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Maximum Principle for Second Order Elliptic Equations and Applications*

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INTRODUCTION 1

The great usefulness of a priori bounds for solutions of (second order elliptic) partial differential equations is well known [1, 7, 9]. Maximum principles are particular a priori bounds. We are interested in maximum principles for second order elliptic equations in nondivergence form when the coefficients of the leading terms are discontinuous [4, 6, 11].

In particular if Ω is a domain (bounded connected open set) of R^n we consider the Dirichlet problem

$$\begin{gathered} \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} = f \\ u|_{\partial\Omega} = 0, \end{gathered} \tag{1.1}$$

where the following conditions are fulfilled by the coefficients and the known term,

$$\begin{aligned} &a_{ij}(x) \in L^{\infty}(\Omega), \qquad \lambda |\xi|^2 \leqslant \sum_{i,j=1}^{n} a_{ij}(x)\, \xi_i \xi_j \leqslant \Lambda |\xi|^2 \\ &a_{ij}(n) = a_{ji}(n) \qquad f \in L^2(\Omega) \end{aligned} \tag{1.2}$$

with λ, Λ positive constants. We look for $u \in W^{2,n}(\Omega) \cap C^0(\bar{\Omega})$ ($\equiv W^{2,n}(\Omega)$ if Ω is regular) and (1.1) is verified a.e.

A current classical result of Alexandrov and Pucci [1, 7, 8] states that

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THEOREM 1.1. *If $f \in L^n(\Omega)$, then for any solution of* (1.1) *the following bound* (*maximum principle*) *holds,*

$$\|u\|_{L^\infty(\Omega)} \leqslant K \|f\|_{L^n(\Omega)}, \tag{1.3}$$

where K is a positive constant depending only on λ, Λ, Ω, n.

In Theorem 1.1, $f \in L^n(\Omega)$ where n is the dimension of the space is fundamental. Nevertheless in many cases obtaining bounds like (1.3) with some power p less than n can be useful.

In this paper we construct a physical case in which the hypothesis $p < n$ is interesting. Then we state a result like Theorem 1.1 for $p < n$ [2, 4] and prove a bound for this p (when $\lambda \to 0^+$).

SECTION 2

Let $n = 2$. We derive the equation for the displacements of a structure covering Ω and clamped on $\partial\Omega$. The structure is a net of steel cables with rectangular mesh with edges parallel to the coordinate axes. The draught of the net can be realized with turnbuckles or with weights and is assigned. The cables parallel to the y axis are stretched by a draught $a_1 \equiv a_1(x)$ that we can suppose is defined in the first projection of Ω, Ω_1 and belongs to $L^\infty(\Omega_1)$, in general continuous a.e. or a step function. Analogously the cables parallel to the x axis are stretched by a draught $a_2 \equiv a_2(y)$ belonging to $L^\infty(\Omega_2)$, where Ω_2 is the second projection.

The structure must support a load $f_1(x, y)$ given by the sum of the dead load and of an accidental load like wind pressure or weight of snow and a load $f_2(x, y)$ given by some assigned, distributed, or concentrated loads.

We emphasize here that in the case of concentrated loads it is very natural to expect $f_2(x, y)$ like $1/(d(P, \bar{P}))^\sigma$ where $\bar{P}(\bar{x}, \bar{y})$ is a fixed point in $\bar{\Omega}$, $d(P, \bar{P})$ is the euclidean distance from $P(x, y)$ to $\bar{P}$, and σ is a positive constant depending on the load. In this case, easily arising from the applications, $f(x, y) = f_1(x, y) + f_2(x, y)$ does not belong to $L^n(\Omega)$ for any $\sigma \geqslant 1$.

We call $u(x, y)$ the displacements along the z axis orthogonal to the plane of Ω. The equation that we derive expresses the balance in the vertical translation of the element $dx\,dy$ of Ω.

We consider displacements belonging to $C^2(\Omega) \cap C^0(\bar{\Omega})$ and call C_x the intersection of the graph of u restricted to $dx\,dy$ with the plane y, z. If dx is small we can suppose that the angle α formed by the tangent in dx to C_x and the y axis is constant. Then the contribution of $a_1(x)$ to the balance is

$$da_1 = -a_1 \sin\alpha\, dx + a_1 \sin\left(\alpha + \frac{d\alpha}{dy}\,dy\right) dx.$$

We use the "small displacements hypothesis" [5] and then we can replace α with $\sin\alpha$ or with $t_{g\alpha}$, from which

$$da_1 = -a_1 \alpha\, dx + a_1(\alpha + d\alpha)\, dx = a_1 \frac{d\alpha}{dy}\, dy\, dx.$$

Otherwise $\operatorname{tg}\alpha = \partial u/\partial y$, and then

$$da_1 = a_1 \frac{\partial^2 u}{\partial y^2}\, dy\, dx. \tag{2.1}$$

Analogously, on the edges parallel to the y axis the contribution of $a_2(y)$ to the balance is

$$da_2 = a_2 \frac{\partial^2 u}{\partial x^2}\, dx\, dy. \tag{2.2}$$

To obtain the balance, the sum of $f(x, y)\, dx\, dy$, (2.1), and (2.2) must be zero. We deduce

$$f\, dx\, dy + a_1 \frac{\partial^2 u}{\partial y^2}\, dy\, dx + a_2 \frac{\partial^2 u}{\partial x^2}\, dx\, dy = 0$$

and then

$$a_2(y) \frac{\partial^2 u}{\partial x^2} + a_1(x) \frac{\partial^2 u}{\partial y^2} = f(x, y). \tag{2.3}$$

Equation (2.3) is of type (1.1) but $f(x, y)$ in many cases does not belong to $L^n(\Omega)$, as we have pointed out. Then we are interested in the case in which (1.3) holds with some power less than n in the right member.

SECTION 3

In this section we deduce some results concerning the estimate (in L' norm) of the displacements of a structure when the load is a function f not belonging to $L^n(\Omega)$, e.g., the inverse of a distance from an assigned point of $\bar{\Omega}$. We return to problem (1.1) with hypotheses (1.2). In a first step we suppose

$$a_{ij}(x) \in C^\infty(\Omega); \qquad \sum_{i=1}^{n} a_{ii}(x) = 1.$$

Related to the operator $Lu = \sum_{i,j=1}^{n} a_{ij}(x)(\partial^2 u/\partial x_i\, \partial x_j)$, we consider its adjoint operator $L^*v = \sum_{i,j=1}^{n} (\partial^2/\partial x_i\, \partial x_j)(a_{ij} v)$.

A function v belonging to $L^1_{\mathrm{loc}}(\Omega)$ is a nonnegative weak solution of $L^*v=0$ if $v \geqslant 0$ and $\int_\Omega vLu=0$ for all $u \geqslant 0$, $u \in C_0^\infty(\Omega)$.

The following lemma is crucial for our considerations. Let v be a nonnegative weak solution of $L^*v=0$.

LEMMA 3.1. *For any $\lambda \in]0, 1/n]$ there exists $\delta > 0$, $\delta = \delta(\lambda)$, such that*

$$\int_{B_r} v(y)\,dy \leqslant \frac{c}{\lambda} \int_{B_{(1-\delta)r}} v(y)\,dy \tag{3.1}$$

with $c=c(n)$, for any ball B_r such that $B_{(1+2\delta)r} \subset \Omega$. Here B_r, $B_{(1+2\delta)r}$, $B_{(1-\delta)r}$ are concentric balls with radii r, $(1+2\delta)r$, $(1-\delta)r$. Moreover there exists a positive constant K such that $\lambda/\delta \leqslant K$, for any $\lambda \in]0, 1/n]$.

Proof. From now on we denote with c a generic positive constant depending only on n and suppose that B_r is centered at the origin. Fix $0<\delta<1$. There exists $g \in C_0^\infty(B_{(1+2\delta)r})$ such that

$$\begin{aligned} Lg &\geqslant 0 && \text{in } B_{(1+\delta)r} - B_{(1-\delta)r} \\ Lg &> k_1 && \text{in } B_r - B_{(1-\delta)r} \\ |Lg| &\leqslant k_2 && \text{in } \Omega \end{aligned} \tag{3.2}$$

for some positive constants k_1, k_2 depending on g. As g depends on δ we obtain $K_1 = \lambda c_1(\delta)$, $k_2 = k_2(\delta)$. If we choose $\delta = \delta(\lambda)$ it is easy to verify that we can pick δ such that

$$\frac{k_2(\delta)}{c_1(\delta)} \leqslant c, \qquad \frac{\lambda}{\delta} \leqslant c \qquad \text{for } \lambda \in \left]0, \frac{1}{n}\right]. \tag{3.3}$$

For example, we can choose $\delta = (\sqrt{1+2\lambda}-1)/(2(\sqrt{1+2\lambda}+1))$ and $g(x) = [(1+\delta)^2 r^2 - |x|^2]^2$. From (3.2), as v is a nonnegative weak solution we deduce

$$\begin{aligned} \int_{B_r - B_{(1-\delta)r}} v(y)\,dy &= \frac{1}{k_1} \int_{B_r - B_{(1-\delta)r}} v(y)\,Lg(y)\,dy \\ &\leqslant \frac{1}{k_1} \int_{B_{(1+v)r} - B_{(1-\delta)r}} v(y)\,Lg(y)\,dy \leqslant \frac{k_2}{k_1} \int_{B_{(1-\delta)r}} v(y)\,dy \end{aligned}$$

from which, using (3.3),

$$\int_{B_r} v(y)\,dy \leqslant \frac{c}{\lambda} \int_{B_{(1-\delta)r}} v(y)\,dy$$

and the lemma is proved.

From Lemma 3.1, using the same arguments as those of [2, 4] and the well known Gehring theorem [3, 10, 12], it is easy to deduce the estimate

$$\|u\|_{L^\infty(\Omega)} \leqslant k\,\|Lu\|_{L^q(\Omega)} \tag{3.4}$$

if u is a solution of problem (1.1) with Lu in place of f, for some $q<n$, where $k=k(\lambda, \Omega, q)$.

The following theorem is a maximum principle for solutions of second order nondivergence form elliptic equations with discontinuous coefficients.

Let us consider the operator $Lu=\sum_{i,j=1}^n a_{ij}(x)(\partial^2 u/\partial x_i\,\partial x_j)$ in Ω with

$$a_{ij}(x)\in L^\infty(\Omega),\quad a_{ij}(n)=a_{ji}(n),\quad \sum_{i=1}^n a_{ii}(n)=1,\quad \lambda|\xi|^2 \leqslant \sum_{i,j=1}^n a_{ij}(x)\,\xi_i\xi_j \tag{3.5}$$

a.e. in Ω, for any $\xi\in R^n$. We prove the following

THEOREM 3.1. *There exist $q<n$ and $k=k(\lambda,\Omega,q)$ such that, under hypotheses* (3.5) *only*, (3.4) *holds for any $u\in W^{2,n}(\Omega)\cap C^0(\bar\Omega)$.*

Proof. Let $(a_{ij}^d(x))_d$ be a sequence of C^∞ functions tending to $a_{ij}(x)$ in $L^{nq/(n-q)}$, for any $1\leqslant i,j\leqslant n$. We note also that we can choose a_{ij}^d in such a way that for any of them (3.5) holds, with the same λ. We fix $u\in W^{2,n}(\Omega)\cap C^0(\bar\Omega)$, $u|_{\partial\Omega}=0$, set $L_d u=\sum_{i,j=1}^n a_{ij}^d(\partial^2 u/\partial x_i\,\partial x_j)$, and note that, by (3.4),

$$\|u\|_{L^\infty(\Omega)} \leqslant k\,\|L_d u\|_{L^q(\Omega)} \tag{3.6}$$

with $k=k(\lambda,\Omega,q)$, independent on d.

We have, using Hölder's inequality,

$$\begin{aligned}
&\sum_{i,j=1}^n \int_\Omega |a_{ij}^d(x)-a_{ij}(x)|^q \left|\frac{\partial^2 u}{\partial x_i\,\partial x_j}\right|^q dx \\
&\qquad\leqslant \sum_{i,j=1}^n \left(\int_\Omega \left|\frac{\partial^2 u}{\partial x_i\,\partial x_j}\right|^n dx\right)^{q/n} \cdot \left(\int_\Omega |a_{ij}^d - a_{ij}|^{nq/(n-q)}\,dx\right)^{(n-q)/n}.
\end{aligned} \tag{3.7}$$

From (3.7) we deduce that $L_d u$ tends to Lu in $L^q(\Omega)$. Then, from (3.6) and (3.7), letting d go to infinity we obtain the theorem.

Remark 3.1. We note that Theorem 3.1 holds for any $u\in W^{2,p}(\Omega)\cap C^0(\bar\Omega)$, $u|_{\partial\Omega}=0$ with $q<p\leqslant n$.

Remark 3.2. Using Lemma 3.1, in [4] the existence of a function $\alpha(\lambda)$ is proved such that (3.4) holds for $\alpha(\lambda)<q\leqslant n$, and $n-\alpha(\lambda)> k\lambda^{(5n-1)/(n-1)}$ for some positive constant k independent on $\lambda\in\,]0,1/n]$.

From Theorem 3.1 and Remarks 3.1 and 3.2 we deduce the following remark concerning the estimates "a priori" for the displacements of a structure subjected to concentrated loads.

Remark 3.3. Consider the possibility of bounds like (3.4) when the coefficients are discontinuous and $f(P)$, the known term, allows singularities like $1/[d(P, \bar{P})]^{\sigma}$, $\sigma \geqslant 1$, $\bar{P} \in \bar{\Omega}$.

Using Theorem 3.1 we note that we can obtain L^{∞} bounds for the displacements in terms of L^q norms, for some $q < n - \sigma$, of the loads. We obtain these bounds for some $\sigma \geqslant 1$ and for displacements belonging to $W^{2,\uparrow}(\Omega) \cap C^0(\bar{\Omega})$ for some $q < p < n - \sigma$.

By Remark 3.2 we note that the bounds worsen if the draught tends to zero and we can estimate how q tends to $n - \sigma$.

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