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Euclidean $N=2$ supergravity

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A R T I C L E   I N F O

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A B S T R A C T

Euclidean special geometry has recently been investigated in the context of Euclidean supersymmetric theories with vector multiplets. In the rigid case, the scalar manifold is described by affine special para-Kähler geometry while the target geometries of Euclidean vector multiplets coupled to supergravity are given by projective special para-Kähler manifolds. In this Letter, we derive the Killing spinor equations of Euclidean $N=2$ supergravity theories coupled to vector multiplets. These equations provide the starting point for finding general supersymmetric instanton solutions.

1. Introduction

Special geometry was first discovered in the study of the coupling of $N=2$ supergravity to vector multiplets [1]. In recent years, this geometry has provided an important ingredient in the understanding of non-perturbative structure in field theory, supergravity, string compactifications (see for example: [2]), as well as in the study and analysis of black hole physics [3]. More recently, the Euclidean version of special geometry has been investigated in the context of Euclidean supersymmetric theories [4–6]. The Euclidean versions of the special geometries can be obtained from their standard counterparts by replacing i by the object e with the properties $e^2=1$ and $\bar{e}=-e$. In the context of finding instanton solutions, this replacement was first done in [7] in the study of D-instantons in type IIB supergravity. Geometrically, this change of i into e, is effectively the replacement of the complex structure by a para-complex structure. Details on para-complex geometry, para-holomorphic bundles, para-Kähler manifolds and affine special para-Kähler manifolds can be found in [4]. In the rigid case, the scalar manifold is described by affine special para-Kähler geometry. Starting from the general five dimensional vector multiplet action, the dimensional reduction over a time-like circle was considered in [4]. The Euclidean action, together with the supersymmetry transformations when expressed in terms of para-holomorphic coordinates, are of the same form as their Minkowskian counterparts.

In [6] the results of the rigid case were generalised by considering the dimensional reduction of the five dimensional supergravity theory of [8]. The dimensional reduction with respect to a time-like and space-like direction, gives respectively the Euclidean and Lorentzian theories in four dimensions. The bosonic action for both types of reductions was obtained in [6]. The target geometries of Euclidean vector multiplets coupled to supergravity are given by projective special para-Kähler manifolds [6]. In this work, we complete the analysis of [6] and determine the associated Killing spinor equations. These will be a step in the direction of the classification of instanton solutions with non-trivial gauge and scalar fields. We organise this work as follows. We review the bosonic reduction [6] in Section 2. This will fix our notation, as well as the relation between the five and four dimensional bosonic fields needed to study the reduction of the Killing spinor equations from five to four dimensions. Section 3 contains the reduction of the Killing spinor equations. Section 4 describes how these equations can be rewritten using an appropriate chiral decomposition, and recast into an ϵ-complex form, or into an adapted co-ordinate form. We conclude in Section 5.

2. Bosonic reduction and special ϵ-Kähler geometry

In this section we review the bosonic reduction of the five dimensional supergravity theory [6]. The Lagrangian of the five dimensional theory is given by [8][1]

$$\hat{\mathbf{e}}^{-1}\hat{\mathcal{L}}_5 = \frac{1}{2}\hat{R} - \frac{1}{2}G_{ij}\partial_{\hat{m}}h^i\partial^{\hat{m}}h^j - \frac{1}{4}G_{ij}(\mathcal{F}^i)_{\hat{m}\hat{n}}(\mathcal{F}^j)^{\hat{m}\hat{n}} + \frac{\hat{\mathbf{e}}^{-1}}{48}C_{ijk}\epsilon^{\hat{n}_1\hat{n}_2\hat{n}_3\hat{n}_4\hat{n}_5}(\mathcal{F}^i)_{\hat{n}_1\hat{n}_2}(\mathcal{F}^j)_{\hat{n}_3\hat{n}_4}(\mathcal{A}^k)_{\hat{n}_5}. \tag{2.1}$$

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[1] This is related to the original Lagrangian via the following identifications:

$$\mathcal{F}^i \to \frac{6^{1/6}}{2}\mathcal{F}^i, \qquad h^i \to 6^{-1/3}h^i, \qquad a_{ij} \to 4.6^{-1/3}G_{ij}.$$

Here $\hat{\mathbf{e}}$ is the determinant of the fünfbein and $\hat{R}$ the space–time Ricci scalar, C_{ijk} are real constants, symmetric in i, j, k. All the physical quantities of the theory are determined in terms of a homogeneous cubic polynomial $\mathcal{V}$ which defines very special geometry,

$$G_{ij} = -\frac{1}{2}\frac{\partial}{\partial h^i}\frac{\partial}{\partial h^j}(\ln \mathcal{V})\Big|_{\mathcal{V}=1} = \frac{9}{2}h_i h_j - \frac{1}{2}C_{ijk}h^k \tag{2.2}$$

where

$$\mathcal{V} = \frac{1}{6}C_{ijk}h^i h^j h^k = h^i h_i = 1, \quad h_i \equiv \frac{1}{6}C_{ijk}h^j h^k. \tag{2.3}$$

In particular we have the relation

$$G_{ij}h^j = \frac{3}{2}h_i. \tag{2.4}$$

The reduction ansatz is given by [6]:

$$\hat{e}^a = e^{-\phi/2}e^a, \qquad \hat{e}^0 = e^{\phi}\big(dt - A^0\big). \tag{2.5}$$

Here $\hat{e}$ are the fünfbeins, e^a are the vielbeins, A^0 and ϕ are, respectively, a gauge field and a scalar field. All fields are independent of the coordinate t, and $e^a_t = 0$, $A^0_t = 0$. The five dimensional flat metric is denoted by $\eta_{\hat{m}\hat{n}} = (-\epsilon, +, +, +, \epsilon)$ while the four dimensional one is denoted by $\eta_{ab} = (+, +, +, \epsilon)$; Roman indices m, n denote $D = 5$ frame indices, whereas $a, b, \ldots$ are $D = 4$ frame indices. Here $\epsilon = -1$ for reduction on a space-like direction and $\epsilon = 1$ for reduction on a time-like direction.

Note that the non-vanishing components of the $D = 5$ spin connection $\hat{\omega}$, written in the frame basis, are given by

$$\begin{aligned}
\hat{\omega}_{0,0\hat{a}} &= -\epsilon e^{\frac{\phi}{2}}\partial_a\phi, \\
\hat{\omega}_{0,\hat{a}\hat{b}} &= -\frac{\epsilon}{2}e^{2\phi}\big(F^0\big)_{ab}, \\
\hat{\omega}_{\hat{a},0\hat{b}} &= -\frac{\epsilon}{2}e^{2\phi}\big(F^0\big)_{ab}, \\
\hat{\omega}_{\hat{a},\hat{b}\hat{c}} &= e^{\frac{\phi}{2}}\left(\omega_{a,bc} + \frac{1}{2}\eta_{ac}\partial_b\phi - \frac{1}{2}\eta_{ab}\partial_c\phi\right)
\end{aligned} \tag{2.6}$$

where indices on the LHS are $D = 5$ frame indices, taken with respect to the basis $\hat{e}$, whereas the indices on the RHS are e^a frame indices, and $F^0 = dA^0$. The spin connection associated with the $D = 4$ basis e^a has components $\omega_{a,bc}$.

The $D = 5$ gauge potentials $\mathcal{A}^i$ ($\mathcal{F}^i = d\mathcal{A}^i$) are decomposed as

$$\mathcal{A}^i = x^i\big(dt - A^0\big) + A^i, \qquad A^i_t = 0 \tag{2.7}$$

where A^i are the $D = 4$ gauge potentials; the scalar fields x^i and gauge potentials A^i are also independent of t. So the components of the $D = 5$ gauge field strengths $\mathcal{F}^i$ in the frame basis are given by

$$\begin{aligned}
\mathcal{F}^i_{0\hat{a}} &= -e^{-\frac{\phi}{2}}\partial_a x^i, \\
\mathcal{F}^i_{\hat{a}\hat{b}} &= e^{\phi}\big(F^i - x^i F^0\big)_{ab}
\end{aligned} \tag{2.8}$$

where $F^i = dA^i$, and on the LHS, the indices are frame indices defined with respect to (2.5), and on the RHS e^a frame indices are used.

Then, after performing the redefinitions:

$$h^i = e^{-\phi}y^i, \qquad G_{ij} = -2\epsilon g_{ij}e^{2\phi}, \tag{2.9}$$

and rescaling the $D = 4$ gauge fields F^0 and F^i by a factor of $\sqrt{2}$, we obtain from (2.1)

$$\begin{aligned}
\mathbf{e}^{-1}\mathcal{L} = {}& \frac{1}{2}R - g_{ij}\big(\partial_a x^i\partial^a x^j - \epsilon\partial_a y^i\partial^a y^j\big) \\
& + Cyyy\bigg[\frac{\epsilon}{24}F^0\cdot F^0 \\
& + \epsilon\frac{1}{6}\big(gxxF^0\cdot F^0 + g_{ij}F^i\cdot F^j - 2(gx)_i F^i\cdot F^0\big)\bigg] \\
& + \frac{1}{12}\big[3(Cx)_{ij}F^i\cdot\tilde{F}^j - 3(Cxx)_i F^i\cdot\tilde{F}^0 \\
& + (Cxxx)F^0\cdot\tilde{F}^0\big]
\end{aligned} \tag{2.10}$$

where R is the Ricci scalar of the $D = 4$ manifold with metric $ds_4^2 = \delta_{ab}e^a e^b$. We have used the notation

$$\begin{aligned}
& Chhh = C_{ijk}h^i h^j h^k, \qquad (Chh)_i = C_{ijk}h^i h^j, \\
& (Cy)_{ij} = C_{ijk}h^i
\end{aligned} \tag{2.11}$$

and $F\cdot F = F_{ab}F^{ab}$. The dual field strength is $\tilde{F}_{ab} = \frac{\epsilon}{2}\epsilon_{abcd}F^{cd}$, and we remark that the relationship between the $D = 5$ and $D = 4$ volume forms is[2]

$$\widehat{\mathrm{dvol}_5} = -e^{-2\phi}\hat{e}^0 \wedge \mathrm{dvol}_4 \tag{2.12}$$

where dvol_4 is the volume form of the $D = 4$ manifold with metric ds_4^2.

The explicit form of g_{ij} is

$$g_{ij} = \epsilon\frac{3}{2}\left(\frac{(Cy)_{ij}}{Cyyy} - \frac{3}{2}\frac{(Cyy)_i(Cyy)_j}{(Cyyy)^2}\right). \tag{2.13}$$

For both values of ϵ, it was demonstrated in [6] that (2.10) can be described by the Lagrangian of the four dimensional $N = 2$ supergravity theory coupled to vector multiplets [9–11]

$$\begin{aligned}
\mathbf{e}^{-1}\mathcal{L} = {}& \frac{1}{2}R - g_{ij}\partial_\mu z^i\partial^\mu\bar{z}^j + \frac{1}{4}\,\mathrm{Im}\,\mathcal{N}_{IJ}F^I\cdot F^J \\
& + \frac{1}{4}\,\mathrm{Re}\,\mathcal{N}_{IJ}F^I\cdot\tilde{F}^J,
\end{aligned} \tag{2.14}$$

with the cubic prepotential

$$F = \frac{1}{6}C_{ijk}\frac{X^i X^j X^k}{X^0}. \tag{2.15}$$

It should be mentioned that the dimensional reduction of (2.1) on a space-like circle was considered before in [8]. The coupling of $N = 2$ vector multiplets to $N = 2$ supergravity is encoded in a holomorphic homogeneous prepotential $F(X)$ of degree two. To demonstrate the equivalence of the reduced theory with the one given by (2.14), (2.15), the so-called ϵ-complex coordinates ($X^I = \mathrm{Re}\,X^I + i_\epsilon\,\mathrm{Im}\,X^I$) were introduced and F is taken to be ϵ-holomorphic, i.e. it depends on ϵ-complex scalar fields. Here i_ϵ satisfies $i_\epsilon = e$, for $\epsilon = 1$ and $i_\epsilon = i$, for $\epsilon = -1$. In the symplectic formulation of the theory, one introduces the symplectic vectors

$$V = \begin{pmatrix} X^I \\ F_I \end{pmatrix} \tag{2.16}$$

satisfying the symplectic constraint

$$i_\epsilon\big(\bar{X}^I F_I - X^I\bar{F}_I\big) = -N_{IJ}X^I\bar{X}^J = 1 \tag{2.17}$$

where

$$N_{IJ} = -i_\epsilon(F_{IJ} - \bar{F}_{IJ}), \tag{2.18}$$

[2] This is the opposite sign convention to that used in [6].

$F_I = \frac{\partial F}{\partial X^I}$ and $F_{IJ} = \frac{\partial^2 F}{\partial X^I \partial X^J}$. The constraint (2.17) can be solved by setting

$$X^I = e^{K(z,\bar{z})/2} X^I(z) \tag{2.19}$$

where $K(z,\bar{z})$ is the Kähler potential. Then we have

$$e^{-K(z,\bar{z})} = -N_{IJ} X^I(z) \bar{X}^J(\bar{z}). \tag{2.20}$$

The resulting geometry of the physical scalar fields z^i of the vector multiplets is then given by a special Kähler manifold with Kähler metric

$$g_{i\bar{j}} = \frac{\partial^2 K(z,\bar{z})}{\partial z^i \partial \bar{z}^j}. \tag{2.21}$$

A convenient choice of inhomogeneous coordinates z^i is the *special* coordinates defined by

$$X^0(z) = 1, \qquad X^i(z) = z^i.$$

The gauge field coupling matrix is

$$\bar{\mathcal{N}}_{IJ} = F_{IJ}(X) + i_\epsilon \epsilon \frac{(N\bar{X})_I (N\bar{X})_J}{\bar{X} N \bar{X}}. \tag{2.22}$$

For theories with cubic prepotentials in (2.15), we obtain

$$g_{ij} = \epsilon \left(\frac{3}{2} \frac{(Cy)_{ij}}{Cyyy} - \frac{9}{4} \frac{(Cyy)_i (Cyy)_j}{(Cyyy)^2} \right) \tag{2.23}$$

and

$$\mathcal{N}_{00} = \frac{1}{3} Cxxx + \epsilon i_\epsilon Cyyy \left(\frac{2}{3} gxx + \frac{1}{6} \right),$$
$$\mathcal{N}_{0i} = -\frac{1}{2} (Cxx)_i - \frac{2}{3} \epsilon i_\epsilon Cyyy (gx)_i,$$
$$\mathcal{N}_{ij} = (Cx)_{ij} + \frac{2}{3} \epsilon i_\epsilon g_{ij} Cyyy. \tag{2.24}$$

Therefore the kinetic term of the scalar fields agrees with the reduced theory where

$$z^i = x^i - i_\epsilon y^i. \tag{2.25}$$

Using (2.24) the gauge part of the action (2.14) gives

$$\frac{1}{6} \epsilon Cyyy \left(\frac{1}{4} F^0 \cdot F^0 + gxx F^0 \cdot F^0 - 2(gx)_i F^i \cdot F^0 + g_{ij} F^i \cdot F^j \right)$$
$$+ \frac{1}{12} \left(Cxxx F^0 \cdot \tilde{F}^0 - 3(Cxx)_i F^i \cdot \tilde{F}^0 + 3(Cx)_{ij} F^i \cdot \tilde{F}^j \right) \tag{2.26}$$

which agrees with the reduced Lagrangian.

3. Reduced Killing spinor equations

In this section we start with the supersymmetry variation of the gravitini and gaugino in the five dimensional supergravity theory and reduce them to four dimensions. The associated Killing spinor equations are

$$\left(\hat{D}_{\hat{m}} + \frac{i}{8} h_i \left(\Gamma_{\hat{m}}{}^{\hat{n}_1 \hat{n}_2} - 4 \delta_{\hat{m}}^{\hat{n}_1} \Gamma^{\hat{n}_2} \right) \mathcal{F}^i_{\hat{n}_1 \hat{n}_2} \right) \hat{\varepsilon} = 0 \tag{3.1}$$

and

$$\left(\left(\mathcal{F}^i - h^i h_j \mathcal{F}^j \right)_{\hat{n}_1 \hat{n}_2} \Gamma^{\hat{n}_1 \hat{n}_2} + 2i \hat{\nabla}_{\hat{m}} h^i \Gamma^{\hat{m}} \right) \hat{\varepsilon} = 0. \tag{3.2}$$

Here $\hat{D}_{\hat{m}} = \partial_{\hat{m}} + \frac{1}{4} \hat{\omega}_{\hat{m}, \hat{n}_1 \hat{n}_2} \Gamma^{\hat{n}_1 \hat{n}_2}$ is the five dimensional covariant derivative. Note that Γ_0 squares to $-\epsilon$, and $\Gamma^0 = -\epsilon \Gamma_0$. We first reduce (3.1) and (3.2) to $D = 4$; throughout what follows the rescaling of the $D = 4$ gauge field strengths by $\sqrt{2}$ is taken into account.

First consider the $\hat{m} = 0$ component of (3.1); this reduces from $D = 5$ to $D = 4$ to give

$$\left(\frac{i}{2} e^{\frac{\phi}{2}} h_i \Gamma^a \left(\partial_a x^i + i \partial_a y^i \Gamma_0 \right) \right.$$
$$\left. + \frac{i}{4\sqrt{2}} e^{2\phi} \Gamma^{ab} \left(h_i \Gamma_0 \left(F^i - x^i F^0 \right)_{ab} + i \epsilon e^{\phi} F^0_{ab} \right) \right) \hat{\varepsilon} = 0. \tag{3.3}$$

Consider also the reduction of the $D = 5$ gaugino equation (3.2); which gives

$$\left(-\frac{1}{\sqrt{2}} e^{\frac{3\phi}{2}} \Gamma_0 \left(\delta^i_j - h^i h_j \right) \left(F^j - x^j F^0 \right)_{ab} \Gamma^{ab} \right.$$
$$\left. + \Gamma^a \left(\partial_a x^i - h^i h_j \partial_a x^j + i \Gamma_0 \partial_a y^i - i e^{\phi} h^i \partial_a \phi \Gamma_0 \right) \right) \hat{\varepsilon} = 0. \tag{3.4}$$

After some calculation, details of which are given in Appendix A, the two conditions (3.3) and (3.4) can be combined into the following expression:

$$\frac{i}{2} e^{K/2} (\mathrm{Im}\, \mathcal{N})_{IJ} \Gamma^{ab} F^J_{ab} \left[\mathrm{Im} \left(g^{i\bar{j}} \mathcal{D}_{\bar{j}} \bar{X}^I \right) + i \epsilon \Gamma_0 \, \mathrm{Re} \left(g^{i\bar{j}} \mathcal{D}_{\bar{j}} \bar{X}^I \right) \right] \hat{\varepsilon}$$
$$+ \Gamma^a \partial_a \left[\mathrm{Re}\, z^i - i \Gamma_0 \, \mathrm{Im}\, z^i \right] \hat{\varepsilon} = 0 \tag{3.5}$$

where

$$\mathcal{D}_{\bar{j}} \bar{X}^I = \partial_{\bar{j}} \bar{X}^I + \partial_{\bar{j}} K \bar{X}^I. \tag{3.6}$$

In particular, one finds that (3.3) is obtained from (3.5) by contracting with h_i, whereas one obtains (3.4) by considering the directions of (3.5) which are orthogonal to h_i.

Next consider the $\hat{m} = \hat{a}$ component of (3.1); this reduces to $D = 4$ to give the following expression:

$$D_a \hat{\varepsilon} + \left(\frac{1}{2\sqrt{2}} e^{\frac{3\phi}{2}} \Gamma_0 \Gamma^b \left(F^0 \right)_{ab} - \frac{1}{4} \Gamma_a{}^b \partial_b \phi - \frac{i}{4} \epsilon \Gamma_0 \Gamma_a{}^b e^{-\phi} h_i \partial_b x^i \right.$$
$$+ \frac{i}{2} \epsilon \Gamma_0 e^{-\phi} h_i \partial_a x^i + \frac{i}{4\sqrt{2}} h_i \Gamma_a{}^{bc} e^{\frac{\phi}{2}} \left(F^i - x^i F^0 \right)_{bc}$$
$$\left. - \frac{i}{\sqrt{2}} h_i e^{\frac{\phi}{2}} \left(F^i - x^i F^0 \right)_{ab} \Gamma^b \right) \hat{\varepsilon} = 0. \tag{3.7}$$

In order to rewrite this expression, we introduce the $U(1)$ (para)-Kähler potential[3]

$$A_a = -\frac{i_\epsilon}{2} \left(\partial_i K \partial_a z^i - \partial_{\bar{i}} K \partial_a z^{\bar{i}} \right). \tag{3.8}$$

This can be recast as

$$A_a = -\frac{3}{2} e^{-\phi} h_i \partial_a x^i. \tag{3.9}$$

Then, using the identities listed in Appendix A, (3.7) can be rewritten as

$$D_a \hat{\varepsilon} + \left(\frac{1}{4} \partial_a \phi - \frac{i}{2} A_a \epsilon \Gamma_0 \right.$$
$$+ \frac{i}{4} e^{\frac{K}{2}} \Gamma^{bc} F^I_{bc} \left(\mathrm{Im}\, X^J + i \epsilon \Gamma_0 \, \mathrm{Re}\, X^J \right) (\mathrm{Im}\, \mathcal{N})_{IJ} \Gamma_a \left. \right) \hat{\varepsilon}$$
$$+ \frac{1}{2} \epsilon \Gamma_a \Gamma_0 e^{-\frac{3\phi}{2}} \left(\frac{i}{4\sqrt{2}} e^{2\phi} \Gamma^{bc} \left(h_i \Gamma_0 \left(F^i - x^i F^0 \right)_{bc} + i \epsilon e^{\phi} F^0_{bc} \right) \right.$$
$$\left. + \frac{i}{2} e^{\frac{\phi}{2}} h_i \Gamma^b \left(\partial_b x^i + i \partial_b y^i \Gamma_0 \right) \right) \hat{\varepsilon} = 0. \tag{3.10}$$

[3] To be distinguished from the gauge potentials A^0, A^i.

Observe that the second and third lines of this expression can be removed by using (3.3) and so on setting

$$\hat{\varepsilon} = e^{-\frac{\phi}{4}}\varepsilon \tag{3.11}$$

it follows that (3.1) and (3.2) can be rewritten as

$$\begin{aligned} &D_a\varepsilon - \frac{i}{2}\epsilon A_a \Gamma_0 \varepsilon \\ &\quad + \frac{i}{4} e^{\frac{K}{2}} \Gamma^{bc} F^I_{bc} \big(\operatorname{Im} X^J + i\epsilon \Gamma_0 \operatorname{Re} X^J\big)(\operatorname{Im}\mathcal{N})_{IJ} \Gamma_a \varepsilon = 0 \end{aligned} \tag{3.12}$$

and

$$\begin{aligned} &\frac{i}{2} e^{K/2} (\operatorname{Im}\mathcal{N})_{IJ} \Gamma^{ab} F^J_{ab} \big[\operatorname{Im}\big(g^{i\bar{j}} \mathcal{D}_{\bar{j}} \bar{X}^I\big) + i\epsilon \Gamma_0 \operatorname{Re}\big(g^{i\bar{j}} \mathcal{D}_{\bar{j}} \bar{X}^I\big)\big]\varepsilon \\ &\quad + \Gamma^a \partial_a \big[\operatorname{Re} z^i - i\Gamma_0 \operatorname{Im} z^i\big]\varepsilon = 0. \end{aligned} \tag{3.13}$$

4. Chiral decomposition

In this section we express the transformations (3.12) and (3.13) in terms of chiral spinors. In order to define the various projections, it is convenient to note that[4]

$$\Gamma_{\hat{n}_1\hat{n}_2\hat{n}_3\hat{n}_4\hat{n}_5} = i(\widehat{\mathrm{dvol}_5})_{\hat{n}_1\hat{n}_2\hat{n}_3\hat{n}_4\hat{n}_5} \tag{4.1}$$

which implies that

$$\Gamma_0 \Gamma_{ab} = \frac{i}{2} \epsilon_{ab}{}^{cd} \Gamma_{cd}. \tag{4.2}$$

In the Minkowski case ($\epsilon = -1$), we decompose the spinor ε in terms of chiral spinors as $\varepsilon = \varepsilon_- + \varepsilon_+$, where we set

$$\begin{aligned} &\Gamma_\pm = \frac{1}{2}(1 \pm \Gamma_0), \\ &\Gamma_\pm \varepsilon_\pm = \varepsilon_\pm, \\ &\Gamma_\pm \varepsilon_\mp = 0 \end{aligned} \tag{4.3}$$

and we also define

$$F^\pm_{ab} = \frac{1}{2}(F_{ab} \pm i\tilde{F}_{ab}). \tag{4.4}$$

Also, (4.2) implies that

$$\Gamma \cdot F = \Gamma \cdot \big(F^- \Gamma_+ + F^+ \Gamma_-\big). \tag{4.5}$$

We find that (3.12) and (3.13) can be rewritten as

$$\begin{aligned} &D_a \varepsilon_\pm \pm \frac{i}{2} A_a \varepsilon_\pm \\ &\quad \pm \frac{1}{4} e^{\frac{K}{2}} \Gamma^{bc} F^{\mp I}_{bc} \big(\operatorname{Re} X^J \pm i \operatorname{Im} X^J\big) \operatorname{Im}\mathcal{N}_{IJ} \Gamma_a \varepsilon_\mp = 0 \end{aligned} \tag{4.6}$$

and

$$\begin{aligned} &\pm\frac{1}{2} e^{\frac{K}{2}} \operatorname{Im}\mathcal{N}_{IJ} \Gamma^{ab} F^{\mp J}_{ab} \big(\operatorname{Re}\big(g^{i\bar{j}} \mathcal{D}_{\bar{j}} \bar{X}^I\big) \pm i \operatorname{Im}\big(g^{i\bar{j}} \mathcal{D}_{\bar{j}} \bar{X}^I\big)\big)\varepsilon_\pm \\ &\quad + \Gamma^a \partial_a \big(\operatorname{Re} z^i \pm i \operatorname{Im} z^i\big)\varepsilon_\mp = 0. \end{aligned} \tag{4.7}$$

This is in agreement with the Killing spinor equations given by [11] (on making the identification $\varepsilon^1 = \varepsilon_+$, $\varepsilon_2 = \varepsilon_-$):

$$\begin{aligned} &D_a \varepsilon^\alpha + \frac{i}{2} A_a \varepsilon^\alpha + \frac{1}{4} (\operatorname{Im}\mathcal{N})_{IJ} X^J(z) e^{K/2} \Gamma \cdot F^{-I} \epsilon^{\alpha\beta} \Gamma_a \epsilon_\beta = 0, \\ &-\frac{1}{2} e^{\frac{K}{2}} (\operatorname{Im}\mathcal{N})_{IJ} g^{i\bar{j}} \mathcal{D}_{\bar{j}} \bar{X}^I(\bar{z}) \gamma \cdot F^{-J} \epsilon_{\alpha\beta} \epsilon^\beta + \Gamma^a \partial_a z^i \epsilon_\alpha = 0. \end{aligned} \tag{4.8}$$

[4] We remark that the sign in (4.1) is fixed by requiring that the integrability conditions of the Killing spinor equations (3.1) and (3.2) should be consistent with the gauge field equations obtained from (2.1).

Next consider the Euclidean case ($\epsilon = 1$). There are two alternative chiral decompositions possible. For the first, we define

$$\begin{aligned} &\Gamma_\pm = \frac{1}{2}(1 \pm i\Gamma_0), \\ &\Gamma_\pm \varepsilon_\pm = \varepsilon_\pm, \\ &\Gamma_\pm \varepsilon_\mp = 0 \end{aligned} \tag{4.9}$$

with

$$F^\pm_{ab} = \frac{1}{2}(F_{ab} \pm \tilde{F}_{ab}), \tag{4.10}$$

and (4.2) implies that

$$\Gamma \cdot F = \Gamma \cdot \big(F^- \Gamma_+ + F^+ \Gamma_-\big). \tag{4.11}$$

We find that (3.12) and (3.13) can be rewritten as

$$\begin{aligned} &D_a \varepsilon_\pm \mp \frac{1}{2} A_a \varepsilon_\pm \\ &\quad \pm \frac{i}{4} e^{\frac{K}{2}} \Gamma^{bc} F^{\mp I}_{bc} \big(\operatorname{Re} X^J \pm \operatorname{Im} X^J\big) \operatorname{Im}\mathcal{N}_{IJ} \Gamma_a \varepsilon_\mp = 0 \end{aligned} \tag{4.12}$$

and

$$\begin{aligned} &\pm\frac{i}{2} e^{\frac{K}{2}} \operatorname{Im}\mathcal{N}_{IJ} \Gamma^{ab} F^{\mp J}_{ab} \big(\operatorname{Re}\big(g^{i\bar{j}} \mathcal{D}_{\bar{j}} \bar{X}^I\big) \pm \operatorname{Im}\big(g^{i\bar{j}} \mathcal{D}_{\bar{j}} \bar{X}^I\big)\big)\varepsilon_\pm \\ &\quad + \Gamma^a \partial_a \big(\operatorname{Re} z^i \pm \operatorname{Im} z^i\big)\varepsilon_\mp = 0. \end{aligned} \tag{4.13}$$

This is the form of the Killing spinor equations expressed in terms of the so-called *adapted coordinates* [6].

For the second chiral decomposition in the Euclidean case, we define [4]

$$\begin{aligned} &\Gamma_\pm = \frac{1}{2}(1 \pm ie\Gamma_0), \\ &\Gamma_\pm \varepsilon_\pm = \varepsilon_\pm, \\ &\Gamma_\pm \varepsilon_\mp = 0 \end{aligned} \tag{4.14}$$

and let

$$F^\pm_{ab} = \frac{1}{2}(F_{ab} \pm e\tilde{F}_{ab}). \tag{4.15}$$

With these conventions, (4.11) holds, and we find that (3.12) and (3.13) can be rewritten as

$$\begin{aligned} &D_a \varepsilon_\pm \mp \frac{e}{2} A_a \varepsilon_\pm \\ &\quad \pm \frac{ie}{4} e^{\frac{K}{2}} \Gamma^{bc} F^{\mp I}_{bc} \big(\operatorname{Re} X^J \pm e \operatorname{Im} X^J\big) \operatorname{Im}\mathcal{N}_{IJ} \Gamma_a \varepsilon_\mp = 0 \end{aligned} \tag{4.16}$$

and

$$\begin{aligned} &\pm\frac{ie}{2} e^{\frac{K}{2}} \operatorname{Im}\mathcal{N}_{IJ} \Gamma^{ab} F^{\mp J}_{ab} \big(\operatorname{Re}\big(g^{i\bar{j}} \mathcal{D}_{\bar{j}} \bar{X}^I\big) \pm e \operatorname{Im}\big(g^{i\bar{j}} \mathcal{D}_{\bar{j}} \bar{X}^I\big)\big)\varepsilon_\pm \\ &\quad + \Gamma^a \partial_a \big(\operatorname{Re} z^i \pm e \operatorname{Im} z^i\big)\varepsilon_\mp = 0. \end{aligned} \tag{4.17}$$

5. Discussion

In this Letter we have derived the Killing spinor equations for Euclidean supergravity theories coupled to Abelian vector multiplets (3.12) and (3.13). We have obtained the four dimensional Killing spinor equations from the reduction of those in the five dimensional theory. We explicitly show how this is achieved by writing the reduced equations in an ϵ-Kähler covariant formalism. These equations were also rewritten, for the Euclidean case, in terms of chiral spinors using both the adapted and para-complex co-ordinates. ϵ-special Kähler geometry in Euclidean theories is expected to play an important role in the analysis of instantons, solitons and cosmological solutions in supergravity and M-theory. The Killing spinor equations given in (3.12) and (3.13) provide the starting point to find general instanton solutions of the effective Euclidean $N=2$ supergravity action coupled to $N=2$ matter multiplets. As for the case of black holes, one also expects that the rich geometric structure of the theory will lead to a simplified approach for finding new instanton solutions.

Spinorial geometry techniques [12] have proven to be a very useful tool in finding all instanton solutions preserving various fractions of supersymmetry. Those techniques were also used recently in finding solutions of Einstein–Maxwell theory with [13] or without [14] a cosmological constant, as well as the supersymmetric solutions of the Euclidean $N=4$ super Yang–Mills theory [15]; where interesting relations to integrable models [13] and the Hitchin equations [15] were found. We will report on the instanton solutions with vector multiplets in a separate publication. Another direction which needs to be investigated is the construction of gauged Euclidean supergravity models.

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Appendix A. ϵ-Kähler special geometry identities

In this appendix, we summarize a number of useful identities. First, consider rewriting the reduction of (3.3) and (3.4) in a special ϵ-Kähler covariant fashion as (3.5).

This is done by making use of the following identities:

$$\operatorname{Im}\mathcal{N}_{I0}\big[\operatorname{Im}\big(g^{i\bar{j}}\mathcal{D}_{\bar{j}}\bar{X}^I\big)+i\epsilon\Gamma_0\operatorname{Re}\big(g^{i\bar{j}}\mathcal{D}_{\bar{j}}\bar{X}^I\big)\big] = \big(2\epsilon e^{4\phi}+2ie^{3\phi}\Gamma_0 h_j x^j\big)h^i - 4ie^{3\phi}\Gamma_0\big(x^i - h_j x^j h^i\big) \tag{A.1}$$

and

$$\operatorname{Im}\mathcal{N}_{I\ell}\big[\operatorname{Im}\big(g^{i\bar{j}}\mathcal{D}_{\bar{j}}\bar{X}^I\big)+i\epsilon\Gamma_0\operatorname{Re}\big(g^{i\bar{j}}\mathcal{D}_{\bar{j}}\bar{X}^I\big)\big] = -2ie^{3\phi}\Gamma_0 h^i h_\ell + 4ie^{3\phi}\Gamma_0\big(\delta^i_\ell - h^i h_\ell\big) \tag{A.2}$$

where we have also used the identities

$$\mathcal{D}_{\bar{j}}\bar{X}^0 = -\frac{3}{2}\epsilon i_\epsilon e^{-\phi}h_j, \qquad \mathcal{D}_{\bar{j}}\bar{X}^i = \delta^i_j - \frac{3}{2}h^j h_i - \frac{3}{2}\epsilon i_\epsilon e^{-\phi}h_j x^i \tag{A.3}$$

and

$$g^{ij}h_j = -\frac{2}{9}\epsilon e^{-\phi}h^i Cyyy \tag{A.4}$$

and

$$Cyyy = 6e^{3\phi}, \qquad e^{-K} = \frac{4}{3}Cyyy = 8e^{3\phi}. \tag{A.5}$$

Another useful identity used to obtain (3.5) is

$$\Gamma^a\partial_a\big(\operatorname{Re}z^i - i\Gamma_0\operatorname{Im}z^i\big) = \Gamma^a\big(\partial_a\big(x^i+i\Gamma_0 y^i\big) - h^i h_j\partial_a x^j - i\Gamma_0 h^i\partial_a\phi e^{\phi}\big) + \Gamma^a h^i\big(h_j\partial_a x^j + i\Gamma_0\partial_a\phi e^{\phi}\big) \tag{A.6}$$

where the expression on the first line of the RHS is projected orthogonal to the direction of h_i, and the second line contains the term parallel to h_i.

A number of useful identities used to obtain (3.10) are

$$\frac{i}{4}e^{\frac{K}{2}}\big[\operatorname{Im}X^J + i\epsilon\Gamma_0\operatorname{Re}X^J\big]\operatorname{Im}\mathcal{N}_{IJ} = \frac{i}{8\sqrt{2}}e^{-\frac{3\phi}{2}}\big(-i\epsilon\Gamma_0\operatorname{Im}\mathcal{N}_{I0} - \big(i\epsilon\Gamma_0 x^j + y^j\big)\operatorname{Im}\mathcal{N}_{Ij}\big) \tag{A.7}$$

and

$$-i\epsilon\Gamma_0\operatorname{Im}\mathcal{N}_{00} - \big(i\epsilon\Gamma_0 x^j + y^j\big)\operatorname{Im}\mathcal{N}_{0j} = 6e^{3\phi}\left(-\frac{i}{6}\Gamma_0 - \frac{1}{2}e^{-\phi}h_i x^i\right) \tag{A.8}$$

and

$$-i\epsilon\Gamma_0\operatorname{Im}\mathcal{N}_{i0} - \big(i\epsilon\Gamma_0 x^j + y^j\big)\operatorname{Im}\mathcal{N}_{ij} = 3e^{2\phi}h_i \tag{A.9}$$

together with

$$(gy)_i = -\frac{3}{4}\epsilon e^{-\phi}h_i. \tag{A.10}$$

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