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On a Class of Random Schrödinger Operators

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Considered are random Schrödinger operators on $L^2(\mathbb{R}^d)$ that are stationary and metrically transitive with respect to a lattice, e.g., $H = H_0 + V_\omega$ with $V_\omega(x) = \sum_{i \in Z^d} q_i(\omega) f(x - x_i)$, $\{q_i\}$ independent identically distributed. A method of carrying over results from the case of potentials metrically transitive with respect to $\mathbb{R}^d$ is presented. Among these results are the Thouless formula and Kotani's theory.

1. INTRODUCTION

Random Schrödinger operators $H_\omega = H_0 + V_\omega$ serve as models for quantum mechanical disordered structures. In the last decade many research articles were devoted to such operators. Most of the authors consider Schrödinger operators with metrically transitive potentials or their discrete analogs. Here metrical transitivity means that there are measure preserving transformations $\{T_y\}_{y \in \mathbb{R}^d}$ on the underlying probability space Ω such that $V_{T_y\omega}(x) = V_\omega(x - y)$ and that any measurable subset A of Ω which is invariant under all T_y has probability zero or one. The above-mentioned discrete analogs are finite difference operators on the sequence space $l^2(Z^d)$. For those operators the potential (a function on Z^d) is metrically transitive with respect to transformations $\{T_i\}$ indexed by Z^d.

Beside these two classes of random operators there are physically interesting operators acting on $L^2(\mathbb{R}^d)$ that are metrically transitive not with respect to a continuous group of measure preserving transformations, but only with respect to a discrete one (see articles in [19, 9, 4, 12, 13]). Let us give an example: Let $\{q_i\}_{i \in Z^d}$ be a metrically transitive random field on Z^d and let $\{x_i\}_{i \in Z^d}$ be a lattice in $\mathbb{R}^d$. We consider particles of random charges $q_i(\omega)$ at the lattice positions x_i. This is a simple model for a disordered alloy. If the particle at x_i produces a potential $q_i(\omega) f(\cdot - x_i)$

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the total potential is given by

$$V_\omega(x) = \sum_{i \in Z^d} q_i(\omega) f(x - x_i). \tag{1}$$

By assumption on the underlying probability space Ω there are measure preserving transformations $\{T_i\}_{i \in Z^d}$ satisfying $q_i(T_j\omega) = q_{i-j}(\omega)$. This implies [12, 14]

$$V_{T_j\omega}(x) = V_\omega(x - x_j). \tag{2}$$

Thus V_ω is "metrically transitive" with respect to a discrete group of measure preserving transformations. More examples of such potentials can be found in [13]. We only mention an extension of (1),

$$V_\omega(x) = \sum_{i \in \mathbf{Z}^d} q_i(\omega) f(x - x_i + \xi_i(\omega)),$$

where $\{q_i\}_{i \in \mathbf{Z}^d}$ and $\{\xi_i(\omega)\}_{i \in \mathbf{Z}^d}$ are stationary and metrically transitive.

In this paper we present and apply a simple embedding technique—the "suspension construction"—that reduces questions on those potentials to corresponding questions on potentials which are metrically transitive in the usual sense. After finishing the first version of this paper we became aware of the fact that different versions of this technique had been known for a long time in various branches of mathematics (see, e.g., Mackey [20] or Smale [26]). Nevertheless we present the suspension construction in Section 2 for the reader's convenience. It consists in the construction of an "artificial" probability space that carries a larger (namely, continuous) group of measure preserving transformations than the original probability space. On the new probability space we can embed a random field V satisfying (2) into a metrically transitive random field $\overline{V}$. The rest of this paper presents applications of this construction.

In Section 3 we investigate the density of states and the Lyapunov exponent for potentials metrically transitive with respect to a lattice. Among other results we prove the Thouless formula that connects the density of states and the Lyapunov exponent for one dimensional operators. We simply apply—via suspension—well known results [2, 7] to our case.

In Section 4 we carry over the Kotani theory [17] to our case, again a simple application of the suspension construction. Potentials of type (1) may be deterministic in Kotani's sense, even if the $\{q_i\}$ are independent. An example of this phenomenon is discussed in Section 4. For those examples we cannot exclude absolutely continuous spectrum by Kotani theory.

2. Definitions and the Basic Construction

Throughout this paper $(\Omega, \mathscr{F}, P)$ will denote a probability space and $\{T_i\}_{i \in Z^d}$ will denote a group of endomorphisms of Ω, i.e., of measurable, measure preserving transformations. A set $A \in \mathscr{F}$ is called invariant (under $\{T_i\}$) if $T_i^{-1}A = A$ for all $i \in Z^d$. $\{T_i\}$ is called ergodic if any invariant set has probability zero or one. If $\{T_i\}_{i \in Z^d}$ is ergodic we call a random field $\{V_\omega(x); x \in \mathbb{R}^d\}$ metrically transitive with respect to Z^d (or Z^d-metrically transitive for short) if

$$V_{T_i\omega}(x) = V_\omega(x - i). \tag{3}$$

If $\{T_y\}_{y \in \mathbb{R}^d}$ is an ergodic group of endomorphisms of Ω we call $V_\omega(x)$ metrically transitive with respect to $\mathbb{R}^d$ if

$$V_{T_y\omega}(x) = V_\omega(x - y). \tag{4}$$

Suppose that we are given a group $\{T_i\}_{i \in Z^d}$ of endomorphisms on Ω. Then we construct a new probability space $(\overline{\Omega}, \overline{\mathscr{F}}, \overline{P})$ as the product of the spaces $(\Omega, \mathscr{F}, P)$ and $(\mathbb{R}^d/Z^d, \mathscr{B}(\mathbb{R}^d/Z^d), \mu)$, where μ is the normalized invariant measure on $\mathbb{R}^d/Z^d$. We may identify $\mathbb{R}^d/Z^d$ with $C_0 = \{x \in \mathbb{R}^d; 0 \leqq x_i < 1, i = 1, \ldots, d\}$ in the obvious way. We may use this identification freely.

Any vector $x \in \mathbb{R}^d$ has a unique decomposition $x = \underline{x} + \dot{x}$ with $\underline{x} \in Z^d$, $\dot{x} \in C_0 \cong \mathbb{R}^d/Z^d$. Now we define for $x \in \mathbb{R}^d$, $\omega \in \Omega$, $\kappa \in C_0$

$$\overline{T}_x(\omega, \kappa) := \left(T_{\underline{x+\kappa}}\omega, (x + \kappa)^{\cdot}\right). \tag{5}$$

Embedding Ω into $\overline{\Omega}$ by $\omega \hookrightarrow (\omega, 0)$ we have $\{T_i\}$ embedded in $\overline{T}_x$ in the sense $T_i \cong \overline{T}_i|_{\Omega \times \{0\}}$.

As soon as topology is concerned the above definition of $\overline{T}_x$ has the disadvantage of being discontinuous in x. To make it continuous we define $\tilde{\Omega} = \Omega \times \mathbb{R}^d$ and $\tilde{T}_x(\omega, y) = (\omega, y + x)$. We then identify (ω, y) and $(T_i^{-1}\omega, y + i)$; i.e., we consider $\overline{\Omega} = \tilde{\Omega}/\mathbb{Z}^d$. Since $\tilde{T}_x$ respects the equivalence relation it can be looked upon as a transformation $\overline{T}_x$ on $\overline{\Omega}$.

What follows will work with both constructions equally well.

Proposition 1. (i) *$\{\overline{T}_x\}_{x \in \mathbb{R}^d}$ is a group of measure preserving transformations on $(\overline{\Omega}, \overline{\mathscr{F}}, \overline{P})$.*

(ii) *If $\{T_i\}_{i \in Z^d}$ is ergodic then $\{\overline{T}_x\}_{x \in \mathbb{R}^d}$ is ergodic.*

Proof. (i) That $\overline{T}_x$ is measure preserving follows from the invariance of the Lebesgue measure (Haar measure on $\mathbb{R}^d/Z^d$) and the assumption that T_i is measure preserving.

$$\begin{aligned}\bar{T}_x\bar{T}_y(\omega,\kappa) &= \bar{T}_x\left(T_{\underline{y+\kappa}}\omega,(y+\kappa)^{\cdot}\right)\\ &= \left(T_{\underline{x+(y+\kappa)^{\cdot}}}T_{\underline{y+\kappa}}\omega,\left(x+(y+\kappa)^{\cdot}\right)^{\cdot}\right)\\ &= \left(T_{\underline{x+(y+\kappa)^{\cdot}+\underline{y+\kappa}}}\omega,(x+y+\kappa)^{\cdot}\right)\\ &= \left(T_{\underline{x+y+\kappa}}\omega,(x+y+\kappa)^{\cdot}\right)\\ &= \bar{T}_{x+y}(\omega,\kappa).\end{aligned}$$

(ii) Let A be invariant under $\{\bar{T}_x\}_{x\in\mathbf{R}^d}$. Define

$$A_\kappa = \{\omega|(\omega,\kappa)\in A\}.$$

Then

$$\begin{aligned}T_i^{-1}A_\kappa &= \{\omega|(T_i\omega,\kappa)\in A\}\\ &= \{\omega|(\omega,\kappa)\in \bar{T}_i^{-1}A\}\\ &= A_\kappa\end{aligned}$$

since A is invariant. Thus for any κ we have $P(A_\kappa)$ equals zero or one. Moreover

$$\begin{aligned}P(A_\kappa) &= P(\{\omega|(\omega,\kappa)\in A\})\\ &= P(\{\omega|(\omega,0)\in \bar{T}_\kappa^{-1}A\})\\ &= P(A_0).\end{aligned}$$

Thus

$$\bar{P}(A) = \int_{C_0} P(A_\kappa)\,d\kappa = P(A_0);$$

hence $\bar{P}(A)$ equals zero or one. □

Given a random field $V_\omega(x)$ on $\mathbb{R}^d$ we define

$$\bar{V}_{(\omega,\kappa)}(x) = V_\omega(x-\kappa). \tag{6}$$

PROPOSITION 2. *If V_ω is metrically transitive with respect to $\{T_i\}_{i\in Z^d}$ then $\bar{V}_{\bar{\omega}}$ is metrically transitive with respect to $\{\bar{T}_x\}_{x\in\mathbf{R}^d}$.*

Proof.

$$\begin{aligned}\bar{V}_{\bar{T}_y(\omega,\kappa)}(x) &= \bar{V}_{(T_{\underline{y+\kappa}}\omega,(y+\kappa)^{\cdot})}(x)\\ &= V_{T_{\underline{y+\kappa}}\omega}\left(x-(y+\kappa)^{\cdot}\right)\\ &= V_\omega\left(x-(y+\kappa)^{\cdot}-\underline{y+\kappa}\right)\\ &= V_\omega(x-y-\kappa)\\ &= \bar{V}_{(\omega,\kappa)}(x-y).\end{aligned}$$

□

Let H_ω be a random operator on $L^2(\mathbb{R}^d)$, satisfying for $i \in Z^d$

$$H_{T_i\omega} = U_i H_\omega U_i^* \tag{7}$$

with $U_y\phi(x) = \phi(x-y)$. We will call such operators Z^d-ergodic (see [14] for general properties of ergodic operators and examples). Then by

$$\bar{H}_{(\omega,\kappa)} = U_\kappa H_\omega U_\kappa^* \tag{8}$$

we define a random operator on the probability space $\bar{\Omega}$. This operator satisfies

$$\bar{H}_{\bar{T}_x\bar{\omega}} = U_x \bar{H}_{\bar{\omega}} U_x^*; \tag{9}$$

i.e., $\bar{H}_{\bar{\omega}}$ is $\mathbb{R}^d$-ergodic.

Let us give an easy example of our strategy for using the above construction. It is well known [23, 18, 14] that the spectrum of ergodic operators is a nonrandom set and that the same is true for the various parts of the spectrum (a.c. spectrum, etc.). For a given ω the operators $\bar{H}_{(\omega,\kappa)}$ are unitary equivalent to $\bar{H}_{(\omega,0)}$ for arbitrary $\kappa \in \mathbb{R}^d/Z^d$. Hence if we prove something on the spectrum of $\bar{H}_{\bar{\omega}}$ we have the same information on H_ω.

3. The Density of States and the Lyapunov Exponent

An important quantity in the study of random Schrödinger operators H_ω is the so-called density of states (or better, integrated density of states) $N(\lambda)$. It is defined by a thermodynamical limit in the following way: Restrict H_ω to a bounded box Λ and define $N_\Lambda(\lambda)$ as the number of eigenvalues (counting multiplicity) of H_ω restricted to Λ with some boundary condition at $\partial\Lambda$. Then $N(\lambda)$ is defined as the limit of $(1/|\Lambda_n|)N_{\Lambda_n}$, where Λ_n is the hypercube of side length n centered at the origin; i.e.,

$$N(\lambda) = \lim_{n\to\infty} \frac{1}{|\Lambda_n|} N_{\Lambda_n}(\lambda). \tag{10}$$

The existence of the limit and its independence of ω (a.s.) and the boundary conditions chosen are shown for quite general random Schrödinger operators [24, 21, 15].

In what follows we will assume that the stochastic potential V_ω is Z^d-metrically transitive (i.e., (2) holds) and satisfies

$$E\left(\int_{C_0} |V_\omega(x)|^p \, dx\right) < \infty, \tag{11}$$

C_0 being the unit cell of the lattice and $p = 1$ for $d = 1$, $p > 1$ for $d = 2$, and $p \geqq d/2$ for $d \geqq 3$. Under these conditions $N(\lambda)$ defined choosing Dirichlet boundary conditions exists and is independent of ω [15]. We can recover this result with the above embedding technique. For our proof of the Thouless formula the proof below will be needed.

THEOREM 1. *Suppose that V_ω is a Z^d-metrically transitive potential satisfying* (11). *Then*

(i) *the density of states $N(\lambda)$ defined with Dirichlet boundary conditions exists and is independent of ω (a.s.). Furthermore it is equal to the density of states $\bar{N}(\lambda)$ of the $\mathbb{R}^d$-ergodic operator $\bar{H}_{\bar{\omega}}$.*

(ii) *If $E(\int_{C_0} e^{-tV_\omega(x)}\,dx) < \infty$ for some $t > 0$ then $N(\lambda)$ is independent of the chosen boundary condition.*

Proof. (i) Define $F_\Lambda(\omega) = N_\Lambda^{\mathrm{D}}(\omega, \lambda)$ for fixed $\lambda \in \mathbb{R}$, where the superscript D indicates the Dirichlet boundary conditions. $\bar{F}_\Lambda(\bar{\omega})$ is the corresponding process for $\bar{H}_{\bar{\omega}} = H_0 + \bar{V}_{\bar{\omega}}$. $\bar{F}_\Lambda$ is a continuous superadditive process in the sense of Akcoglu and Krengel [1] if $\bar{E}(|\bar{V}_{\bar{\omega}}(0)|^p) < \infty$ ($p = 1$ for $d = 1$, $p > 1$ for $d = 2$, $p = d/2$ for $d \geqq 3$) (see [15]). But $\bar{E}(|\bar{V}_{\bar{\omega}}(0)|^p) = E(\int_{C_0}|V_\omega(x)|^p\,dx) < \infty$ by assumption. Hence $\lim_{N\to\infty}(1/|\Lambda_N|)\bar{F}_{\Lambda_N}$ exists, by the superadditive ergodic theorem of Akcoglu and Krengel [1] (for details see [15]). Hence we can conclude that for P-almost all ω the set $K_\omega := \{\kappa; \lim_{N\to\infty}(1/|\Lambda_N|)\bar{F}_{\Lambda_N}(\omega, \kappa) \text{ exists}\}$ has full measure.

Fix such an ω and choose $\kappa \in C_0$ arbitrary and $\kappa_0 \in K_\omega$. Then $\bar{F}_{\Lambda_{N-1}}(\omega, \kappa_0) \leqq \bar{F}_{\Lambda_N}(\omega, \kappa) \leqq \bar{F}_{\Lambda_{N+1}}(\omega, \kappa_0)$ by monotonicity of N_Λ^{D} in Λ. Therefore $\lim(1/|\Lambda_N|)\bar{F}_{\Lambda_N}(\omega, \kappa)$ exists and we have shown $P(K_\omega = C_0) = 1$. This implies that $N_\omega(\lambda) = \bar{N}_{(\omega,0)}(\lambda)$ exists for almost all ω and $N_\omega(\lambda) = \bar{N}_{(\omega,\kappa)}(\lambda)$ for almost all ω and *all* $\kappa \in C_0$.

The proof of (ii) follows the same lines. We use that $\bar{N}_{\bar{\omega}}$ is independent of the boundary conditions if $\bar{E}(e^{-t\bar{V}_{\bar{\omega}}(0)}) < \infty$. But $\bar{E}(e^{-t\bar{V}_{\bar{\omega}}(0)}) = E(\int_{C_0} e^{-tV_\omega(x)})$ (see [15]). □

For one dimensional operators the Lyapunov exponent is another important quantity (see, e.g., [23, 6, 7, 17]). Let $\phi(\omega, x)$ denote the fundamental matrix of $(-d^2/dx^2 + V_\omega)\psi = \lambda\psi$. $\phi(\omega, x)$ is defined in the following way: $\begin{pmatrix} u(x) \\ u'(x) \end{pmatrix} = \phi_\lambda(\omega, x)\begin{pmatrix} u(0) \\ u'(0) \end{pmatrix}$ for any solution u of $(-d^2/dx^2 + V_\omega(x) - \lambda)u = 0$ (for initial value problems for noncontinuous V_ω see Neumark [22, Chap. V]).

The Lyapunov exponent $\gamma_\lambda^\pm(\omega)$ is defined by

$$\gamma_\lambda^\pm(\omega) = \overline{\lim_{x\to\pm\infty}} \frac{1}{|x|}\ln\|\phi_\lambda(\omega, x)\|. \tag{12}$$

For $\mathbb{R}$-metrically transitive V_ω it is easy to see that $\gamma_\lambda^+(\omega) = \gamma_\lambda^-(\omega)$ for fixed λ and P-almost all ω and moreover the $\overline{\lim}$ actually is a limit (lim). As mentioned by Craig and Simon [7] this follows from the subadditive ergodic theorem under the condition that $E(|V_\omega(0)|) < \infty$ (for details see the Appendix). In a way similar to that for $N(\lambda)$ we carry over this result to the Z-metrically transitive case:

THEOREM 2. *If V_ω is Z-metrically transitive and satisfies $E(\int_0^1 |V_\omega(x)|dx) < \infty$, then*

$$\gamma_\lambda(\omega) = \lim_{|x|\to\infty} \frac{1}{|x|} \ln\|\phi_\lambda(\omega, x)\| \tag{13}$$

exists for fixed λ and P-almost every $\omega \in \Omega$ and is independent of ω (fixed λ and ω-a.s.) Moreover $\gamma_\lambda(\omega)$ is the Lyapunov exponent of $\overline{H}_{(\omega,\kappa)}$ for almost all ω and all $\kappa \in [0,1]$.

Some remaining details of the proof are indicated in the Appendix.

Define $\gamma(\lambda) := E(\gamma_\lambda(\omega))$ $(= \gamma_\lambda(\omega)$ for fixed λ a.s.). Then we have:

COROLLARY. *If V_ω satisfies the conditions of Theorem 2, then $\gamma(\lambda)$ is subharmonic and*

$$\gamma_\lambda^\pm(\omega) \leqq \gamma(\lambda) \tag{14}$$

for almost all ω and all λ.

Proof. By Craig and Simon [7] and Hermann [10] we know these results for the Lyapunov exponent of $\overline{H}_{\overline{\omega}}$, which is the same as the one for H_ω. Craig and Simon state the subharmonicity of $\gamma(\lambda)$ under the assumption that V_ω is continuous. This condition, however, is not used in the proof. They only need that the matrix $\phi_\lambda(\omega, x)$ is analytic in λ. This can be seen by an inspection of the proofs of Theorems 1 and 2 in Neumark [22, Sect. 15]. □

On the basis of arguments by Thouless [27], Avron and Simon [2] and Craig and Simon [7] (see also Kotani [17]) proved an important relation between the Lyapunov exponent and the density of states, the so-called Thouless formula. To formulate this let us denote by $\gamma_0(\lambda)$ and $N_0(\lambda)$ the Lyapunov exponent of the free Hamiltonian H_0 and its density of states, respectively, i.e., $\gamma_0(\lambda) = (\max(0, -E))^{1/2}$, $N_0(\lambda) = (1/\pi)(\max(0, E))^{1/2}$. Then for $\mathbb{R}$-metrically transitive bounded continuous V_ω the Thouless formula reads

$$\gamma(\lambda) - \gamma_0(\lambda) = \int_{-\infty}^{+\infty} \ln|\lambda - \lambda'|(N(d\lambda) - N_0(d\lambda)) \tag{15}$$

for all λ and almost all ω.

If V_ω is Z-metrically transitive bounded and continuous the corresponding γ and N at the same time are the (averaged) Lyapunov exponent and the density of states of the $\mathbb{R}$-metrically transitive potential $\bar{V}_{\bar{\omega}}$. Hence (15) holds in the Z-metrically transitive case, too.

From (15) it follows (see Craig and Simon [6, 7], Kotani [17]) that N is log-Hölder continuous; i.e., for any R, there is a $C > 0$, such that

$$|N(\lambda) - N(\mu)| \leqq C\left(\ln|\lambda - \mu|^{-1}\right)^{-1} \tag{16}$$

for all $|\lambda - \mu| < \frac{1}{2}$ and $|\lambda| < R$.

4. Kotani Theory

Ishii [11] and Pastur [23] proved that the positivity of the Lyapunov exponent implies absence of absolutely continuous spectrum (see also Deift and Simon [8] for a different proof). Kotani [17] proved the converse, i.e., that vanishing of γ implies absolutely continuous spectrum (see also Simon [25] for finite difference operators). We give a precise statement in the case of Z-metrically transitive potentials:

Theorem 3. *Let V_ω be a Z-metrically transitive continuous and bounded potential. Denote by E_ω^{ac} the absolutely continuous component of the spectral projection of H_ω and by E_ω^{sing} its singular component.*

(i) *If $\gamma(\lambda) > 0$ for $\lambda \in A$ then $E_\omega^{ac}(A) = 0$ a.s.*

(ii) *If $\gamma(\lambda) = 0$ for $\lambda \in A$, A a set of positive Lebesgue measure, then $E_\omega^{ac}(A) \neq 0$ a.s.*

(iii) *If $\gamma(\lambda) = 0$ on (a, b) then $E_\omega^{sing}((a, b)) = 0$ a.s.*

(iv) *If $\gamma(\lambda) = 0$ on a set of positive Lebesgue measure then V_ω is deterministic.*

We recall the notion of a deterministic process: Denote by $\mathscr{F}_I$ the σ-algebra on Ω generated by the random variables $\{V_\omega(x); x \in I\}$. Then V_ω is called deterministic, if $\bigcap_{j=1}^{\infty}\mathscr{F}_{(-\infty,-j)} = \mathscr{F}_{(-\infty,\infty)}$ up to measure zero sets.

Proof. We have already seen that γ is the Lyapunov exponent of $\bar{H}_{\bar{\omega}}$. Furthermore it is easy to see that $\bar{E}_{\bar{\omega}}^{ac}$ (this is the a.c. component of the spectral projection of $\bar{H}_{\bar{\omega}}$) satisfies: $\bar{E}_{(\omega,\kappa)}^{ac}(A)$ is unitary equivalent to $\bar{E}_{(\omega,0)}^{ac}(A) = E_\omega^{ac}(A)$ and the same holds for $\bar{E}_{(\omega,\kappa)}^{sing}$. Therefore (i) to (iii) follow from Pastur [23] and Kotani [17]. If $\gamma(\lambda) = 0$ on a set of positive Lebesgue measure we know by Kotani theory that $\bar{V}_{\bar{\omega}}$ is deterministic. Now we observe that the σ-algebra $\bar{\mathscr{F}}_I$ generated by $\{\bar{V}(x) | x \in I\}$ is contained in $\mathscr{F}_{I+[0,1)}$. Therefore V_ω is deterministic. □

Let us now consider the example

$$V_\omega(x) = \sum q_i(\omega) f(x - i). \tag{17}$$

If the $\{q_i\}$ are independent identically distributed and f has compact support (and is not identically zero), then $V_\omega(x)$ is nondeterministic. Thus under these assumptions H_ω has purely singular spectrum. Bentosela, Carmona, Duclos, Simon, Souillard, and Weder [4] proved that H_ω has pure point spectrum if $\operatorname{supp} f \subset [0,1]$, $f \leqq 0$, and the distribution of the q_i has a continuous density with compact support.

If f does not have compact support $V_\omega(x)$ may be deterministic even if the $\{q_i\}$ are independent. To see this choose a sequence $\{A_i\}_{i \in Z}$ of mutually disjoint open subsets of $(0,1)$. Take a continuous function f with $\phi \neq \operatorname{supp} f \cap [i, i+1] \subset A_i + i$. If $\{q_i\}_{i \in Z}$ is an arbitrary metrically transitive sequence of random variables and if V_ω is defined by (17) we can recover $V_\omega(x)$ for all $x \in R$ from the knowledge of $V_\omega(x)$ for $x \in [N, N+1]$ with $N \in Z$ arbitrary. Therefore V_ω is deterministic and we cannot exclude absolutely continuous spectrum by Kotani theory.

Let us call $A = \{E; \gamma(E) = 0\}$. As remarked by Deift and Simon [8] it follows by Kotani theory [17] that A is the essential support of the a.c. part of the spectral measure of H_ω (see [8] or [3] for definitions). As we saw above the Lebesgue measure $|A|$ of A may be nonzero if $\operatorname{supp} f$ is not compact. Nevertheless $|A|$ is very small near the bottom of the spectrum:

THEOREM 4. *Let f be a continuous nonnegative function satisfying $\sum_{i \in Z} \sup_{x \in [i, i+1]} \{|f(x)|\} < \infty$. Furthermore, let $\{q_i\}_{i \in Z}$ be independent random variables with common distribution μ satisfying* $\inf(\operatorname{supp} \mu) = 0$. *Then*

$$\lim_{\varepsilon \to 0^+} -\varepsilon^{1/2} \ln |A \cap (0, \varepsilon)| \geqq 0.$$

Remark. We expect that this theorem holds without $f \geqq 0$ and $\inf(\operatorname{supp} \mu) = 0$.

Proof. Deift and Simon [8] proved that $N(\varepsilon)^2 \geqq |A \cap (0, \varepsilon)|$ for the $\mathbb{R}$-metrically transitive case. This can be carried over to Z-metrically transitive potentials by the above method. In [16] it was shown that $\lim_{\varepsilon \to 0^+} -\varepsilon^{1/2} \ln N(\varepsilon) \geqq 0$. From this the statement follows. □

APPENDIX

In this appendix we give some details concerning the Lyapunov exponent.

Proposition. *If V_ω is $\mathbb{R}$-metrically transitive and if $E(|V_\omega(0)|) < \infty$, then for fixed λ*

$$\gamma_\lambda(\omega) = \lim_{x \to \infty} \frac{1}{x} \ln \|\phi_\lambda(\omega, x)\| \tag{18}$$

exists P-almost surely.

Proof. By the constancy of the Wronskian we have $\det \phi_\lambda(\omega, x) = 1$, hence $\|\phi_\lambda(\omega, x)\| \geqq 1$. Therefore the process $F_x(\omega) = \ln\|\phi_\lambda(\omega, x)\|$ is positive ($\geqq 0$) and subadditive. Moreover after introducing the Prüfer transform we get (see, e.g., Carmona [5], formula (2.8))

$$F_x(\omega) \leqq \frac{1}{2} \int_0^x |1 + V_\omega(y) - \lambda| \, dy \leqq Cx + \int_0^x |V_\omega(y)| \, dy. \tag{19}$$

Formula (19) shows that $E(\sup_{0 \leqq x \leqq 1} F_x(\omega)) < \infty$; thus we can apply the ergodic theorem of Akcoglu and Krengel [1] to establish the lim in (18) for x running through the rationals (e.g.). Now let x_n be an arbitrary sequence diverging monotonically to $+\infty$. Then $F_{\lceil x_n \rceil}(\omega) \leqq F_{x_n}(\omega) \leqq F_{\lfloor x_n \rfloor}(\omega)$, where $\lfloor x_n \rfloor$ is the largest integer smaller than x_n, $\lceil x_n \rceil$ the smallest integer larger than x_n. Therefore $(1/x_n) F_{x_n}(\omega)$ converges for all ω for which $(1/n) F_n(\omega)$ converges. Hence $\lim_{x \to \infty} (1/x) F_x(\omega)$ exists almost surely.

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