino in the five dimensional supergravity theory and reduce them to four dimensions. The associated Killing spinor equations are

\[ e^{\tilde{h}_i} \left( \Gamma_m^{\tilde{h}_i} \hat{\partial}_m - 4\delta^i_m \bar{\Gamma}_m^{\tilde{h}_i} \right) \hat{\partial}_m \hat{\psi} = 0 \] (3.1)

and

\[ \left( (F^i - h^i h_j F^j) \hat{\Gamma}_m^{\tilde{h}_i} \hat{\Gamma}_m^{\tilde{h}_i} + 2i \hat{\Gamma}_m^{\tilde{h}_i} h^i F^0 \right) \hat{\psi} = 0. \] (3.2)

Here \( \hat{\partial}_m = \partial_m + i \frac{1}{2} \bar{\partial}_m \gamma_{\tilde{h}_i} \hat{\Gamma}_m^{\tilde{h}_i} \) is the five dimensional covariant derivative. Note that \( F_0 \) squares to \(-\epsilon\), and \( F^0 = -\epsilon F_0 \). We first reduce (3.1) and (3.2) to \( D = 4 \): throughout what follows the rescaling of the \( D = 4 \) gauge field strengths by \( \sqrt{2} \) is taken into account.

First consider the \( \tilde{m} = 0 \) component of (3.1); this reduces from \( D = 5 \) to \( D = 4 \) to give

\[ \left( \frac{i}{2} e^{\hat{h}_i} \hat{\Gamma}_m (\partial_m \hat{\psi} + i \hat{\partial}_m \hat{\psi}) \right) + \frac{i}{4\sqrt{2}} e^{2\hat{h}_i} \hat{\Gamma}^{ab} (h_1 \hat{\Gamma}_0 (F^i - \chi^i F_0)_{ab} + i \epsilon e^{\hat{h}_i} \hat{F}^0_{ab}) \hat{\psi} = 0. \] (3.3)

Consider also the reduction of the \( D = 3 \) gaugino equation (3.2); which gives

\[ \left( -\frac{1}{4} e^{\hat{h}_i} \hat{\Gamma}_0 (F^i - \chi^i) F^0_{ab} \hat{\Gamma}^{ab} \right) + \hat{\Gamma}^a \hat{\partial}_a (h^i \hat{\partial}_i \hat{\psi}) = 0. \] (3.4)

After some calculations, details of which are given in Appendix A, the two conditions (3.3) and (3.4) can be combined into the following expression:

\[ \frac{i}{2} e^{K/2} (\text{Im} N)_{IJ} F^i (F^0_{ab}) + \frac{i}{4} e^{\hat{h}_i} \hat{\Gamma}_0 e^{\hat{h}_i} \hat{\Gamma}_0 (F^i - \chi^i) F^0_{ab} \hat{\Gamma}^{ab} \]

\[ + \hat{\Gamma}^a \hat{\partial}_a [\text{Re} z^i - i \hat{\Gamma}_0 \text{Im} z^i] \hat{\psi} = 0 \] (3.5)

where

\[ D_f X^i = \partial_f X^i + \partial_f K X^i. \]

In particular, one finds that (3.3) is obtained from (3.5) by contracting with \( h_i \), whereas one obtains (3.4) by considering the directions of (3.5) which are orthogonal to \( h_i \).

Next consider the \( \tilde{m} = \tilde{a} \) component of (3.1); this reduces to \( D = 4 \) to give the following expression:

\[ D_{ab} \hat{\psi} + \left( \frac{1}{2\sqrt{2}} e^{\hat{h}_i} \hat{\Gamma}_0 \hat{\Gamma}^{ab} (F^0_{ab}) - \frac{1}{4} e^{\hat{h}_i} \hat{\Gamma}_0 e^{\hat{h}_i} \hat{\Gamma}_0 e^{\hat{h}_i} \hat{\Gamma}_0 e^{\hat{h}_i} \hat{\Gamma}_0 \right) \hat{\psi} \]

\[ \left( \frac{i}{2} e^{\hat{h}_i} \hat{\Gamma}_0 e^{\hat{h}_i} \hat{\Gamma}_0 (F^i - \chi^i) F^0_{ab} \hat{\Gamma}^{ab} \right) + \hat{\Gamma}^a \hat{\partial}_a (h^i \hat{\partial}_i \hat{\psi}). \] (3.7)

In order to rewrite this expression, we introduce the \( U(1) \) (para)-Kähler potential\(^{1}\)

\[ A_0 = \frac{i}{2} \left( \partial_i \hat{K} \partial_i z^i - \partial_i K \partial_i \bar{z}^i \right). \] (3.8)

This can be recast as

\[ A_0 = -\frac{i}{2} e^{\hat{h}_i} \hat{\partial}_i \hat{\partial}_i \hat{\psi}. \] (3.9)

Then, using the identities listed in Appendix A, (3.7) can be rewritten as

\[ D_{ab} \hat{\psi} + \left( \frac{i}{2} e^{\hat{h}_i} \hat{\partial}_i \hat{\psi} \right) - \frac{1}{2} A_0 \epsilon \hat{\Gamma}_0 \]

\[ + \frac{i}{4} e^{\hat{h}_i} \hat{\Gamma}^{bc} (F^0_{bc}) + \frac{i}{4\sqrt{2}} e^{\hat{h}_i} \hat{\Gamma}_0 e^{\hat{h}_i} \hat{\Gamma}_0 (F^i - \chi^i) F^0_{bc} + i \epsilon e^{\hat{h}_i} \hat{F}^0_{bc} \]

\[ + \frac{i}{2} e^{\hat{h}_i} \hat{\Gamma}_0 e^{\hat{h}_i} \hat{\Gamma}_0 e^{\hat{h}_i} \hat{\Gamma}_0 e^{\hat{h}_i} \hat{\Gamma}_0 \hat{\psi} = 0. \] (3.10)

1 To be distinguished from the gauge potentials \( A^0, A^i \).
Observe that the second and third lines of this expression can be removed by using (3.3) and so on setting
\[ \hat{\epsilon} = e^{-\frac{\xi}{2}} \epsilon \] (3.11)

it follows that (3.1) and (3.2) can be rewritten as
\[
D_0 \epsilon - \frac{i}{2} \epsilon A_\mu \Gamma^\mu_0 \epsilon
+ \frac{i}{4} \epsilon^2 \Gamma^{bc} F^I_{bc} \left( \text{Im} X^I + i \epsilon \Gamma_0 \text{Re} X^I \right) \left( \text{Im} \mathcal{N} \right)_I \Gamma^\mu_0 \epsilon = 0
\] (3.12)

and
\[
\frac{i}{2} \epsilon \Gamma^{I} \hat{a}_I \left( \text{Re} z^I - i \Gamma_0 \text{Im} z^I \right) \hat{\epsilon} = 0.
\] (3.13)

4. Chiral decomposition

In this section we express the transformations (3.12) and (3.13) in terms of chiral spinors. In order to define the various projections, it is convenient to note that
\[
\Gamma_{(i} \hat{a}_j \hat{a}_k \hat{a}_l = \epsilon (d\text{Vol})_{(i} \hat{a}_j \hat{a}_k \hat{a}_l\hat{a}_s)
\] (4.1)

which implies that
\[
\Gamma_0 \Gamma_{ab} = \frac{i}{2} \epsilon \hat{a}_a \hat{a}_b \Gamma_{cd}.
\] (4.2)

In the Minkowski case (\( \epsilon = -1 \)), we decompose the spinor \( \epsilon \) in terms of chiral spinors as \( \epsilon = \epsilon_+ + \epsilon_- \), where we set
\[
\Gamma_0 = \frac{1}{2} \left( 1 \pm i \Gamma_0 \right).
\] (4.3)

and we also define
\[
P_{ab} = \frac{1}{2} \left( F_{ab} \pm i \tilde{F}_{ab} \right).
\] (4.4)

Also, (4.2) implies that
\[
\Gamma^+ \cdot F = \Gamma^+ \cdot (F^- \Gamma^+ + F^+ \Gamma^-).
\] (4.5)

We find that (3.12) and (3.13) can be rewritten as
\[
D_0 \epsilon_{\pm} \pm \frac{i}{2} A_\mu \epsilon_{\pm}
\pm \frac{i}{4} \epsilon^2 \Gamma^{bc} F^I_{bc} \left( \text{Re} X^I + \epsilon \text{Im} X^I \right) \left( \text{Im} \mathcal{N} \right)_I \Gamma^\mu_0 \epsilon_{\mp} = 0
\] (4.6)

and
\[
\frac{i}{2} \epsilon \Gamma^{I} \hat{a}_I \left( \text{Re} z^I - \epsilon \text{Im} z^I \right) \epsilon_{\pm} = 0.
\] (4.7)

This is in agreement with the Killing spinor equations given by [11] (on making the identification \( \epsilon_1 = \epsilon_+ \), \( \epsilon_2 = \epsilon_- \)):
\[
D_0 \epsilon_+ + \frac{i}{2} A_\mu \epsilon_+ + \frac{1}{4} \left( \text{Im} \mathcal{N} \right)_I \left( \text{Im} \mathcal{N} \right)_I J \left( \text{Re} X^I + \epsilon \text{Im} X^I \right) \text{Re} \mathcal{N}_J \Gamma^\mu_0 \epsilon_+ = 0,
\]
\[
\frac{1}{2} \epsilon \Gamma^{I} \hat{a}_I \left( \text{Re} z^I - \epsilon \text{Im} z^I \right) \epsilon_+ = 0.
\] (4.8)

Next consider the Euclidean case (\( \epsilon = 1 \)). There are two alternative chiral decompositions possible. For the first, we define
\[
\Gamma_0 = \frac{1}{2} \left( 1 \pm i \Gamma_0 \right),
\] (4.9)

with
\[
F_{ab} = \frac{1}{2} \left( F_{ab} \pm \tilde{F}_{ab} \right),
\] (4.10)

and (4.2) implies that
\[
\Gamma^+ \cdot F = \Gamma^+ \cdot (F^- \Gamma^+_+ + F^+ \Gamma^++).
\] (4.11)

We find that (3.12) and (3.13) can be rewritten as
\[
D_0 \epsilon_{\pm} \pm \frac{i}{2} A_\mu \epsilon_{\pm}
\pm \frac{i}{4} \epsilon^2 \Gamma^{bc} F^I_{bc} \left( \text{Re} X^I + \epsilon \text{Im} X^I \right) \left( \text{Im} \mathcal{N} \right)_I \Gamma^\mu_0 \epsilon_{\mp} = 0
\] (4.12)

and
\[
\frac{i}{2} \epsilon \Gamma^{I} \hat{a}_I \left( \text{Re} z^I - \epsilon \text{Im} z^I \right) \epsilon_{\pm} = 0.
\] (4.13)

This is the form of the Killing spinor equations expressed in terms of the so-called adapted coordinates [6].

For the second chiral decomposition in the Euclidean case, we define [4]
\[
\Gamma_0 = \frac{1}{2} \left( 1 \pm i e \Gamma_0 \right),
\] (4.14)

and let
\[
F_{ab} = \frac{1}{2} \left( F_{ab} \pm e \tilde{F}_{ab} \right).
\] (4.15)

With these conventions, (4.11) holds, and we find that (3.12) and (3.13) can be rewritten as
\[
D_0 \epsilon_+ \pm \frac{i}{2} A_\mu \epsilon_+
\pm \frac{i}{4} \epsilon^2 \Gamma^{bc} F^I_{bc} \left( \text{Re} X^I + \epsilon \text{Im} X^I \right) \left( \text{Im} \mathcal{N} \right)_I \Gamma^\mu_0 \epsilon_+ = 0
\] (4.16)

and
\[
\frac{i}{2} \epsilon \Gamma^{I} \hat{a}_I \left( \text{Re} z^I + \epsilon \text{Im} z^I \right) \epsilon_+ = 0.
\] (4.17)

5. Discussion

In this Letter we have derived the Killing spinor equations for Euclidean supergravity theories coupled to Abelian vector multiplets (3.12) and (3.13). We have obtained the four dimensional Killing spinor equations from the reduction of those in the five dimensional theory. We explicitly show how this is achieved by writing the reduced equations in an \( \epsilon \)-Kähler covariant formalism. These equations were also rewritten, for the Euclidean case, in terms of chiral spinors using both the adapted and para-complex co-ordinates. \( \epsilon \)-special Kähler geometry in Euclidean theories is
expected to play an important role in the analysis of instantons, solitons and cosmological solutions in supergravity and M-theory. The Killing spinor equations given in (3.12) and (3.13) provide the starting point to find general instanton solutions of the effective Euclidean $N = 2$ supergravity action coupled to $N = 2$ matter multiplets. As for the case of black holes, one also expects that the rich geometric structure of the theory will lead to a simplified approach for finding new instanton solutions.

Spinorial geometry techniques [12] have proven to be a very useful tool in finding all instanton solutions preserving various fractions of supersymmetry. Those techniques were also used recently in finding solutions of Einstein–Maxwell theory with [13] or without [14] a cosmological constant, as well as the supersymmetric solutions of the Euclidean $N = 4$ super Yang–Mills theory [15]; where interesting relations to integrable models [13] and the Hitchin equations [15] were found. We will report on the instanton solutions with vector multiplets in a separate publication.

Another useful identity used to obtain (3.5) is

$$\Gamma^a \partial_a (Re z^\dagger - i I_3 \, Im z^\dagger) = \Gamma^a (\delta_a (x^\dagger + i I_0 y^\dagger) - h^i j \partial_a x^j - i I_0 h^i j \partial_a \phi^j e^{\phi^j})$$

$$+ \Gamma^a (h^i j \partial_a x^j + i I_0 \partial_a \phi^j e^{\phi^j})$$  \hspace{1cm} (A.6)

where the expression on the first line of the RHS is projected orthogonal to the direction of $h$, and the second line contains the term parallel to $h$.

A number of useful identities used to obtain (3.10) are

$$i \frac{e}{4} \left[ Im X^j + i e I_0 Re X^j \right] Im X_{ij}$$

$$= \frac{i}{8 \sqrt{2}} e^{-\phi} \left( -i e \Gamma_0 \, Im N_{0j} - (i e \Gamma_0 x^j + y^j) \, Im X_{ij} \right)$$  \hspace{1cm} (A.7)

and

$$-i e \Gamma_0 \, Im N_{0j} - (i e \Gamma_0 x^j + y^j) \, Im X_{ij} = 3 e^{2\phi} h_i$$

together with

$$(gy)_i = -\frac{3}{4} e^{-\phi} h_i.$$ \hspace{1cm} (A.10)

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