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Hankel Matrices and Their Applications to the Numerical Factorization of Polynomials

Mohammad A. Hasan and Ali A. Hasan

Department of Electrical Engineering, Colorado State University, Fort Collins, Colorado 80523

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Globally convergent algorithms for the numerical factorization of polynomials are presented. When the zeros of a polynomial are all simple and of different modulus, these procedures are useful in the simultaneous determination of all zeros. These methods are derived based on the algebraic properties of sums of powers of complex numbers and Hankel matrices. The remainder and quotient polynomials which arise from applying the Euclidean and a version of Householder's algorithms are investigated in terms of their convergence properties which turn out to be useful in the splitting of a polynomial into a product of two factors.

1. INTRODUCTION

The computation of zeros of a polynomial is a classical problem in science and engineering. For example, classical linear system theory is based on the properties of rational functions (ratio of polynomials). The curves and surfaces used in computational geometry and computer graphics involve polynomials, and many procedures for numerical integration and statistical curve fitting are based on polynomials.

The Fundamental Theorem of Algebra states that every polynomial of positive degree has at least one zero over $\mathscr{C}$, the field of complex numbers. For convenience, the polynomial to be considered will be taken as monic. Thus let $p(z) = z^m + c_1 z^{m-1} + \cdots + c_m$ be a polynomial of degree $m > 0$ with coefficients in $\mathscr{C}$ and $c_m \neq 0$. If $p(z)$ has a zero z_0 of multiplicity s, then $p(z_0) = p'(z_0) = \cdots = p^{(s-1)}(z_0) = 0$ and $p^{(s)}(z_0) \neq 0$. The Euclidean algorithm furnishes information about the multiplicity of the zeros. If z_0 is a zero of multiplicity s of $p(z)$, it is a zero of multiplicity $s-1$ of $p'(z)$. Therefore, without loss of generality, it can be assumed

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that all zeros of $p(z)$ are simple for otherwise one can divide $p(z)$ by $d(x)$, the greatest common divisor of $p(z)$ and $p'(z)$, to obtain a polynomial having the same zeros of $p(z)$ but with simple multiplicities.

There are numerous algorithms for factoring a polynomial. In [10], Sebastiao é Silva proposed an algorithm for finding the dominant zeros of polynomials where polynomials of degree at most $m - 1$ are obtained by applying the Euclidean algorithm to $w_n(z) = z^n$ and $p(z)$ for each $n \geq m$. In [11], Stewart used a power method approach to prove the convergence of Silva's method. To improve the convergence of this method, Householder [6] generalized the algorithm of Sebastiao é Silva by using $w_n(z) = q(z)^n$ for some nonconstant polynomial $g(z)$ of degree at most $m - 1$. Householder's algorithm is a method for computing polynomials $p_{m-r}^{(n)}(z)$ which converge to $\prod_{j=r+1}^{m}(z - z_j)$, where $\{z_j\}_{j=r+1}^{m}$ are the roots of $p(z)$ ordered such that $|g(z_i)| > |g(z_j)|$ for $1 \leq i \leq$ r and $r + 1 \leq j \leq m$. Recently these methods were revived in [1] where the parallel complexity of the simultaneous approximations to all zeros of a polynomial based on Householder's generalization was investigated. In all the aforementioned methods, a polynomial can be factored into two polynomials if there is a sufficient gap between the magnitudes of zeros of $p(z)$ or the set $\{|g(z_i)|\}_{i=1}^{m}$. Other well-known methods which apply the same principle are those of Graeffe, Bernoulli, and the *qd* algorithm. The method of Graeffe and the *qd* algorithm are not normally thought of as methods of factorization, but each provides, in principle, factors of the given polynomial with zeros being zeros of equal modulus of the given polynomial. For a survey of some of these methods the reader is referred to [4, 5] and the references therein.

In this paper, we will develop a class of new methods for factoring a polynomial over $\mathscr{C}$. The essence of these methods is a process whereby a sequence of polynomials of degrees less than m which in the limit have some zero(s) of $p(z)$ is generated. These methods, like the methods of Bernoulli, Householder, and Graeffe, are based on root powering. As in Householder's algorithm, the main features of these methods are simplicity and global convergence. They are simple in that they use just the well-known Euclidean algorithm for polynomials and are globally convergent since they do not require initial conditions to start them. Furthermore, the derivation of the methods in this work gives deeper insight into known methods such as Bernoulli's method and the *qd* algorithm. In particular, Householder's algorithm is further analyzed and extended where approximations of factors of $p(z)$ are extracted by means of the coefficients of the remainder and the quotient polynomials obtained by applying the Euclidean algorithm. Namely, assume that $p_{m-1}^{(n)}(z) + q_n(z)p(z) = g(z)^n$ for $n = 1, \ldots, \infty$, where $p_{m-1}^{(n)}(z)$ and $q_n(z)$ are polynomials of degrees at most $m - 1$ and $n - m$. The polynomials $p_{m-1}^{\infty}(z)$ and their coefficients are then used to generate two sets of polynomials $\{p_r^{(n)}(z)\}_{n=1}^{\infty}$ and $\{p_{m-r}^{(n)}(z)\}_{n=1}^{\infty}$ of degrees r and $m-r$ for $r = 1, \ldots, m-1$, where it will be shown that $\{p_r^{(n)}(z)\}_{n=1}^{\infty}$ converges to $\prod_{j=1}^{r}(z - w_j)$ with $w_i = g(z_i)$ while $\{p_{m-r}^{(n)}(z)\}_{n=1}^{\infty}$ converges to $\prod_{j=r+1}^{m}(z - z_j)$. Thus, when $g(z) = z$, two factors the product of which is $p(z)$ can simultaneously be determined. We will also investigate the convergence properties of the quotient polynomials, $q_n(z)$.

Throughout this development, the notation $\{p_{m-j}^{(n)}(z)\}_{n=m}^{\infty}$, for $j = 1, 2, \ldots, m$ will refer to a sequence of polynomials of degree at most $m-j$. The notation $V(z_1, \ldots, z_k)$ will denote a Vandermonde matrix of dimension k where

$$V(z_1, \ldots, z_k) = \begin{bmatrix} z_1^{k-1} & z_1^{k-2} & \cdots & z_1 & 1 \\ z_2^{k-1} & z_2^{k-2} & \cdots & z_2 & 1 \\ z_3^{k-1} & z_3^{k-2} & \cdots & z_3 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ z_k^{k-1} & z_k^{k-2} & \cdots & z_k & 1 \end{bmatrix}$$

and $|V|(z_1, \ldots, z_k)$ is its determinant which is called the Vandermondian of the z_i and is equal to $\prod_{i<j}(z_i - z_j)$ [2]. The notation $|A|$ will denote the determinant of A if A is a square matrix and the absolute value of A if A is a complex number. The superscript which appears in the quantity $b^{(n)}$ will only mean that the quantity is an element of a sequence in the integer n, unless otherwise specified. The symbol π_r^m denotes all distinct combinations of choosing r integers from the set of integers $\{1, 2, \ldots, m\}$.

2. ASYMPTOTIC PROPERTIES OF LINEAR COMBINATIONS OF POWERS OF COMPLEX NUMBERS

For expediency and convenience of presentation, we present in this section a number of preliminary and technical results which are used repeatedly in our analysis. These include mainly results concerning the asymptotic properties of linear combinations of powers of complex numbers. These properties will be used later to develop methods for computing zeros and factorization of polynomials. The next lemma provides the main algebraic properties of a finite linear combination of powers of complex numbers.

LEMMA 1. *Let $z_1, z_2, \ldots, z_m$ be a set of distinct nonzero complex numbers and let $\{d_i\}_{i=1}^m$ be a set of nonzero complex numbers. For each positive integer n, define $U_n = \sum_{j=1}^m d_j z_j^n$. Let $f(z) = e_0 z^s + e_1 z^{s-1} + \cdots + e_s$ where s is a positive integer. Then*

$$\sum_{k=0}^{s} U_{n+k} e_{s-k} = \sum_{i=1}^{m} d_i f(z_i) z_i^n \tag{2.1}$$

holds for $n = 1, 2, \ldots$. Therefore, if for each $1 \leq r \leq m$, the constants $\{c_i\}_{i=1}^r$ are defined by

$$\prod_{i=1}^{r} (z - z_i) = z^r + c_1 z^{r-1} + \cdots + c_r,$$

then the relation

$$\sum_{k=0}^{r-1} U_{n+k} c_{r-k} + U_{n+r} = \sum_{i=r+1}^{m} \prod_{j=1}^{r} (z_i - z_j) d_i z_i^n \tag{2.2}$$

holds for each positive integer n.

Proof. Equation (2.1) follows from the observation that

$$\begin{aligned}\sum_{k=0}^{s} U_{n+k} e_{s-k} &= \sum_{k=0}^{s} e_{s-k} \sum_{i=1}^{m} d_i z_i^{n+k} \\ &= \sum_{i=1}^{m} d_i \sum_{k=0}^{s} z_i^k e_{s-k} = \sum_{i=1}^{m} d_i z_i^n f(z_i).\end{aligned}$$

Clearly, when $f(z) = \prod_{i=1}^r (z - z_i)$, Eq. (2.1) simplifies to Eq. (2.2). Q.E.D.

In the following, we will repeatedly deal with two types of matrices which are denoted by $H_r^{(n)}$ and $B_r^{(n)}$, and are defined as follow.

DEFINITION 1 [1]. Given a sequence $\{U_n\}_{n=0}^\infty$ of complex numbers, then a *Hankel matrix* of order s is defined as

$$H_r^{(n)} = \begin{bmatrix} U_n & U_{n+1} & \cdots & U_{n+r-1} \\ U_{n+1} & U_{n+2} & \cdots & U_{n+r} \\ \cdots & \cdots & \cdots & \cdots \\ U_{n+r-1} & U_{n+r} & \cdots & U_{n+2r-2} \end{bmatrix}. \tag{2.3}$$

The reader is referred to [4, 5] for more information concerning these matrices and their determinants.

DEFINITION 2. Let $\{z_i\}_{i=1}^m$ be a set of distinct nonzero complex numbers and let $C = [c_{ij}]$ be a nonsingular $m \times m$ complex matrix. Set $U_i^{(u)} = \sum_{j=1}^m c_{ij} z_j^n$, for $i = 1, \dots, m$. A matrix $B_r^{(n)}$ will be called a C-matrix of order r if it has the form

$$B_r^{(n)} = \begin{bmatrix} U_1^{(n)} & U_2^{(n)} & \cdots & U_r^{(n)} \\ U_1^{(n+1)} & U_2^{(n+1)} & \cdots & U_r^{(n+1)} \\ \cdots & \cdots & \cdots & \cdots \\ U_1^{(n+r-1)} & U_2^{(n+r-1)} & \cdots & U_r^{(n+r-1)} \end{bmatrix}. \tag{2.4}$$

In the next lemma an expression for the determinant of $B_r^{(n)}$ is given.

LEMMA 2. *Let* $\{z_i\}_{i=1}^m$, $\{U_i^n\}_{n=1}^\infty$, $i = 1, \dots, m$, *and* $B_r^{(n)}$ *be as in Definition* 2. *Then*

$$|B_r^{(n)}| = \sum_{(i_1 < i_2 < \cdots < i_r)} C_{i_1 i_2 \dots i_r} z_{i_1}^n z_{i_2}^n \dots z_{i_r}^n |V| (z_{i_1}, z_{i_2}, \dots, z_{i_r}), \tag{2.5}$$

where

$$C_{i_1 i_2 \dots i_r} = \begin{vmatrix} c_{1i_1} & c_{1i_2} & \cdots & c_{1i_r} \\ c_{2i_1} & c_{2i_2} & \cdots & c_{2i_r} \\ \cdots & \cdots & \cdots & \cdots \\ c_{ri_1} & c_{ri_2} & \cdots & c_{ri_r} \end{vmatrix},$$

and where summation is taken with respect to all combinations π_r^m *consisting of* $(i_1, \dots, i_r)$ *of the set* $\{1, 2, \dots, m\}$. *Moreover, if* $|z_1| \geq |z_2| \geq \cdots \geq |z_r| > |z_i|$ *for* $i = r + 1, \dots, m$, *and* $C_{12\dots r} \neq 0$, *where* $1 \leq r \leq m$, *then* $|B_r^{(n)}|$ *is nonzero for all sufficiently large* n. *Additionally*, $|B_r^{(n)}| = 0$ *for* $r > m$.

Proof. Since $U_i^n = \sum_{j=1}^m c_{ij} z_j^n$, the matrix $B_r^{(n)}$ can be expressed as

$$B_r^{(n)} = \begin{bmatrix} z_1^n & z_2^n & \cdots & z_m^n \\ z_1^{n+1} & z_2^{n+1} & \cdots & z_m^{n+1} \\ \cdots & \cdots & \cdots & \cdots \\ z_1^{n+r-1} & z_2^{n+r-1} & \cdots & z_m^{n+r-1} \end{bmatrix} \begin{bmatrix} c_{11} & c_{21} & \cdots & c_{r1} \\ c_{12} & c_{22} & \cdots & c_{r2} \\ \cdots & \cdots & \cdots & \cdots \\ c_{1m} & c_{2m} & \cdots & c_{rm} \end{bmatrix}.$$

Applying the Binet–Cauchy theorem [5], the determinant $|B_r^{(n)}|$ can be written in the form

$$|B_r^{(n)}| = \sum_{(i_1,\ldots,i_r)\in \pi_r^m} \begin{vmatrix} c_{1i_1} & c_{1i_2} & \cdots & c_{1i_r} \\ c_{2i_1} & c_{2i_2} & \cdots & c_{2i_r} \\ \cdots & \cdots & \cdots & \cdots \\ c_{ri_1} & c_{ri_2} & \cdots & c_{ri_r} \end{vmatrix} \times \begin{vmatrix} z_{i_1}^n & z_{i_2}^n & \cdots & z_{i_r}^n \\ z_{i_1}^{n+1} & z_{i_2}^{n+1} & \cdots & z_{i_r}^{n+1} \\ \cdots & \cdots & \cdots & \cdots \\ z_{i_1}^{n+r-1} & z_{i_2}^{n+r-1} & \cdots & z_{i_r}^{n+r-1} \end{vmatrix}$$

$$= \sum_{(i_1,\ldots,i_r)\in \pi_r^m} z_{i_1}^n \cdots z_{i_r}^n \begin{vmatrix} c_{1i_1} & c_{1i_2} & \cdots & c_{1i_r} \\ c_{2i_1} & c_{2i_2} & \cdots & c_{2i_r} \\ \cdots & \cdots & \cdots & \cdots \\ c_{ri_1} & c_{ri_2} & \cdots & c_{ri_r} \end{vmatrix} \times \begin{vmatrix} 1 & 1 & \cdots & 1 \\ z_{i_1} & z_{i_2} & \cdots & z_{i_r} \\ \cdots & \cdots & \cdots & \cdots \\ z_{i_1}^{r-1} & z_{i_2}^{r-1} & \cdots & z_{i_r}^{r-1} \end{vmatrix}$$

$$= \sum_{(i_1,\ldots,i_r)\in \pi_r^m} C_{i_1\ldots i_r} z_{i_1}^n \ldots z_{i_r}^n |V|(z_{i_1}, z_{i_2}, \ldots, z_{i_r}). \qquad (2.6)$$

In the case where $|z_1| \geq |z_2| \geq \cdots \geq |z_r| > |z_i|$ for $i = r+1, \ldots, m$, and if $C_{12\ldots r} \neq 0$ the term $C_{12\ldots r} z_i^n \ldots z_r^n |V|(z_1, \ldots, z_r)$ is dominant and therefore $|B_r^{(n)}| \neq 0$ for all sufficiently large n.

To show that $B_r^{(n)}$ is singular for $r > m$, we first show that $B_{m+1}^{(n)}$ is singular. If in Eq. (2.1) of Lemma 1 we set $f(z) = \prod_{i=1}^m (z - z_i) = z^m + c_1 z^{m-1} + \cdots + c_m$, then the equation

$$\sum_{k=0}^{m-1} U_i^{(n+k)} c_{m-k} + U_i^{(n+m)} = 0$$

holds for each positive integer n and for $1 \leq i \leq m$. Thus the matrix $B_{m+1}^{(n)}$ is singular since $(c_m, c_{m-1}, \ldots, c_1, 1)^T$ is a zero eigenvector of this matrix. By considering polynomials of degrees $r > m$ of the form $f(z) = z^{r-m} p(z) = z^r + c_1 z^{r-1} + \cdots + c_m z^{r-m}$, it can easily be seen that $B_r^{(n)}$ is singular since the r-dimensional vector $(0, \ldots, 0, c_m, c_{m-1}, \ldots, c_1, 1)^T$ is a zero eigenvector of $B_r^{(n)}$. In fact, it can easily be seen that the following

$r - m$ vectors,

$$(0, \ldots, 0, c_m, c_{m-1}, \ldots, c_1, 1)^T, (0, \ldots, 0, c_m, c_{m-1}, \ldots, c_1, 1, 0)^T, \\ \ldots, (c_m, c_{m-1}, \ldots, c_1, 1, 0, \ldots, 0)^T \in \mathscr{C}^r \text{ span the null space of } B_r^{(n)}.$$

Q.E.D.

COROLLARY 3 [1]. *Let* $\{z_i\}_{i=1}^m$, $\{d_i\}_{i=1}^m$, *and* $\{U_n\}_{n=1}^\infty$ *be as defined in Lemma* 1. *Then for each* $1 \le r \le m$,

$$|H_r^{(n)}| = \sum_{(i_1, \ldots, i_r)} d_{i_1} \ldots d_{i_r} z_{i_1}^n \ldots z_{i_r}^n |V|^2 (z_{i_1}, \ldots, z_{i_r}), \tag{2.7}$$

where $(i_1, i_2, \ldots, i_r)$ *runs for all* r *combinations of the set* $\{1, 2, \ldots, m\}$. *Moreover, if* $|z_1| \ge |z_2| \ge \cdots \ge |z_r| > |z_i|$ *for* $i = r + 1, \ldots, m$, *where* $1 \le r \le m$, *then* $|H_r^{(n)}|$ *is nonzero for all sufficiently large* n. *Additionally,* $|H_r^{(n)}| = 0$ *for* $r > m$.

Proof. This result follows directly from Lemma 2 by setting $c_{ij} = d_j z_j^{i-1}$. Q.E.D.

Having stated these results, several comments are in order. In view of (2.5), we have $|B_m^{(n)}| = |C| z_1^n \ldots z_m^n |V|(z_1, \ldots, z_m)$. Similarly, $|H_m^{(n)}| = d_1 \ldots d_m z_1^n \ldots z_m^n |V|^2 (z_1, \ldots, z_m)$. Consequently, $|B_m^{(n+1)}|/|B_m^{(n)} = |H_m^{(n+1)}/|H_m^{(n)}| = z_1 z_2 \ldots z_m$ for each $n \ge 1$. A particularly important consequence of the last lemma is that when $|z_1| \ge |z_2| \ge \cdots \ge |z_r| > |z_{r+1}| \ge |z_i|$ for $i = r + 2, \ldots, m$, it is easily established from (2.5) that $||B_r^{(n+1)}|/|B_r^{(n)}| - z_1 z_2 \ldots z_r| \le K |z_{r+1}/z_r|^n$ for some $K \ge 0$. This implies that $\lim_{n \to \infty} |B_r^{(n+1)}/|B_r^{(n)}| = z_1 z_2 \ldots z_r$. Analogous relations hold for $H_r^{(n)}$, in which case $||H_r^{(n+1)}/|H_r^{(n)}| - z_1 z_2 \ldots z_r| \le K |z_{r+1}/z_r|^n$ for some $K \ge 0$ and thus $\lim_{n \to \infty} |H_r^{(n+1)}|/|H_r^{(n)}| = z_1 z_2 \ldots z_r$. The last equation constitutes the basis of the *qd* algorithm [4]. Thus, if all $\{z_i\}_{i=1}^m$ are of different modulus, one can apply the last equation to compute all zeros of $p(z) = \prod_{i=1}^m (z - z_i)$. In this case the sequences $\{B_j^{(n)}\}_{n=1}^\infty$ and $\{H_j^{(n)}\}_{n=1}^\infty$ for $j = 1, \ldots, m$ are to be computed. Then

$$z_j = \lim_{n \to \infty} \frac{|B_j^{(n)}| \, |B_{j-1}^{(n+1)}|}{|B_{j-1}^{(n)}| \, |B_j^{(n+1)}|} = \lim_{n \to \infty} \frac{|H_j^{(n)}| \, |H_{j-1}^{(n+1)}|}{|H_{j-1}^{(n)}| \, |H_j^{(n+1)}|}.$$

In many of the results of the following sections we make use of the following result.

LEMMA 4. *Let A be an $r \times r$ nonsingular matrix and let b and c be r-dimensional vectors. Let $Z = (1, z, \ldots, z^{r-1})^T$, where $z \in \mathscr{C}$. Then $A^{-1}b = -c$ if and only if*

$$\frac{\begin{vmatrix} A^T & Z \\ b^T & z^r \end{vmatrix}}{|A|} = z^r + c^T Z$$

for each complex number z.

Proof. Since A is nonsingular,

$$\begin{vmatrix} A^T & Z \\ b^T & z^r \end{vmatrix} = |A|\left(z^r - b^T (A^T)^{-1} Z\right).$$

Hence if $A^{-1}b = -c$, then

$$\frac{\begin{vmatrix} A^T & Z \\ b^T & z^r \end{vmatrix}}{|A|} = z^r + c^T Z.$$

Conversely, if

$$\frac{\begin{vmatrix} A^T & Z \\ b^T & z^r \end{vmatrix}}{|A|} = z^r + c^T Z,$$

then $c^T Z = -b^T (A^T)^{-1} Z$ for each $z \in \mathscr{C}$ from which it follows that $A^{-1}b = -c$. Q.E.D.

The next result shows how to generate approximations of polynomials having zeros of maximum modulus among the set $\{z_i\}_{i=1}^m$.

THEOREM 5. *Let $\{z_i\}_{i=1}^m$ be a set of nonzero distinct complex numbers such that $|z_i| \geq |z_2| \geq \cdots \geq |z_r| > |z_{r+1} \geq |z_i|$ for $i = r+2, \ldots, m$, where $1 \leq r \leq m$. Let $\{U_i^n\}_{n=1}^\infty$ be as defined above and assume that $C_{12\ldots r} \neq 0$ and let $\prod_{i=1}^r (z - z_i) = z^r + c_1 z^{r-1} + \cdots + c_r$, then*

$$\text{(i)} \qquad \lim_{n\to\infty} \frac{\begin{vmatrix} U_1^{(n)} & U_2^{(n)} & \cdots & U_r^{(n)} & 1 \\ U_1^{(n+1)} & U_2^{(n+1)} & \cdots & U_r^{(n+1)} & z \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ U_1^{(n+r-1)} & U_2^{(n+r-1)} & \cdots & U_r^{(n+r-1)} & z^{r-1} \\ U_1^{(n+r)} & U_2^{(n+r)} & \cdots & U_r^{(n+r)} & z^r \end{vmatrix}}{|B_r^{(n)}|}$$

$$= z^r + c_1 z^{r-1} + \cdots + c_r, \tag{2.8}$$

with $O(|z_{r+1}/z_r|^n)$ order of convergence.

$$\text{(ii)} \qquad \lim_{n\to\infty} \begin{bmatrix} U_1^{(n)} & U_1^{(n+1)} & \cdots & U_1^{(n+r-1)} \\ U_2^{(n)} & U_2^{(n+1)} & \cdots & U_2^{(n+r-1)} \\ \cdots & \cdots & \cdots & \cdots \\ U_r^{(n)} & U_r^{(n+1)} & \cdots & U_r^{(n+r-1)} \end{bmatrix}^{-1} \begin{bmatrix} U_1^{n+r} \\ U_2^{n+r} \\ \vdots \\ U_r^{n+r} \end{bmatrix} = -\begin{bmatrix} c_r \\ \vdots \\ c_1 \end{bmatrix} \tag{2.9}$$

with $O(|z_{r+1}/z_r|^n)$ *order of convergence.*

Proof. From Lemma 2, $|B_r^{(n)}| \neq 0$ for all sufficiently large n. Let

$$p_r^{(n)}(z) = \frac{\begin{vmatrix} U_1^{(n)} & U_2^{(n)} & \cdots & U_r^{(n)} & 1 \\ U_1^{(n+1)} & U_2^{(n+1)} & \cdots & U_r^{(n+1)} & z \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ U_1^{(n+r-1)} & U_2^{(n+r-1)} & \cdots & U_r^{(n+r-1)} & z^{r-1} \\ U_1^{(n+r)} & U_2^{(n+r)} & \cdots & U_r^{(n+r)} & z^r \end{vmatrix}}{|B_r^{(n)}|},$$

then $p_r^{(n)}(z)$ is a monic polynomial of degree r since $|B_r^{(n)}| \neq 0$. It can easily be checked that

$$\begin{bmatrix} U_1^{(n)} & U_2^{(n)} & \cdots & U_r^{(n)} & 1 \\ U_1^{(n+1)} & U_2^{(n+1)} & \cdots & U_r^{(n+1)} & z \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ U_1^{(n+r-1)} & U_2^{(n+r-1)} & \cdots & U_r^{(n+r-1)} & z^{r-1} \\ U_1^{(n+r)} & U_2^{(n+r)} & \cdots & U_r^{(n+r)} & z^r \end{bmatrix}$$

$$= \begin{bmatrix} z_1^n & z_2^n & \cdots & z_m^n & 1 \\ z_1^{n+1} & z_2^{n+1} & \cdots & z_m^{n+1} & z \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ z_1^{n+r} & z_2^{n+r} & \cdots & z_m^{n+r} & z^r \end{bmatrix} \begin{bmatrix} c_{11} & c_{21} & \cdots & c_{r1} & 0 \\ c_{12} & c_{22} & \cdots & c_{r2} & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ c_{1m} & c_{2m} & \cdots & c_{rm} & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix}.$$

Applying the Binet–Cauchy theorem [5] yields that

$$p_r^{(n)}(z) = \frac{1}{|B_r^{(n)}|} \sum_{(i_1,\ldots,i_r)\in \pi_r{}^m} z_{i_1}^n \ldots z_{i_r}^n$$

$$\times \begin{vmatrix} c_{1i_1} & c_{1i_2} & \cdots & c_{1i_r} \\ c_{2i_1} & c_{2i_2} & \cdots & c_{2i_r} \\ \cdots & \cdots & \cdots & \cdots \\ c_{ri_1} & c_{ri_2} & \cdots & c_{ri_r} \end{vmatrix} \begin{vmatrix} 1 & 1 & \cdots & 1 & 1 \\ z_{i_1} & z_{i_2} & \cdots & z_{i_r} & z \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ z_{i_1}^r & z_{i_2}^r & \cdots & z_{i_r}^r & z^r \end{vmatrix}$$

$$= \frac{1}{|B_r^{(n)}|} \sum_{(i_1,\ldots,i_r)\in \pi_r{}^m} C_{i_1\ldots i_r} z_{i_1}^n \ldots z_{i_r}^n |V|(z_{i_1}\ldots z_{i_r})$$

$$\times \prod_{j=1}^{r} (z - z_{i_j})|V|(z_{i_1}\ldots z_{i_r}, z).$$

Since $C_{12\ldots r} z_1^n z_2^n \ldots z_r^n |V|(z_1 \ldots z_r)$ is the dominant term, this implies that $|p_r^{(n)}(z) - \prod_{j=1}^{r}(z - z_j)| \leq K|z_{r+1}/z_r|^n$ *for some* $K \geq 0$.

This proves the first conclusion of the theorem. The second conclusion follows directly from Lemma 4. Q.E.D.

COROLLARY 6 [1]. *Let* $\{z_i\}_{i=1}^m$, $\{d_i\}_{i=1}^m$, *and* $\{U_n\}_{n=1}^\infty$ *be as defined in Lemma* 1 *and assume that* $|z_1| \geq |z_2| \geq \cdots \geq |z_r| > |z_{r+1} \geq |z_i|$ *for* $i = r + 2, \ldots, m$, *where* $1 \leq r \leq m$. *Let* $\prod_{i=1}^{r}(z - z_i) = z^r + c_1 z^{r-1} + \cdots + c_r$, *then*

$$\text{(i)} \qquad \lim_{n\to\infty} \frac{\begin{vmatrix} U_n & U_{n+1} & \cdots & U_{n+1-1} & 1 \\ U_{n+1} & U_{n+2} & \cdots & U_{n+r} & z \\ \vdots & & & & \vdots \\ U_{n+r-1} & U_{n+r} & \cdots & U_{n+2r-2} & z^{r-1} \\ U_{n+r} & U_{n+r+1} & \cdots & U_{n+2r-1} & z^r \end{vmatrix}}{|H_r^{(n)}|}$$

$$= z^r + c_1 z^{r-1} + \cdots + c_r \tag{2.10}$$

with $O(|z_{r+1}/z_r|^n)$ *order of convergence.*

$$\text{(ii)} \qquad \lim_{n\to\infty} \begin{bmatrix} U_n & U_{n+1} & \cdots & U_{n+r-1} \\ U_{n+1} & U_{n+2} & \cdots & U_{n+r} \\ \vdots & & & \vdots \\ U_{n+r-1} & U_{n+r} & \cdots & U_{n+2r-2} \end{bmatrix}^{-1} \begin{bmatrix} U_{n+r} \\ U_{n+r+1} \\ \vdots \\ U_{n+2r-1} \end{bmatrix}$$

$$= -\begin{bmatrix} c_r \\ \vdots \\ c_1 \end{bmatrix} \tag{2.11}$$

with $O(|z_{r+1}/z_r|^n)$ *order of convergence.*

Note that Eqs. (2.8) and (2.10) are mainly of theoretical interest since it is not obvious how to compute them efficiently. However, their equivalent forms (2.9) and (2.11) can be efficiently implemented since they only required the inversion of structured matrices as shown in the following remarks.

Remark 1. Suppose that $B_r^{(n)}$ is nonsingular and let

$$x_i^{(n)} = \begin{bmatrix} U_1^{(n)} \\ U_2^{(n)} \\ \vdots \\ U_r^{(n)} \end{bmatrix}, \qquad X_2^{(n)} = \begin{bmatrix} U_1^{(n+1)} & \cdots & U_1^{(n+r-1)} \\ U_2^{(n+1)} & \cdots & U_2^{(n+r-1)} \\ \cdots & \cdots & \cdots \\ U_r^{(n+1)} & \cdots & U_r^{(n+r-1)} \end{bmatrix},$$

and

$$x_3^{(n)} = \begin{bmatrix} U_1^{(n+r)} \\ U_2^{(n+r)} \\ \vdots \\ U_r^{(n+r)} \end{bmatrix}$$

so that $B_r^{(n)} = [x_1^{(n)} X_2^{(n)}]$ and $B_r^{(n+1)} = [X_2^{(n)} x_3^{(n)}]$. Assume that

$$\{B_r^{(n)}\}^{-1} = \begin{bmatrix} a_1^{(n)^T} \\ A_2^{(n)} \end{bmatrix},$$

then

$$\{B_r^{(n+1)}\}^{-1} = \begin{bmatrix} X_2^{(n)} & x_3^{(n)} \end{bmatrix}^{-1} = \begin{bmatrix} A_2^{(n)} - \dfrac{A_2^{(n)} x_3^{(n)} a_1^{(n)^T}}{x_3^{(n)^T} a_1^{(n)}} \\ \dfrac{a_1^{(n)}}{x_3^{(n)^T} a_1^{(n)}} \end{bmatrix}.$$

It can be shown that $\lim_{n \to \infty} x_3^{(n)^T} a_1^{(n)} = c_r = z_1 z_2 \ldots z_r \neq 0$, i.e., the inverse above is well defined for sufficiently large n. Thus an updating equation for the inverse of $\{B_r^{(n)}\}$ can be applied to develop a recursive solution of (2.9) and (2.11).

Remark 2. It should be observed that if $B_r^{(n)}$ is invertible, then

$$\begin{bmatrix} U_1^{(n)} & U_1^{(n+1)} & \cdots & U_1^{(n+r-1)} \\ U_2^{(n)} & U_2^{(n+1)} & \cdots & U_2^{(n+r-1)} \\ \cdots & \cdots & \cdots & \cdots \\ U_r^{(n)} & U_r^{(n+1)} & \cdots & U_r^{(n+r-1)} \end{bmatrix}^{-1} \begin{bmatrix} U_1^{(n+i)} \\ U_2^{(n+i)} \\ \vdots \\ U_r^{(n+i)} \end{bmatrix} = \mathbf{e}_{i+1}$$

for $i = 1, \ldots, r-1$, where $\mathbf{e}_i$ is the ith column of an $r \times r$ identity matrix. Thus Eqs. (2.8) and (2.9) can be rewritten as

$$\lim_{n\to\infty} \left\{B_r^{(n)}\right\}^{-1} B_r^{(n+1)} = C_r$$

and (2.10) and (2.11) as $\lim_{n\to\infty}\{H_r^{(n)}\}^{-1}H_r^{(n+1)} = C_r$, with $O(|z_{r+1}/z_r|^n)$ order of convergence, where

$$C_r = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & -c_r \\ 1 & 0 & 0 & \cdots & 0 & -c_{r-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 & -c_1 \end{bmatrix}. \tag{2.12}$$

A particularly important special case is that when $r = m$, we have $\{B_m^{(n)}\}^{-1}B_m^{(n+1)} = C_m$, and that $\{H_m^{(n)}\}^{-1}H_m^{(n+1)} = C_m$ hold for each positive integer n. These observations can be summarized in the following theorem.

THEOREM 7. *Let $B_r^{(n)}$ and $H_r^{(n)}$ be a C-matrix and a Hankel matrix for the set $\{z_i\}_{i=1}^m$, respectively, and let C_r be as defined in* (2.12). *Then the following results hold*:

(i) $B_m^{(n)} = C_m^n B_m^{(0)}$ *and* $H_m^{(n)} = C_m^n H_m^{(0)}$.

(ii) *For* $r \geq m$, $|B_r^{(n+1)} - zB_r^{(n)}| = 0$ *iff* $z^{r-m}p(z) = 0$ *for each* $n > 1$.

(iii) *For* $r \geq m$, $|H_r^{(n+1)} - zH_r^{(n)}| = 0$ *iff* $z^{r-m}p(z) = 0$ *for each* $n > 1$.

(iv) *Assume that* $|z_1| \geq |z_2| \geq \cdots \geq |z_r| > |z_{r+1} \geq |z_i|$ *for* $i = r+2, \ldots, m$, *where* $1 \leq r \leq m$ *and let* $p_r(z) = \prod_{i=1}^r (z - z_i)$. *Then for* $r < m$,

$$\frac{|H_r^{(n+1)} - zH_r^{(n)}|}{|H_r^{(n)}|} = 0 \qquad \textit{iff}\ p_r(z) = 0$$

and

$$\frac{|B_r^{(n+1)} - zB_r^{(n)}|}{|B_r^{(n)}|} = 0 \qquad \textit{iff}\ p_r(z) = 0$$

provided that $C_{12\ldots r} \neq 0$.

(v) *For* $l \geq 0$, $\{B_m^{(n)}\}^{-1}B_m^{(n+l)} = C_m^l$ *and* $\{H_m^{(n)}\}^{-1}H_m^{(n+l)} = C_m^l$.

Remark 3. The two sequences of matrices $\{B_r^{(n)}\}_{n=1}^\infty$ and $\{H_r^{(n)}\}_{n=1}^\infty$ are of finite rank since $|H_r^{(n)}| = 0$ and $|B_r^{(n)}| = 0$ for $r > m$. Theorem 7 shows that C-matrices and Hankel matrices of finite rank behave very much like powers of companion matrices. The significance of this result is that it provides a direct method of generating C-matrices by simply considering submatrices of powers of C_m, the companion matrix of $p(z)$.

3. APPLICATIONS TO POLYNOMIALS

When a polynomial has two factors having zeros of different magnitude ranges, then Theorem 5 and Corollary 6 can be applied to extract one of these factors. The main goal of this section is to state some methods of generating sequences for which these results can be applied. There are many ways to generate such sequences like the Newton identities and power series expansions. However, we will place special emphasis on the Euclidean and Householder's algorithms which are described next.

3.1. *A Modified Householder's Algorithm*

Consider the following variation of Householder's algorithm. Let $p(z)$ and $g(z)$ be polynomials over $\mathscr{C}$ such that $p(z) = \prod_{i=1}^{m}(z - z_i)$ and degree $g(z) < m$. Next, generate a sequence of polynomials $p_{m-1}^{(n)}(z) = b_{m-1}^{(n)} z^{m-1} + b_{m-2}^{(n)} z^{m-2} + \cdots + b_0^{(n)}$ as follows.

ALGORITHM 3.1. (i) For each positive integer $n = 2^{l_0}$, compute $p_{m-1}^{(n)}(z)$ as

$$p_{m-1}^{(1)}(z) = g(z),$$

$$p_{m-1}^{(2l)}(z) = \left(p_{m-1}^{(l)}(z)\right)^2 \bmod p(z) \qquad \text{for } l = 1, \ldots, l_0, \tag{3.1}$$

and form $p_{m-1}^{(n)}(z), p_{m-1}^{(n+1)}(z), \ldots, p_{m-1}^{(n+m-1)}(z)$ by applying

$$p_{m-1}^{(n+1)}(z) = g(z) p_{m-1}^{(n)}(z) \bmod p(z)$$

or

$$p_{m-1}^{(n+1)}(z) = g(z) p_{m-1}^{(n)}(z) + q_n(z) p(z). \tag{3.2}$$

(ii) Apply the Euclidean algorithm to generate a new set of polynomials $p_{m-r}^{(n)}(z)$ of degrees $m - r$ for $r = 2, \ldots, m-1$ as follows.

$$p_{m-j-1}^{(n+i)}(z) = \frac{p_{m-j}^{(n+i)}(z)}{a_{m-j}^{(n+i)}} - \frac{p_{m-j}^{(n)}(z)}{a_{m-j}^{(n)}} \tag{3.3}$$

for $j = 1, \ldots, m-1$ and $i = j, j+1, \ldots, m$, where $a_{m-j}^{(n)}$ is the leading coefficient of $p_{m-j}^{(n)}(z)$.

In connection with this algorithm, Bini and Gemignani [1] used a different method of generating polynomials of degree $m-r$ which converges to $\prod_{i=r+1}^{m}(z-z_i)$. For a given n, they apply the Euclidean algorithm to the polynomials $p_{m-r}^{(n)}(z)$ and $p_{m-r-1}^{(n)}(z)$ for $r=1,2,\ldots,m-2$, where $p_m^{(n)}(z)=p(z)$. In Algorithm 3.1, we used polynomials generated for different n's, namely $n, n+1,\ldots,n+m-1$, to obtain the sequence $\{p_{m-r}^{(n+i)}(z)\}_{r=1}^{m-1}$ for $i=0,1,\ldots,m-1$. Note that these two approaches are generally different although asymptotically they have the same rate of convergence as shown in Section 4.

In this section we explore this algorithm in greater depth and show how it applies when there is more than one dominant zero. Specifically, the coefficients of these polynomials will be used to extract some factors of $p(z)$.

In the following, we assume that $w_i=g(z_i)\neq 0$ and the $\{z_i\}$'s ordered so that $|w_{i+1}/w_i|\leq 1$. When all zeros are different in magnitude, it will be shown in the next section that $a_{m-j}^{(n)}\neq 0$, $j=1,\ldots,m-1$ for all sufficiently large n and thus factors of all degrees can be determined. However, when some zeros have equal modulus, these methods should be modified, applying the Euclidean algorithm and shift (if necessary), to determine all roots of a polynomial or just to factor it to polynomials of lower degrees.

The case $g(z)=z$ is particularly important since it allows one to generate two factors of $p(z)$ whose product is $p(z)$. A recursive formula for computing the remainder and quotient polynomials $\{p_{m-1}^{(n)}(z)\}_{n=m}^{\infty}$ and $\{q_n(z)\}_{n=m}^{\infty}$, that arise from dividing $w(z)=z^n$ by $p(z)$ for $n\geq m$ can be shown to satisfy

$$p_{m-1}^{(n+1)}(z) = b_{m-1}^{(n)} p_{m-1}^{(m)}(z) + b_{m-2}^{(n)} z^{m-1} + \cdots + b_0^{(n)} z$$

and

$$q_{n+1}(z) = z q_n(z) + b_{m-1}^{(n)} q_m(z), \tag{3.4}$$

where $q_m(z)=1$ and $p_{m-1}^{(m)}(z)$ satisfies $z^m = q_m(z)p(z) + p_{m-1}^{(m)}(z)$, and

$$p_{m-1}^{(m)}(z) = -\sum_{j=1}^{m} a_{m-j} z^{m-j}.$$

Thus a recursive formula for $b_{m-j}^{(n)}$, for $n\geq m$, has the following form:

$$\begin{aligned} b_{m-j}^{(m)} &= -a_{m-j}, \qquad \text{for } j=1,2,\ldots,m \\ b_{m-j}^{(n+1)} &= -a_{m-j} b_{m-1}^{(n)} + b_{m-j-1}^{(n)} \qquad \text{for } j=1,2,\ldots,m-1 \\ b_0^{(n+1)} &= -a_0 b_{m-1}^{(n)}. \end{aligned} \tag{3.5}$$

Having determined the sequence $b_{m-1}^{(k)}$, for $k = m, m+1, \ldots, n-1$, the quotient polynomials $\{q_n(z)\}_{n=m}^{\infty}$ are determined by

$$q_n(z) = z^n + b_{m-1}^{(m)} z^{n-1} + b_{m-1}^{(m+1)} z^{n-2} + \cdots + b_{m-1}^{(n-1)}. \qquad (3.6)$$

It can easily be verified that for each positive integer $n \geq m$, the polynomials $q_n(z)$, $p(z)$, and $p_{m-1}^{(n)}(z)$ satisfy the identity $q_n(z)p(z) + p_{m-1}^{(n)}(z) = z^n$.

In the next theorem it will be shown that each element of the sequence $\{b_{m-r}^{(n)}\}_{r=1}^{m}$ is a linear combination of powers of complex numbers. But before establishing that, we need the following lemma which states some properties of the entries of the inverse of a Vandermonde matrix.

LEMMA 8. *Let $z_1, z_2, \ldots, z_m$ be a set of distinct nonzero complex numbers. Let*

$$V = \begin{bmatrix} z_1^{m-1} & z_1^{m-2} & \cdots & z_1 & 1 \\ z_2^{m-1} & z_2^{m-2} & \cdots & z_2 & 1 \\ z_3^{m-1} & z_3^{m-2} & \cdots & z_3 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ z_m^{m-1} & z_m^{m-2} & \cdots & z_m & 1 \end{bmatrix}$$

and

$$C = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1m} \\ c_{21} & c_{22} & \cdots & c_{2m} \\ \cdots & \cdots & \cdots & \cdots \\ c_{m1} & c_{m2} & \cdots & c_{mm} \end{bmatrix} = V^{-1}. \qquad (3.7)$$

Then

(i) $c_{1j} \neq 0$ *and* $c_{mj} \neq 0$, *for each* $1 \leq j \leq m$.

(ii) *For each* $1 \leq r \leq m$, *the leading principal submatrix*

$$C_r = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1r} \\ c_{21} & c_{22} & \cdots & c_{2r} \\ \cdots & \cdots & \cdots & \cdots \\ c_{r1} & c_{r2} & \cdots & c_{rr} \end{bmatrix} \qquad (3.8)$$

is nonsingular.

(iii) *For each* $1 \leq r \leq m$, *the submatrix*

$$\begin{bmatrix} c_{rr} & c_{rr+1} & \cdots & c_{rm} \\ c_{r+1r} & c_{r+1r+1} & \cdots & c_{r+1m} \\ \cdots & \cdots & \cdots & \cdots \\ c_{mr} & c_{mr+1} & \cdots & c_{mm} \end{bmatrix}$$

is nonsingular.

Therefore, for each $1 \le j \le m$, *the number of zero entries in the jth row does not exceed* $\min\{j-1, m-j\}$.

Proof. (i) Clearly, for $1 \le j \le m$, we have

$$c_{1j} = (-1)^{j+1} \frac{|V|(z_1, \dots, z_{j-1}, z_{j+1}, \dots, z_m)}{|V|(z_1, \dots, z_m)}.$$

Thus $c_{1j} \neq 0$ since it is the ratio of nonzero Vandermondians. Similarly,

$$c_{mj} = \frac{|V|(z_1, \dots, z_{j-1}, z_{j+1}, \dots, z_m)}{|V|(z_1, \dots, z_m)} \prod_{i \neq j} z_i.$$

Hence $c_{mj} \neq 0$, since $z_i \neq 0$ for $i = 1, \dots, m$.

(ii) The proof is by contradiction. Assume that C_r is singular. Then there exists a nonzero vector $\xi \in \mathscr{C}^r$ such that $\xi^T C_r = 0$. Let C and V be partitioned so that

$$C = \begin{bmatrix} C_r & C_{12} \\ C_{21} & C_{22} \end{bmatrix} \quad \text{and} \quad V = \begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix}$$

with V_{11} being an $r \times r$ matrix. Since $CV = I$, it follows that

$$C_r V_{12} + C_{12} V_{22} = 0. \tag{3.9}$$

Premultiplying both sides of Eq. (3.9) by ξ^T yields $\xi^T C_{12} V_{22} = 0$. The nonsingularity of V_{22}, being the Vandermonde matrix $V(z_{r+1}, \dots, z_m)$, gives that $\xi^T C_{12} = 0$. Therefore, $\xi^T [C_r \; C_{12}] = 0$. This implies that the first rows of C are not linearly independent, which contradicts the nonsingularity of C.

The proof of (iii) is similar to that of (ii). To prove the last conclusion, assume for some j that $j - 1 \le m - j$ and the number of zeros entries in the jth row of C exeeds $j - 1$. Let P be a permutation matrix such that the first $j - 1$ entries of CP are zero. Since $(CP)^{-1} = P^{-1}V$ is a Vandermonde matrix, it follows that the $(j-1)$th leading submatrix is singular. This contradicts (ii). Similar argument holds if $m - j \le j - 1$. Q.E.D.

Remark 4. It should be noted that part (i) of the last theorem does not apply for $j \neq 1, m$. For example, if $m = 3$, $z_1 = 4$, $z_2 = 2$, and $z_3 = -2$, then $c_{21} = 0$.

The following result shows that for each $1 \le j \le m$, $b_{m-j}^{(n)}$ is a linear combination of the n-power of the $\{g(z_i)\}_{i=1}^m$, where $\{z_i\}_{i=1}^m$ are the zeros of $p(z)$.

LEMMA 9. *Let* $p(z) = \prod_{i=1}^{m}(z - z_i)$ *and let* $p_{m-1}^{(n)}(z) = b_{m-1}^{(n)} z^{m-1} + b_{m-2}^{(n)} z^{m-2} + \cdots + b_0^{(n)}$ *be as generated in Algorithm* 3.1. *Let* $C = [c_{ij}] = V(z_1, \ldots, z_m)^{-1}$, *then* $b_{m-i}^{(n)} = \sum_{j=1}^{m} c_{ij} g(z_j)^n$ *for* $i = 1, \ldots, m$.

Proof. The identity $g(z)^n - q_n(z)p(z) \equiv p_{m-1}^{(n)}(z) = \sum_{j=1}^{m} b_{m-j}^{(n)} z^{m-j}$ yields that $g(z_j)^n = p_{m-1}^{(n)}(z_j)$ for $j = 1, \ldots, m$. This leads to the system of equations

$$\begin{bmatrix} z_1^{m-1} & z_1^{m-2} & \cdots & z_1 & 1 \\ z_2^{m-1} & z_2^{m-2} & \cdots & z_2 & 1 \\ z_3^{m-1} & z_3^{m-2} & \cdots & z_3 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ z_m^{m-1} & z_m^{m-2} & \cdots & z_m & 1 \end{bmatrix} \begin{bmatrix} b_{m-1}^{(n)} \\ b_{m-2}^{(n)} \\ b_{m-3}^{(n)} \\ \vdots \\ b_0^{(n)} \end{bmatrix} = \begin{bmatrix} g(z_i)^n \\ g(z_2)^n \\ g(z_3)^n \\ \vdots \\ g(z_m)^n \end{bmatrix},$$

from which it follows that $b_{m-i}^{(n)} = \sum_{j=1}^{m} c_{ij} g(z_j)^n$. Q.E.D.

3.2. *Factorization of Polynomials*

In this section, we take up the general problem of numerical factorization of polynomials. This goal can be established by applying the results of the previous sections to sequences generated by Algorithm 3.1. Specifically, if $p(z)$ is a polynomial that has at least two zeros of different modulus, then the results of Section 3 can be applied to extract a polynomial factor having zeros which are zeros of $p(z)$ of largest modulus. In particular, Theorem 5 and Corollary 6 applied to the sequence $\{b_{m-j}^{(n)}\}_{n=m}^{\infty}$, yield the following.

THEOREM 10. *Let* $p(z) = \prod_{i=1}^{m}(z - z_i)$ *be a polynomial of degree m and let the* $b_{m-j}^{(n)}$*'s be as defined by Algorithm* 3.1. *Let* $\{z_j\}_{j=1}^{m}$ *be the zeros of* $p(z)$ *such that the* w_i*'s are nonzero, distinct and* $|w_1| \geq |w_2| \geq \cdots \geq |w_r| > |w_{r+1}| \geq |w_i|$ *for* $i = r + 2, \ldots, m$, *for some* $1 \leq r < m$. *Let l be the number of nonzero entries of the jth row of C, defined in Lemma* 8. *Then*

(i) *the matrix*

$$A_{m,j}^{(n)} = \begin{bmatrix} b_{m-j}^{(n)} & b_{m-j}^{(n+1)} & \cdots & b_{m-j}^{(n+m-1)} \\ b_{m-j}^{(n+1)} & b_{m-j}^{(n+2)} & \cdots & b_{m-j}^{(n+m)} \\ \cdots & \cdots & \cdots & \cdots \\ b_{m-j}^{(n+m-1)} & b_{m-j}^{(n+m)} & \cdots & b_{m-j}^{(n+2m-1)} \end{bmatrix}$$

is of rank l. In particular, $A_{m,j}^{(n)}$ *is full rank for* $j = 1, m$.

(ii) *For* $1 \le j \le m$, *assume that the jth row of C defined in* (3.7) *contains l nonzero entries, say* $c_{ji_1}, \dots, c_{ji_l}$, *and assume that there exists* $1 \le s \le l$ *such that* $|w_{i_1}| \ge |w_{i_2}| \ge \cdots \ge |w_{i_s}| > |w_{i_{s+1}}| \ge |w_{i_j}|$ *for* $j = s + 2, \dots, l$, *where* $1 \le s < l$. *Let* $\prod_{i=1}^{s}(z - w_{i_j}) = z^s + c_1 z^{s-1} + \cdots + c_s$, *then*

$$\lim_{n\to\infty} \frac{\begin{vmatrix} b_{m-j}^{(n)} & b_{m-j}^{(n+1)} & \cdots & b_{m-j}^{(n+s-1)} & 1 \\ b_{m-j}^{(n+1)} & b_{m-j}^{(n+2)} & \cdots & b_{m-j}^{(n+s)} & z \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ b_{m-j}^{(n+s-1)} & b_{m-j}^{(n+s)} & \cdots & b_{m-j}^{(n+2s-1)} & z^{s-1} \\ b_{m-j}^{(n+s)} & b_{m-j}^{(n+s+1)} & \cdots & b_{m-j}^{(n+2s-1)} & z^s \end{vmatrix}}{|A_{s,j}^{(n)}|} = \prod_{i=1}^{s} (z - w_{i_j}) \tag{3.10}$$

or equivalently

$$\lim_{n\to\infty} \begin{bmatrix} b_{m-j}^{(n)} & b_{m-j}^{(n+1)} & \cdots & b_{m-j}^{(n+s-1)} \\ b_{m-j}^{(n+1)} & b_{m-j}^{(n+2)} & \cdots & b_{m-j}^{(n+s)} \\ \cdots & \cdots & \cdots & \cdots \\ b_{m-j}^{(n+s-1)} & b_{m-j}^{(n+s)} & \cdots & b_{m-j}^{(n+2s-1)} \end{bmatrix}^{-1} \begin{bmatrix} b_{m-j}^{(n+s)} \\ b_{m-j}^{(n+s+1)} \\ \vdots \\ b_{m-j}^{(n+2s-1)} \end{bmatrix} = - \begin{bmatrix} c_s \\ c_{s-1} \\ \vdots \\ c_1 \end{bmatrix} \tag{3.11}$$

with $O(|w_{i_{s+1}}/w_{i_s}|^n)$ *order of convergence. Particularly, if* $j = 1, m$, *then* $l = m$ *and the indices* i_j *can be arranged so that* $i_j = j$ *for* $j = 1, \dots, m$.

Proof. Applying Lemma 1, it can easily be seen that rank $A_{m,j}^{(n)}$ is the number of nonzero elements of the jth row of C. For $j = 1, m$, we have $c_{1j} \ne 0$ and $c_{mj} \ne 0$. From Lemma 2, it follows that $|A_{m,j}^{(n)}| = \prod_{i=1}^{m} c_{ji} |V|(w_1, \dots, w_m) w_1^n \dots w_m^n \ne 0$. Thus the matrix $A_{m,j}^{(n)}$, $j = 1, m$, is nonsingular. The proofs of parts (i) and (ii) follow directly from Theorem 5. Q.E.D.

We remark that if for some j, $A_{m,j}^{(n)}$ is of rank $l < m$, then a factor of degree l can be readily computed exactly without the resort to the limiting process stated in the last theorem. In this case

$$\begin{bmatrix} b_{m-j}^{(n)} & b_{m-j}^{(n+1)} & \cdots & b_{m-j}^{(n+l-1)} \\ b_{m-j}^{(n+1)} & b_{m-j}^{(n+2)} & \cdots & b_{m-j}^{(n+l)} \\ \cdots & \cdots & \cdots & \cdots \\ b_{m-j}^{(n+l-1)} & b_{m-j}^{(n+l)} & \cdots & b_{m-j}^{(n+2l-1)} \end{bmatrix}^{-1} \begin{bmatrix} b_{m-j}^{(n+l)} \\ b_{m-j}^{(n+l+1)} \\ \vdots \\ b_{m-j}^{(n+2l-1)} \end{bmatrix} = - \begin{bmatrix} c_l \\ c_{l-1} \\ \vdots \\ c_1 \end{bmatrix},$$

where $\prod_{k\in S}(z-w_k) = z^l + c_1 z^{l-1} + \cdots + c_l$ and S is the set of positive integers $1 \le k \le m$ such that $c_{jk} \neq 0$.

In Theorem 10, a polynomial factor of $p(z)$ of order r is extracted using the sequence $b_{m-1}^{(n)}, \dots, b_{m-1}^{n+2r-1}$. In the next theorem a similar result can be obtained by considering the set $\{b_{m-j}^{(n)}, b_{m-j}^{(n+1)}, \dots, b_{m-j}^{(n+r)}\}$, for $j = 1, \dots, r$. To achieve this goal, we consider the matrix

$$B_r^{(n)} = \begin{bmatrix} b_{m-1}^{(n)} & b_{m-1}^{(n+1)} & \cdots & b_{m-1}^{(n+r-1)} \\ b_{m-1}^{(n)} & b_{m-2}^{(n+1)} & \cdots & b_{m-2}^{(n+r-1)} \\ \cdots & \cdots & \cdots & \cdots \\ b_{m-r}^{(n)} & b_{m-r}^{(n+1)} & \cdots & b_{m-r}^{(n+r-1)} \end{bmatrix}.$$

Note that $B_r^{(n)}$ is a transpose of a C-matrix (see Definition 2), and thus by applying Lemma 4, the determinant of $|B_r^{(n)}|$ can be expressed as

$$|B_r^{(n)}| = \sum_{(i_1,\dots,i_r)\in \pi_r^m} C_{i_1\dots i_r} w_{i_1}^n \dots w_{i_r}^n |V|(w_{i_1}, w_{i_2}, \dots, w_{i_r}).$$

The application of Theorem 5 to the factorization of polynomials is included in the following result.

THEOREM 11. *Let $p(z) = \prod_{j=1}^m (z - z_j)$ such that the w_i's are nonzero, distinct, and $|w_1| \ge |w_2| \ge \cdots \ge |w_r| > |w_{r+1} \ge |w_i|$ for $i = r+2, \dots, m$, where $1 \le r \le m$. Let $\{b_{m-j}^{(n)}\}_{n=m}^{\infty}$ be as generated in Algorithm 3.1. Let $\prod_{i=1}^r (z - w_i) = z^r + c_1 z^{r-1} + \cdots + c_r$. Then $|B_r^{(n)}| \neq 0$ for all sufficiently large n, and*

$$\lim_{n\to\infty} \frac{\begin{vmatrix} b_{m-1}^{(n)} & b_{m-1}^{(n+1)} & \cdots & b_{m-1}^{(n+r-1)} & b_{m-1}^{(n+r)} \\ b_{m-2}^{(n)} & b_{m-2}^{(n+1)} & \cdots & b_{m-2}^{(n+r-1)} & b_{m-2}^{(n+r)} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ b_{m-r}^{(n)} & b_{m-r}^{(n+1)} & \cdots & b_{m-r}^{(n+r-1)} & b_{m-r}^{(n+r)} \\ 1 & z & \cdots & z^{r-1} & z^r \end{vmatrix}}{|B_r^{(n)}|} = \prod_{j=1}^r (z - w_j) \quad (3.12)$$

or equivalently

$$\lim_{n\to\infty} \begin{bmatrix} b_{m-1}^{(n)} & b_{m-1}^{(n+1)} & \cdots & b_{m-1}^{(n+r-1)} \\ b_{m-2}^{(n)} & b_{m-2}^{(n+1)} & \cdots & b_{m-2}^{(n+r-1)} \\ \cdots & \cdots & \cdots & \cdots \\ b_{m-r}^{(n)} & b_{m-r}^{(n+1)} & \cdots & b_{m-r}^{(n+r-1)} \end{bmatrix}^{-1} \begin{bmatrix} b_{m-1}^{(n+r)} \\ b_{m-2}^{(n+r)} \\ \vdots \\ b_{m-r}^{(n+r)} \end{bmatrix} = -\begin{bmatrix} c_r \\ c_{r-1} \\ \vdots \\ c_1 \end{bmatrix} \quad (3.13)$$

with $O(|w_{r+1}/w_r|^n)$ order of convergence.

Proof. To show that $|B_r^{(n)}| \neq 0$, we have by virtue of Lemma 2 that under the stated assumptions the term $C_{1\ldots r} w_1^n \ldots w_r^n |V|(w_1 \ldots w_r)$ in the expansion of $|B_r^{(n)}|$ is dominant, since $C_{1\ldots r} \neq 0$ (Lemma 8). Thus the result (i) follows directly from Theorem 5. Q.E.D.

Although Eq. (3.12) seems to be only of theoretical interest, its equivalent form (3.13) can be implemented efficiently as in the Remark 1.

In Theorems 10 and 11 and if $g(z) = z$, we are able to approximate factors of $p(z)$ of maximal modulus. Next we consider a different method of extracting a factor of $p(z)$. To establish that, the following lemma is needed.

LEMMA 12. *Let $\{p_n(z)\}_{n=1}^{\infty}$ be a sequence of monic polynomials of degrees m and let $z_1, \ldots, z_m$ be distinct complex numbers. If for each $1 \leq j \leq m$, the sequence $\{p_n(z_j)\}_{n=1}^{\infty}$ converges to zero, then* $\lim_{n\to\infty} p_n(z) = \prod_{i=1}^{m}(z - z_i)$.

Proof. Let $p_n(z) = z^m + c_{m-1}^{(n)} z^{m-2} + \cdots + c_0^{(n)}$ and let $\prod_{i=2}^{m}(z - z_i) = z^m + c_{m-1} z^{m-1} + \cdots + c_0$. We will show that $\lim_{n\to\infty} c_i^{(n)} = c_i$ for $i = 1, \ldots, m-1$. Set $g_n(z) = p_n(z) - \prod_{i=1}^{m-1}(z - z_i)$, then $g_n(z_j) = p_n(z_j)$ converges to zero for $j = 1, \ldots, m$. Therefore

$$\begin{bmatrix} z_1^{m-1} & z_1^{m-2} & \cdots & z_1 & 1 \\ z_2^{m-1} & z_2^{m-2} & \cdots & z_2 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ z_m^{m-1} & z_m^{m-2} & \cdots & z_m & 1 \end{bmatrix} \begin{bmatrix} c_{m-1}^{(n)} - c_{m-1} \\ c_{m-2}^{(n)} - c_{m-2} \\ \vdots \\ c_0^{(n)} - c_0 \end{bmatrix} = \begin{bmatrix} p_n(z_1) \\ p_n(z_2) \\ \vdots \\ p_n(z_m) \end{bmatrix}.$$

Since all the z_j are distinct, the last system is uniquely solvable with $c_{m-i}^n - c_{m-i} = \sum_{j=1}^{m} c_{ij} p_n(z_j)$, where $[c_{ij}] = V(z_1, \ldots, z_m)^{-1}$. Hence $\lim_{n\to\infty} c_i^n = c_i$ for $i = 1, \ldots, m-1$. Q.E.D.

Assuming that $|w_1| \geq |w_2| \geq \cdots \geq |w_r| > |w_i|$ for $i = r+1, \ldots, m$, where $1 \leq r \leq m$, one can consider the polynomials $p_{m-1}^{(n)}(z), \ldots, p_{m-1}^{(n+r-1)}(z)$ to eliminate $z^{m-1}, \ldots, z^{m-r+1}$. Thus we obtain a polynomial $p_{m-r}^{(n)}(z)$ of degree $m - r$ which is shown to converge to a polynomial having $m - r$ zeros such that $\{w_i\}_{i=r+1}^{m}$ are of smallest modulus. The computation of such a polynomial is described in Algorithm 3.1. In the following theorem an approximation of factors of $p(z)$ of minimum modulus together with a convergence analysis is presented.

THEOREM 13. *Let $p(z) = \prod_{i=1}^{m}(z - z_i)$ and let $p_{m-r}^{(n)}$ be as generated in Algorithm 3.1. Let $\{z_j\}_{j=1}^{m}$ be the zeros of $p(z)$ such that the w_i's are nonzero, distinct, and $|w_1| \geq |w_2| \geq \cdots \geq |w_r| > |w_{r+1}| \geq |w_i|$ for $i = r+2, \ldots, m$, where $1 \leq r \leq m$. Then*

(i) $p_{m-r}^{(n)}$

$$= \frac{\sum_{j=r}^{m} \begin{vmatrix} b_{m-1}^{(n)} & b_{m-2}^{(n)} & \cdots & b_{m-r+1}^{(n)} & b_{m-j}^{(n)} \\ b_{m-1}^{(n+1)} & b_{m-2}^{(n+1)} & \cdots & b_{m-r+1}^{(n+1)} & b_{m-j}^{(n+1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ b_{m-1}^{(n+r-1)} & b_{m-2}^{(n+r-1)} & \cdots & b_{m-r+1}^{(n+r-1)} & b_{m-j}^{(n+r-1)} \end{vmatrix} z^{m-j}}{|B_r^{(n)}|}$$

(ii)
$$\lim_{n\to\infty} p_{m-r}^{(n)}(z) = \prod_{i=r+1}^{m} (z - z_i) \tag{3.14}$$

with $O(|w_{r+1}/w_r|^n)$ order of convergence.

(iii)
$$\lim_{n\to\infty} \frac{\begin{vmatrix} b_{m-1}^{(n)} & b_{m-2}^{(n)} & \cdots & b_{m-r+1}^{(n)} & q_n(z) \\ b_{m-1}^{(n+1)} & b_{m-2}^{(n+1)} & \cdots & b_{m-r+1}^{(n+1)} & q_{n+1-m}(z) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ b_{m-1}^{(n+r-1)} & b_{m-2}^{(n+r-1)} & \cdots & b_{m-r+1}^{(n+r-1)} & q_{n+r-1-m}(z) \end{vmatrix}}{|B_r^{(n)}|}$$
$$= -\frac{1}{\prod_{i=1}^{r}(z - z_i)} \tag{3.15}$$

with $O(|w_{r+1}/w_r|^n)$ order of convergence.

Proof. Consider the system of equations

$$b_{m-1}^{(n+i-1)} z^{m-1} + b_{m-2}^{(n+i-1)} z^{m-2} + \cdots + b_0^{(n+i-1)} = g(z)^{n+i-1} - q_{n+i-1-m}(z) p(z) \tag{3.16}$$

for $i = 1, 2, \ldots, r$. In view of Theorem 5, $|B_r^{(n)}| \neq 0$ for sufficiently large n. Therefore the matrix $B_r^{(n)}$ is full rank. Without loss of generality assume that the first $r - 1$ rows are linearly independent. Solving the first $r - 1$

equations of (3.16) for $z^{m-1}, \ldots, z^{m-r+1}$ yields

$$\begin{bmatrix} z^{m-1} \\ z^{m-2} \\ \vdots \\ z^{m-r+1} \end{bmatrix} = \begin{bmatrix} b_{m-1}^{(n)} & b_{m-2}^{(n)} & \cdots & b_{m-r+1}^{(n)} \\ b_{m-1}^{(n+1)} & b_{m-2}^{(n+1)} & \cdots & b_{m-r+1}^{(n+1)} \\ \cdots & \cdots & \cdots & \cdots \\ b_{m-1}^{(n+r-2)} & b_{m-2}^{(n+r-2)} & \cdots & b_{m-r+1}^{(n+r-2)} \end{bmatrix}^{-1}$$

$$\times \left\{ - \begin{bmatrix} b_{m-r}^{(n)} & b_{m-r-1}^{(n)} & \cdots & b_0^{(n)} \\ b_{m-r}^{(n+1)} & b_{m-r-1}^{(n+1)} & \cdots & b_0^{(n+1)} \\ \cdots & \cdots & \cdots & \cdots \\ b_{m-r}^{(n+r-2)} & b_{m-r-1}^{(n+r-2)} & \cdots & b_0^{(n+r-2)} \end{bmatrix} \begin{bmatrix} z^{m-r} \\ z^{m-r-1} \\ \vdots \\ z^0 \end{bmatrix} \right.$$

$$\left. + \begin{bmatrix} g(z)^n - q_n(z)p(z) \\ g(z)^{n+1} - q_{n+1}(z)p(z) \\ \vdots \\ g(z)^{n+r-2} - q_{n+r-2}(z)p(z) \end{bmatrix} \right\}. \quad (3.17)$$

Substituting this solution into the rth equation yields

$$\begin{bmatrix} b_{m-1}^{(n+r-1)} & b_{m-2}^{(n+r-1)} & \cdots & b_{m-r+1}^{(n+r-1)} \end{bmatrix}$$

$$\times \begin{bmatrix} b_{m-1}^{(n)} & b_{m-2}^{(n)} & \cdots & b_{m-r+1}^{(n)} \\ b_{m-1}^{(n+1)} & b_{m-2}^{(n+1)} & \cdots & b_{m-r+1}^{(n+1)} \\ \cdots & \cdots & \cdots & \cdots \\ b_{m-1}^{(n+r-2)} & b_{m-2}^{(n+r-2)} & \cdots & b_{m-r+1}^{(n+r-2)} \end{bmatrix}^{-1}$$

$$\times \left\{ - \begin{bmatrix} b_{m-r}^{(n)} & b_{m-r-1}^{(n)} & \cdots & b_0^{(n)} \\ b_{m-r}^{(n+1)} & b_{m-r-1}^{(n+1)} & \cdots & b_0^{(n+1)} \\ \cdots & \cdots & \cdots & \cdots \\ b_{m-r}^{(n+r-2)} & b_{m-r-1}^{(n+r-2)} & \cdots & b_0^{(n+r-2)} \end{bmatrix} \begin{bmatrix} z^{m-r} \\ z^{m-r-1} \\ \vdots \\ z^0 \end{bmatrix} \right.$$

$$\left. + \begin{bmatrix} g(z)^n - q_n(z)p(z) \\ g(z)^{n+1} - q_{n+1}(z)p(z) \\ \vdots \\ g(z)^{n+r-2} - q_{n+r-2}(z)p(z) \end{bmatrix} \right\}$$

$$+ b_{m-r}^{(n+r-1)} z^{m-r} + b_{m-r-1}^{(n+r-1)} z^{m-r-1} + \cdots + b_0^{(n+r-1)}$$

$$= g(z)^{n+r-1} - q_{n+r-1} p(z), \quad (3.18)$$

which can be written in the form

$$
-\sum_{j=r}^{m}\left[b_{m-1}^{(n+r-1)} \;\cdots\; b_{m-r+1}^{(n+r-1)}\right]\begin{bmatrix} b_{m-1}^{(n)} & b_{m-2}^{(n)} & \cdots & b_{m-r+1}^{(n)} \\ b_{m-1}^{(n+1)} & b_{m-2}^{(n+1)} & \cdots & b_{m-r+1}^{(n+1)} \\ \cdots & \cdots & \cdots & \cdots \\ b_{m-1}^{(n+r-2)} & b_{m-2}^{(n+r-2)} & \cdots & b_{m-r+1}^{(n+r-2)} \end{bmatrix}^{-1}
$$

$$
\times\begin{bmatrix} b_{m-j}^{(n)} \\ b_{m-j}^{(n+1)} \\ \vdots \\ b_{m-j}^{(n+r-1)} \end{bmatrix} z^{m-j} + \sum_{j=r}^{m} b_{m-j}^{(n+r-1)} z^{m-j}
$$

$$
+\left[b_{m-1}^{(n+r-1)} \;\cdots\; b_{m-r+1}^{(n+r-1)}\right]\begin{bmatrix} b_{m-1}^{(n)} & b_{m-2}^{(n)} & \cdots & b_{m-r+1}^{(n)} \\ b_{m-1}^{(n+1)} & b_{m-2}^{(n+1)} & \cdots & b_{m-r+1}^{(n+1)} \\ \cdots & \cdots & \cdots & \cdots \\ b_{m-1}^{(n+r-2)} & b_{m-2}^{(n+r-2)} & \cdots & b_{m-r+1}^{(n+r-2)} \end{bmatrix}^{-1}
$$

$$
\times\begin{bmatrix} g(z)^{n} - q_n(z)p(z) \\ g(z)^{n+1} - q_{n+1}(z)p(z) \\ \vdots \\ g(z)^{n+r-2} - q_{n+r-2}(z)p(z) \end{bmatrix} = g(z)^{n+r-1} - q_{n+r-1}(z)p(z). \tag{3.19}
$$

Note that if A is nonsingular, then $\begin{vmatrix} A & b \\ c & d \end{vmatrix} = |A|\{d - cA^{-1}b\}$. Thus multiplying both sides of (3.19) by $|B_{r-1}^{(n)}|$ yields that

$$
\sum_{j=r}^{m}\begin{vmatrix} b_{m-1}^{(n)} & \cdots & b_{m-r+1}^{(n)} & b_{m-j}^{(n)} \\ b_{m-1}^{(n+1)} & \cdots & b_{m-r+1}^{(n+1)} & b_{m-j}^{(n+1)} \\ \vdots & \vdots & \vdots & \vdots \\ b_{m-1}^{(n+r-1)} & \cdots & b_{m-r+1}^{(n+r-1)} & b_{m-j}^{(n+r-1)} \end{vmatrix} z^{m-j}
$$

$$
=\begin{vmatrix} b_{m-1}^{(n)} & \cdots & b_{m-r+1}^{(n)} & g(z)^{n} - q_n(z)p(z) \\ b_{m-1}^{(n+1)} & \cdots & b_{m-r+1}^{(n+1)} & g(z)^{n+1} - q_{n+1}(z)p(z) \\ \vdots & \vdots & \vdots & \vdots \\ b_{m-1}^{(n+r-1)} & \cdots & b_{m-r+1}^{(n+r-1)} & g(z)^{n+r-1} \end{vmatrix}
$$

$$= \begin{vmatrix} b_{m-1}^{(n)} & \cdots & b_{m-r+1}^{(n)} & g(z)^n \\ b_{m-1}^{(n+1)} & \cdots & b_{m-r+1}^{(n+1)} & g(z)^{n+1} \\ \vdots & \vdots & \vdots & \vdots \\ b_{m-1}^{(n+r-1)} & \cdots & b_{m-r+1}^{(n+r-1)} & g(z)^{n+r-1} \end{vmatrix}$$

$$- p(z) \begin{vmatrix} b_{m-1}^{(n)} & \cdots & b_{m-r+1}^{(n)} & q_n(z) \\ b_{m-1}^{(n+1)} & \cdots & b_{m-r+1}^{(n+1)} & q_{n+1}(z) \\ \vdots & \vdots & \vdots & \vdots \\ b_{m-1}^{(n+r-1)} & \cdots & b_{m-r+1}^{(n+r-1)} & q_{n+r-1}(z) \end{vmatrix}. \tag{3.20}$$

Next, for each positive integer $n \geq m$, define

$$p_{m-r}^{(n)}(z) = \frac{\sum_{j=r}^{m} \begin{vmatrix} b_{m-1}^{(n)} & b_{m-2}^{(n)} & \cdots & b_{m-r+1}^{(n)} & b_{m-j}^{(n)} \\ b_{m-1}^{(n+1)} & b_{m-2}^{(n+1)} & \cdots & b_{m-r+1}^{(n+1)} & b_{m-j}^{(n+1)} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ b_{m-1}^{(n+r-1)} & b_{m-2}^{(n+r-1)} & \cdots & b_{m-r+1}^{(n+r-1)} & b_{m-j}^{(n+r-1)} \end{vmatrix} z^{m-j}}{|B_r^{(n)}|}.$$

Then $p_{m-r}^{(n)}(z)$ is a monic polynomial of degree $m - r$. The identity (3.20) implies that for each $r + 1 \leq j \leq m$,

$$p_{m-r}^{(n)}(z_j) = \frac{\begin{vmatrix} b_{m-1}^{(n)} & b_{m-2}^{(n)} & \cdots & b_{m-r+1}^{(n)} & w_j^n \\ b_{m-1}^{(n+1)} & b_{m-2}^{(n+1)} & \cdots & b_{m-r+1}^{(n+1)} & w_j^{n+1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ b_{m-1}^{(n+r-1)} & b_{m-2}^{(n+r-1)} & \cdots & b_{m-r+1}^{(n+r-1)} & w_j^{n+r-1} \end{vmatrix}}{|B_r^{(n)}|}.$$

Therefore it follows from the proof of Theorem 5 that for all sufficiently large n, $p_n(z_j) = O(w_j^n / w_r^n)$ for $j = r + 1, \ldots, m$. Thus by Lemma 12, $\lim_{n \to \infty} p_n(z) = \prod_{i=r+1}^{m}(z - z_i)$. The conclusion (ii) follows by dividing both sides of (3.20) by $p(z)B_r^{(n)}$ and noting that

$$\lim_{n \to \infty} \frac{p_{m-r}^{(n)}(z)}{p(z) B_r^{(n)}} = \frac{1}{\prod_{i=1}^{r}(z - z_i)}. \qquad \text{Q.E.D.}$$

It should be noted that the $a_{m-j}^{(n)}$'s in Algorithm 3.1 for a polynomial of degree $m - j$ are equal to the $B_r^{(n)}$ of the last theorem which under the stated assumptions are nonzero for all sufficiently large n.

The following corollary is important when all the w_i's have different modulus.

COROLLARY 14. *Let* $p(z) = \prod_{i=1}^{m}(z - z_i)$ *be a polynomial of simple zeros. Let* $p_{m-1}^{(n)}(z)$, $q_n(z)$, *and* $b_{m-j}^{(n)}$ *be as generated in Algorithm* 3.1. *Assume that* $p(z)$ *has a zero,* z_1, *so that* $|w_1| > |w_2| \geq |w_j|$ *for* $j = 3, \ldots, m$. *Assume also that* $w_i \neq 0$ *for* $j = 1, \ldots, m$ *and that the* w_i*'s are pairwise distinct. Then*

(i) *For all sufficiently large* n, $b_{m-1}^{(n)} \neq 0$ *and* $b_0^{(n)} \neq 0$.

(ii) $\lim_{n\to\infty}(b_{m-1}^{(n+1)}/b_{m-1}^{(n)}) = \lim_{n\to\infty}(b_0^{(n+1)}/b_0^{(n)}) = w_1$ *with* $O(|w_2/w_1|^n)$ *order of convergence.*

(iii) $\lim_{n\to\infty}(p_{m-1}^{(n)}(z)/b_{m-1}^{(n)}) = \prod_{i=2}^{m}(z - z_i)$ *and* $\lim_{n\to\infty}(p_{m-1}^{(n)}(z)/b_0^{(n)}) = \prod_{i=2}^{m}(1 - z/z_i)$.

(iv) $\lim_{n\to\infty}(q_n(z)/b_{m-1}^{(n)}) = 1/(z_1 - z)$.

Proof. (i) follows directly from Theorem 11. To show (ii), let $[c_{ij}] = V(z_1, \ldots, z_m)^{-1}$. Then $b_{m-1}^{(n)} = \sum_{j=1}^{m} c_{1j} w_j^n$ and $b_0^n = \sum_{j=1}^{m} c_{mj} w_j^n$. Since $c_{11} \neq 0$ and $c_{m1} \neq 0$ (Lemma 8), it follows that for $r = 1, m$,

$$\lim_{n\to\infty} \frac{b_{m-r}^{(n+1)}}{b_{m-r}^{(n)}} = \lim_{n\to\infty} \frac{\sum_{j=1}^{m} c_{rj} w_j^{n+1}}{\sum_{j=1}^{m} c_{rj} w_j^n} = \lim_{n\to\infty} \frac{\sum_{j=1}^{m} c_{rj} w_j \left(\dfrac{w_j}{w_1}\right)^n}{\sum_{j=1}^{m} c_{rj} \left(\dfrac{w_j}{w_1}\right)^n} = w_1$$

since $|w_j|/w_1 < 1$ for $j = 2, \ldots, m$.

(iii) For sufficiently large n, $b_{m-1}^{(n)} \neq 0$ and $b_0^n \neq 0$. Thus the polynomial $p_n(z) = p_{m-1}^{(n)}(z)/b_{m-1}^{(n)}$ is well defined. It is easily established that $p_n(z_j) = w_j^n/b_{m-1}^{(n)}$ for $j = 2, \ldots, m$. Since $|w_1| > |w_j|$ for $j = 2, \ldots, m$ and $c_{11} \neq 0$, it follows that $\lim_{n\to\infty} p_n(z_j) = 0$ for $j = 2, \ldots, m$. The conclusion follows from Lemma 12. A similar argument shows that $\lim_{n\to\infty} p_{m-1}^{(n)}(z)/b_0^n = \prod_{i=2}^{m}(1 - z/z_i)$.

(iv) Clearly, $\lim_{n\to\infty} p_{m-1}^{(n)}(z)/p(z)b_{m-1}^{(n)} = 1/(z - z_1)$. The identity $g(z)^n/p(z)b_{m-1}^{(n)} = q_n(z)/b_{m-1}^{(n)} + p_{m-1}^{(n)}(z)/p(z)b_{m-1}^{(n)}$ holds for all $z \neq z_i$, $i = 1, \ldots, m$. There exists a positive number S and $0 < \rho_1, \rho_2 < 1$ such that for each complex number z inside the annulus $\rho_2|w_2| \leq |w| \leq \rho_1 |w_1|$, the inequality $|p_{m-1}^{(n)}(z)/p(z)b_{m-1}^{(n)} + q_n(z)/b_{m-1}^{(n)}| = |w^n/p(z)b_{m-1}^{(n)}| < S\rho_1^n$ holds for large n. The conclusion follows directly from the observation that $p_{m-1}^{(n)}(z)/p(z)b_{m-1}^{(n)} + q_n(z)/b_{m-1}^{(n)}$ converges to zero uniformly in the above annulus. Q.E.D.

The next result deals with the case where $p(z)$ has two roots such that two of the w_i's have equal modulus (e.g., complex conjugate zeros of real polynomials).

COROLLARY 15. *Let $p(z)$, $p_{m-1}^{(n)}(z)$, $q_n(z)$, and $\{z_j\}_{j=1}^m$ be as in Corollary 14 with $m > 2$ so that the w_i's are nonzero and pairwise distinct. Assume that $|w_1| = |w_2| > |w_3| \geq |w_j|$ for $1 = 4, \ldots, m$ and let $(z - w_1)(z - w_2) = z^2 + c_1 z + c_2$. Define $p_n(z) = c_{m-2}^{(n)} z^{m-2} + c_{m-3}^{(n)} z^{m-3} + \cdots + c_0^{(n)}$, where*

$$c_{m-j}^{(n)} = \begin{vmatrix} b_{m-1}^{(n)} & b_{m-j}^{(n)} \\ b_{m-1}^{(n+1)} & b_{m-j}^{(n+1)} \end{vmatrix}.$$

Then

(i) $c_{m-2}^{(n)} \neq 0$ *for all sufficiently large n.*

(ii) $\lim_{n\to\infty}(p_n(z)/c_{m-2}^n) = \prod_{j=3}^m (z - z_j)$.

(iii)
$$\lim_{n\to\infty} \frac{\begin{vmatrix} q_n(z) & q_{n+1}(z) \\ b_{m-1}^{(n)} & b_{m-1}^{(n+1)} \end{vmatrix}}{c_{m-2}^{(n)}} = \frac{1}{(z - z_1)(z - z_2)}.$$

(iv)
$$c_1 = \lim_{n\to\infty} \frac{\left(b_{m-1}^{(n+2)} b_{m-1}^{(n+1)} - b_{m-1}^{(n+3)} b_{m-1}^{(n)}\right)}{\left(b_{m-1}^{(n+2)} b_{m-1}^{(n)} - \left(b_{m-1}^{(n+1)}\right)^2\right)}$$

and

$$c_2 = \lim_{n\to\infty} \frac{b_{m-1}^{(n+3)} b_{m-1}^{(n+1)} - \left(b_{m-1}^{(n+2)2}\right)}{b_{m-1}^{(n+2)} b_{m-1}^{(n)} - \left(b_{m-1}^{(n+1)}\right)^2}$$

with $O(|w_3/w_1|^n)$ order of convergence.

Proof. Since $c_{m-2}^{(n)} = |B_2^{(n)}|$, (i) follows directly from Theorem 13. (ii) is a direct application of Theorem 13. To prove (iii), define

$$\begin{aligned} p_n(z) &= p_{m-1}^{(n)}(z) b_{m-1}^{(n+1)} - p_{m-1}^{(n+1)}(z) b_{m-1}^{(n)} \\ &= c_{m-2}^{(n)} z^{m-2} + c_{m-3}^{(n)} z^{m-3} + \cdots + c_0^{(n)}, \end{aligned}$$

where

$$c_{m-j}^{(n)} = \begin{vmatrix} b_{m-1}^{(n)} & b_{m-j}^{(n)} \\ b_{m-1}^{(n+1)} & b_{m-j}^{(n+1)} \end{vmatrix}.$$

Dividing the identity

$$p_n(z) \equiv \left\{q_n(z)b_{m-1}^{(n+1)} - q_{n+1}(z)b_{m-1}^{(n)}\right\}p(z) - g(z)^n\left(b_{m-1}^{(n+1)} - g(z)b_{m-1}^{(n)}\right)$$

by $p(z)c_{m-2}^n$ yields

$$\frac{p_n(z)}{p(z)c_{m-2}^{(n)}} - \frac{\begin{vmatrix} q_n(z) & q_{n+1}(z) \\ b_{m-1}^{(n)} & b_{m-1}^{(n+1)} \end{vmatrix}}{c_{m-2}^{(n)}} = -\frac{g(z)^n\left(b_{m-1}^{(n+1)} - g(z)b_{m-1}^{(n)}\right)}{p(z)c_{m-2}^{(n)}}$$

which holds for all $z \neq z_i$, $i = 1, \ldots, m$, and large n. There exists a positive number S and $0 < \rho_1, \rho_2 < 1$ such that

$$\left|\frac{g(z)^n\left(b_{m-1}^{(n+1)} - g(z)b_{m-1}^{(n)}\right)}{p(z)c_{m-2}^{(n)}}\right| \leq S\rho_1^n$$

for all z in the annulus $\rho_2|w_3| \leq |w| \leq \rho_1|w_1|$. Hence

$$\left|\frac{p_n(z)}{p(z)c_{m-2}^{(n)}} - \frac{q_n(z)b_{m-1}^{(n+1)} - q_{n+1}(z)b_{m-1}^{(n)}}{c_{m-2}^{(n)}}\right| \leq S\rho_1^n.$$

This implies that

$$\frac{p_n(z)}{p(z)c_{m-2}^{(n)}} - \frac{q_n(z)b_{m-1}^{(n+1)} - q_{n+1}(z)b_{m-1}^{(n)}}{c_{m-2}^{(n)}}$$

converges to zero uniformly in the above annulus.

From Theorem 13, $\lim_{n\to\infty}(p_n(z)/p(z)c_{m-2}^{(n)}) = 1/(z - z_1)(z - z_2)$. Therefore

$$\lim_{n\to\infty} \frac{q_n(z)b_{m-1}^{(n+1)} - q_{n+1}(z)b_{m-1}^{(n)}}{c_{m-2}^{(n)}} = \frac{1}{(z - z_1)(z - z_2)}$$

for all z in the annulus $\rho_2|w_3| \leq |w| \leq \rho_1|w_1|$.

Note that

$$\lim_{n\to\infty} \frac{q_n(z)b_{m-1}^{(n+1)} - q_{n+1}(z)b_{m-1}^{(n)}}{c_{m-2}^{(n)}}$$

$$= \lim_{n\to\infty} \frac{\sum_{j=1}^{\infty} \begin{vmatrix} b_{m-1}^{(n-j)} & b_{m-1}^{(n-j+1)} \\ b_{m-1}^{(n)} & b_{m-1}^{(n+1)} \end{vmatrix} z^{j-1}}{B_2^{(n)}} = \frac{1}{(z - z_1)(z - z_2)}.$$

The proof of (iv) follows directly from Theorem 13(ii). Q.E.D.

Remark 5 (On Multiple Zeros). By applying an argument similar to that of [11], it can be shown that the results of this work hold true even when some zeros of $p(z)$ are not simple. The proof is more complicated, but it can be done by a continuity argument. In this case, the polynomial $p(z)$ is approximated by a polynomial $f(z, \epsilon)$ of same degree having simple zeros and such that $\lim_{\epsilon \to 0} f(z, \epsilon) = p(z)$ uniformly in any bounded region containing the zeros of $p(z)$. Then the results of this section are to be applied to the polynomial f which by letting $\epsilon \to 0$ reduces to results concerning the polynomial $p(z)$. It should be noted that the convergence is not any longer geometric as shown in the following example.

Let $p(z) = (z-1)^3$. It can be shown that $b_0^{(n)} = n(n-1)$. We will apply Theorems 5, 11, and 13 to recover linear and quadratic factors of $p(z)$. Since the sequence $b_0^{(n+1)}/b_0^{(n)} = n(n+1)/n(n-1)$ converges to one, the factor $z-1$ is obtained. Note that $|b_0^{(n+1)}/b_0^{(n)}) - 1| = |n(n+1)/n(n-1) - 1| = 2/(n+1)$. Applying Corollary 15 to recover the quadratic factor $z^2 - 2z + 1$ yields $c_2 = \lim_{n\to\infty}(n+2)/n = 1$ and $c_1 = \lim_{n\to\infty} -2(n+2)/(n+1) = -2$. Note that $|(n+2)/n - c_2| = |(n+2)/n - 1| = 2/n$ and $|-2(n+2)/(n+1) - c_1| = |-2(n+2)/(n+1) + 2| = 2/(n+1)|$. This implies that the order of convergence is $O(1/n)$ in both cases.

To illustrate the various methods proposed in previous sections we present the following simple example.

EXAMPLE. For comparison purposes we consider a polynomial with known factors. Let $p(z) = (z^2-9)(z-1) = z^3 - z^2 - 9z + 9$. Algorithm 3.1 is next applied with $g(z) = z$ to generate the first 10 terms of each of the sequences $\{b_0^{(n)}\}$, $\{b_1^{(n)}\}$, and $\{b_2^{(n)}\}$, i.e.,

$$\{b_0^{(n)}\}_{n=3}^{10} = \{-9, -9, -90, -90, -819, -819, -7380, -7380\}$$

$$\{b_1^{(n)}\}_{n=3}^{10} = \{9, 0, 81, 0, 729, 0, 6561, 0\}$$

$$\{b_2^{(n)}\}_{n=3}^{10} = \{1, 10, 10, 91, 91, 820, 820, 7381\}.$$

Applying Theorem 10 to the sequence $\{b_0^{(n)}\}_{n=3}^{10}$ yields

$$\begin{bmatrix} b_2^{(n)} & b_2^{(n+1)} \\ b_2^{(n+1)} & b_2^{(n+2)} \end{bmatrix}^{-1} \begin{bmatrix} b_2^{(n+2)} \\ b_2^{(n+3)} \end{bmatrix} = \begin{bmatrix} 91 & 820 \\ 820 & 820 \end{bmatrix}^{-1} \begin{bmatrix} 820 \\ 7381 \end{bmatrix} = \begin{bmatrix} 9.0 \\ 0.0012195 \end{bmatrix}.$$

Thus $c_2 = -9$ and $c_1 = -0.0012195$. Similarly, when Theorem 10 is applied to the sequence $\{b_2^{(n)}\}_{n=3}^{10}$ we get

$$\begin{bmatrix} b_0^{(n)} & b_0^{(n+1)} \\ b_0^{(n+1)} & b_0^{(n+2)} \end{bmatrix}^{-1} \begin{bmatrix} b_0^{(n+2)} \\ b_0^{(n+3)} \end{bmatrix} = \begin{bmatrix} -819 & -819 \\ -819 & -7380 \end{bmatrix}^{-1} \begin{bmatrix} -7380 \\ -7380 \end{bmatrix}$$
$$= \begin{bmatrix} 9.01099 \\ 0.0 \end{bmatrix},$$

i.e., $c_2 = -9.01099$ and $c_1 = 0.0$. It should be observed that when Theorem 10 is applied to the sequence $b_1^{(n)}$ we obtain the exact factor $z^2 - 9$ for each n since

$$\begin{bmatrix} b_1^{(n)} & b_1^{(n+1)} \\ b_1^{(n+1)} & b_1^{(n+2)} \end{bmatrix}^{-1} \begin{bmatrix} b_1^{(n+2)} \\ b_1^{(n+3)} \end{bmatrix} = \begin{bmatrix} 9 \\ 0 \end{bmatrix}.$$

Applying Theorem 13 yields

$$\begin{bmatrix} b_2^{(n)} & b_2^{(n+1)} \\ b_1^{(n)} & b_1^{(n+1)} \end{bmatrix}^{-1} \begin{bmatrix} b_2^{(n+2)} \\ b_1^{(n+2)} \end{bmatrix} = \begin{bmatrix} 820 & 820 \\ 0 & 6561 \end{bmatrix}^{-1} \begin{bmatrix} 77381 \\ 0 \end{bmatrix} = \begin{bmatrix} 9.0012195 \\ 0.0 \end{bmatrix}.$$

To extract a linear factor, Theorem 13 is to be applied so that

$$\frac{\begin{vmatrix} 820 & 7381 \\ 6561 & 0 \end{vmatrix} z + \begin{vmatrix} 820 & 7381 \\ -7380 & -7380 \end{vmatrix}}{\begin{vmatrix} 820 & 7381 \\ 6561 & 0 \end{vmatrix}} = z - 0.9998645$$

which is an approximation to the factor $z - 1$.

Remark 6. It should be clear that there is a real possibility of overflow which can be handled by scaling the zeros of $p(z)$ by a suitable factor $\alpha > 0$ and then applying the results of this section to the polynomial $p(\alpha z)$. The other concern is that arising systems such as 3.11 and 3.13 become ill-conditioned for large n. Thus there is a need for a robust linear system solver. A treatment of efficient solutions of the Hankel system of equations can be found in [8].

Remark 7. When the roots of the polynomial $p(z)$ are all simple, it is always possible to find a complex number z_0 such that the polynomial $p(z - z_0)$ has simple zeros with different modulus. Therefore, one can apply Corollary 14 to the polynomial $p(z - z_0)$ using complex arithmetic to determine all zeros of $p(z)$. If $p(z)$ is a real polynomial and real arithmetic has to be used, then one can determine a real number z_0 such that $p(z - z_0)$ has zeros of distinct modulus except for complex conjugate zeros. In this case, Corollary 15 can be applied to determine all zeros of $p(z)$.

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REFERENCES

1. D. Bini and L. Gemignani, On the complexity of polynomial zeros, *SIAM J. Comput.* **21** (1992), 781–799.
2. G. H. Golub and C. G. Van Loan, "Matrix Computations," 2nd ed., The John Hopkins University Press, Baltimore, 1989.
3. G. Heinig and P. Jankowski, Parallel and superfast algorithms for Hankel systems of equations, *Numer. Math.* **58** (1990), 109–127.
4. P. Henrici, "Applied and Computational Complex Analysis," Vol. 1, Wiley, New York, 1974.
5. A. S. Householder, "The Numerical Treatment of a Single Nonlinear Equation," McGraw–Hill, New York, 1970.
6. A. S. Householder, Generalization of an algorithm of Sebastiao e Silva, *Numer. Math.* **16** (1971), 375–382.
7. M. Marden, "Geometry of Polynomials," Mathematical Surveys, Vol. 3, Amer. Math. Soc., Providence, RI, 1966.
8. C. Moler and C. G. Van Loan, Nineteen dubious ways to compute the exponential of a matrix, *SIAM Rev.* **20** (1978), 801–836.
9. A. Ostrowski, "Solution of Equations and Systems of Equations," 2nd ed., Academic Press, New York, 1966.
10. J. Sebastiao é Silva, Sur une methode d'approximation semblable a celle de Graeffe, *Portugal Math.* **2** (1941), 271–279.
11. G. W. Stewart, On the convergence of Sebastiao e Silva method for finding a zero of a polynomial, *SIAM Rev.* **12** (1970), 458–460.