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Integrating factors, adjoint equations and Lagrangians

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Abstract

Integrating factors and adjoint equations are determined for linear and non-linear differential equations of an arbitrary order. The new concept of an adjoint equation is used for construction of a Lagrangian for an arbitrary differential equation and for any system of differential equations where the number of equations is equal to the number of dependent variables. The method is illustrated by considering several equations traditionally regarded as equations without Lagrangians. Noether's theorem is applied to the Maxwell equations.

Keywords: Integrating factor for higher-order equations; Adjoint equation to non-linear equations; Lagrangian; Noether's theorem

1. Introduction

It is a traditional custom to associate adjoint equations exclusively with linear equations. It is also customary to discuss integrating factors for non-linear ordinary differential equations only in the case of first-order equations. Recall that Noether's theorem provides a connection between conservation laws for variational problems with symmetries of the Euler–Lagrange equations. In this introduction, we outline the corresponding definitions and results.

1.1. Integrating factor

The usual approach to integrating factors is as follows. A first-order ordinary differential equation

$$a(x, y)y' + b(x, y) = 0, \tag{1.1}$$

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where $y' = dy/dx$, is written in the differential form:

$$a(x, y)\,dy + b(x, y)\,dx = 0. \tag{1.2}$$

Equation (1.2) is said to be *exact* if its left-hand side is the differential, i.e.,

$$a(x, y)\,dy + b(x, y)\,dx = d\Phi(x, y) \tag{1.3}$$

with some function $\Phi(x, y)$. If Eq. (1.2) is exact, its solution is defined implicitly by $\Phi(x, y) = C = \text{const}$.

In general, Eq. (1.2) is not exact but it becomes exact upon multiplying by a certain function $\mu(x, y)$:

$$\mu(a\,dy + b\,dx) = d\Phi \equiv \Phi_y\,dy + \Phi_x\,dx, \tag{1.4}$$

where

$$\Phi_y = \frac{\partial \Phi}{\partial y}, \qquad \Phi_x = \frac{\partial \Phi}{\partial x}\cdot$$

The function $\mu(x, y)$ is called an *integrating factor* for Eq. (1.2). It follows from (1.4) that

$$\Phi_y = \mu a, \qquad \Phi_x = \mu b. \tag{1.5}$$

The integrability condition for the system (1.5) is written $\Phi_{xy} = \Phi_{yx}$ and yields the following equation for determining the integrating factors:

$$\frac{\partial(\mu a)}{\partial x} = \frac{\partial(\mu b)}{\partial y}\cdot \tag{1.6}$$

Theoretically, Eq. (1.6) provides an infinite number of integrating factors for Eq. (1.2). Practically, however, the integration of Eq. (1.6) is not usually simpler than the integration of the differential equation (1.2) in question. Nevertheless, the concept of an integrating factor gives us a useful tool since integrating factors for certain particular equations can be found by *ad hoc* methods. If one knows two linearly independent integrating factors, $\mu_1(x, y)$ and $\mu_2(x, y)$, for (1.2) then the general solution of (1.2) is obtained without additional quadratures from the equation

$$\frac{\mu_1(x, y)}{\mu_2(x, y)} = C. \tag{1.7}$$

1.2. Adjoint linear differential operators

Let $x = (x^1, \ldots, x^n)$ be n independent variables and $u = (u^1, \ldots, u^m)$ be m dependent variables with the partial derivatives $u_{(1)} = \{u_i^\alpha\}$, $u_{(2)} = \{u_{ij}^\alpha\}, \ldots$ of the first, second, etc. orders, where $u_i^\alpha = \partial u^\alpha/\partial x^i$, $u_{ij}^\alpha = \partial^2 u^\alpha/\partial x^i \partial x^j$. Denoting

$$D_i = \frac{\partial}{\partial x^i} + u_i^\alpha \frac{\partial}{\partial u^\alpha} + u_{ij}^\alpha \frac{\partial}{\partial u_j^\alpha} + \cdots \tag{1.8}$$

the total differentiation with respect to x^i, we have

$$u_i^\alpha = D_i\bigl(u^\alpha\bigr), \quad u_{ij}^\alpha = D_i\bigl(u_j^\alpha\bigr) = D_i D_j\bigl(u^\alpha\bigr), \quad \ldots.$$

Recall the definition of the adjoint linear operator. Let us consider, e.g., the scalar (i.e., $m = 1$) second-order linear partial differential equations

$$L[u] \equiv a^{ij}(x)u_{ij} + b^i(x)u_i + c(x)u = f(x), \tag{1.9}$$

where L is the following linear differential operator:

$$L = a^{ij}(x) D_i D_j + b^i(x) D_i + c(x). \tag{1.10}$$

The summation convention is used throughout the paper. Here, for example, the summation is assumed over $i, j = 1, \dots, n$. The coefficients $a^{ij}(x)$ are symmetric, i.e., $a^{ij} = a^{ji}$.

The *adjoint operator* to L is a second-order linear differential operator L^* such that

$$vL[u] - uL^*[v] = D_i\left(p^i\right) \equiv \operatorname{div} P(x) \tag{1.11}$$

for all functions u and v, where $P(x) = (p^1(x), \dots, p^n(x))$ is any vector. The adjoint operator L^* is uniquely determined and has the form

$$L^*[v] = D_i D_j\left(a^{ij} v\right) - D_i\left(b^i v\right) + cv. \tag{1.12}$$

The operator L is said to be *self-adjoint* if $L[u] = L^*[u]$ for any function $u(x)$. Recall that the operator (1.10) is self-adjoint if and only if

$$b^i(x) = D_j\left(a^{ij}\right), \quad i = 1, \dots, n. \tag{1.13}$$

The linear homogeneous equation

$$L^*[v] \equiv D_i D_j\left(a^{ij} v\right) - D_i\left(b^i v\right) + cv = 0 \tag{1.14}$$

is called the *adjoint equation* to the linear differential equation (1.9), $L[u] = f(x)$.

The definitions of the adjoint operator and the adjoint equation are the same for systems of second-order equations. They are obtained by assuming in Eq. (1.9) that u is an m-dimensional vector-function and that the coefficients $a^{ij}(x)$, $b^i(x)$ and $c(x)$ of the operator (1.10) are $m \times m$-matrices.

If $n = m = 1$ we have the definition of the adjoint operator to linear ordinary differential equations. Let us set $u = y$ and consider the first-order equation

$$L[y] \equiv a_0(x) y' + a_1(x) y = f(x). \tag{1.15}$$

The *adjoint operator* $L^*[z]$ to $L[y]$ has the form

$$L^*[z] = -(a_0 z)' + a_1 z. \tag{1.16}$$

The definition of the adjoint operator to higher-order equations is similar. For example, in the case of the second-order equation

$$L[y] \equiv a_0 y'' + a_1 y' + a_2 y = f(x) \tag{1.17}$$

with variable coefficients $a_0(x), a_1(x), a_2(x)$, the adjoint operator $L^*[z]$ to $L[y]$ is

$$L^*[z] = (a_0 z)'' - (a_1 z)' + a_2 z. \tag{1.18}$$

Likewise, in the case of the third-order equation

$$L[y] \equiv a_0 y''' + a_1 y'' + a_2 y' + a_3 y = f(x), \tag{1.19}$$

the adjoint operator $L^*[z]$ to $L[y]$ is given by

$$L^*[z] = -(a_0 z)''' + (a_1 z)'' - (a_2 z)' + a_3 z. \tag{1.20}$$

The homogeneous equation $L^*[z] = 0$ is called the adjoint equation to $L[y] = f(x)$.

1.3. Noether's theorem

Noether's theorem [9] manifests a connection between symmetries and conservation laws for variational problems and provides a simple procedure for construction of conservation laws for Euler–Lagrange equations with known symmetries. The main steps of this procedure are as follows.

For the sake of brevity, consider Lagrangians $\mathcal{L}(x, u, u_{(1)})$ involving, along with the independent variables $x = (x, \ldots, x^n)$ and the dependent variables $u = (u, \ldots, u^m)$, the first-order derivatives $u_{(1)} = \{u_i^\alpha\}$ only. Then the Euler–Lagrange equations have the form

$$\frac{\delta \mathcal{L}}{\delta u^\alpha} \equiv \frac{\partial \mathcal{L}}{\partial u^\alpha} - D_i\left(\frac{\partial \mathcal{L}}{\partial u_i^\alpha}\right) = 0, \quad \alpha = 1, \ldots, m. \tag{1.21}$$

They are obtained by variation of the integral $\int \mathcal{L}(x, u, u_{(1)})\,\mathrm{d}x$ taken over an arbitrary n-dimensional domain in the space of the independent variables.

Noether's theorem states that if the variational integral is invariant under a continuous transformation group G with a generator

$$X = \xi^i(x, u)\frac{\partial}{\partial x^i} + \eta^\alpha(x, u)\frac{\partial}{\partial u^\alpha}, \tag{1.22}$$

then the vector field $C = (C^1, \ldots, C^n)$ defined by

$$C^i = \xi^i \mathcal{L} + \left(\eta^\alpha - \xi^j u_j^\alpha\right)\frac{\partial \mathcal{L}}{\partial u_i^\alpha}, \quad i = 1, \ldots, n, \tag{1.23}$$

provides a conservation law for the Euler–Lagrange equations (1.21), i.e., obeys the equation $\operatorname{div} C \equiv D_i(C^i) = 0$ for all solutions of (1.21).

The invariance of the variational integral implies that the Euler–Lagrange equations (1.21) admit the group G. Therefore, in order to apply Noether's theorem, one has first of all to find the symmetries of Eqs. (1.21). Then one should single out the symmetries leaving invariant the variational integral (1.21). This can be done by means of the following infinitesimal test for the invariance of the variational integral (proved in [5], see also [6]):

$$X(\mathcal{L}) + \mathcal{L} D_i\left(\xi^i\right) = 0, \tag{1.24}$$

where the generator X is prolonged to the first derivatives $u_{(1)}$ by the formula

$$X = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha} + \left[D_i\left(\eta^\alpha\right) - u_j^\alpha D_i\left(\xi^j\right)\right]\frac{\partial}{\partial u_i^\alpha}. \tag{1.25}$$

If Eq. (1.24) is satisfied, then the vector (1.23) provides a conservation law.

The invariance condition (1.24) can be replaced by the divergence condition

$$X(\mathcal{L}) + \mathcal{L} D_i\left(\xi^i\right) = D_i\left(B^i\right). \tag{1.26}$$

Then Eq. (1.21) has a conservation law $D_i(C^i) = 0$, where (1.23) is replaced by

$$C^i = \xi^i \mathcal{L} + \left(\eta^\alpha - \xi^j u_j^\alpha\right)\frac{\partial \mathcal{L}}{\partial u_i^\alpha} - B^i. \tag{1.27}$$

It is a common belief that the applicability of Noether's theorem is severely restricted because Lagrangians exists only for very special types of differential equations. The aim of the present paper is to dispel this myth.

2. Main constructions

Here, the notion of an integrating factor is extended to higher-order ordinary differential equations. Furthermore, an adjoint equation is defined for non-linear ordinary and partial differential equations of an arbitrary order. Then, using the new concept of an adjoint equation, I obtain a Lagrangian for any ordinary and partial differential equation. It follows that Noether type conservation theorems can be applied to any differential equation as well as to any system where the number of differential equations is equal to the number of the dependent variables.

2.1. Preliminaries

We will use the calculus in the space $\mathcal{A}$ of differential functions introduced in [4] (see also [5, Section 19.1] and [6, Section 8.2]). Let us denote by z the sequence

$$z=(x,u,u_{(1)},u_{(2)},\ldots) \tag{2.1}$$

with elements z^ν $(\nu\geqslant 1)$, where $z^i=x^i$ $(1\leqslant i\leqslant n)$, $z^{n+\alpha}=u^\alpha$ $(1\leqslant\alpha\leqslant m)$ and the remaining elements represent the derivatives of u. Finite subsequences of z are denoted by $[z]$.

A *differential function* f is a locally analytic function $f([z])$ (i.e., locally expandable in a Taylor series with respect to all arguments) of a finite number of variables (2.1). The highest order of derivatives appearing in a differential function f is called the order of f and is denoted by $\operatorname{ord}(f)$. Thus, $\operatorname{ord}(f)=s$ means that $f=f(x,u,u_{(1)},\ldots,u_{(s)})$. The set of all differential functions of finite order is denoted by $\mathcal{A}$. The set $\mathcal{A}$ is a vector space endowed with the usual multiplication of functions. In other words, if $f([z])\in\mathcal{A}$ and $g([z])\in\mathcal{A}$ and if a and b any constants, then

$$\begin{aligned}
&af+bg\in\mathcal{A}, \quad \operatorname{ord}(af+bg)\leqslant\max\bigl\{\operatorname{ord}(f),\operatorname{ord}(g)\bigr\},\\
&fg\in\mathcal{A}, \quad \operatorname{ord}(fg)=\max\bigl\{\operatorname{ord}(f),\operatorname{ord}(g)\bigr\}.
\end{aligned}$$

Furthermore, the space $\mathcal{A}$ is closed under the total derivation: if $f\in\mathcal{A}$, then

$$D_i(f)\in\mathcal{A}, \quad \operatorname{ord}\bigl(D_i(f)\bigr)=\operatorname{ord}(f)+1.$$

The *Euler–Lagrange operator* in $\mathcal{A}$ is defined by the formal sum

$$\frac{\delta}{\delta u^\alpha}=\frac{\partial}{\partial u^\alpha}-D_i\frac{\partial}{\partial u_i^\alpha}+D_iD_j\frac{\partial}{\partial u_{ij}^\alpha}+\cdots, \quad \alpha=1,\ldots,m, \tag{2.2}$$

where, for every s, the summation is presupposed over the repeated indices $i,j,\ldots$ running from 1 to n. The operator $\delta/\delta u^\alpha$ is termed also the *variational derivative*.

The operator (2.2) with one independent variable x is written

$$\frac{\delta}{\delta u^\alpha}=\frac{\partial}{\partial u^\alpha}-D_x\frac{\partial}{\partial u_x^\alpha}+D_x^2\frac{\partial}{\partial u_{xx}^\alpha}-D_x^3\frac{\partial}{\partial u_{xxx}^\alpha}+\cdots. \tag{2.3}$$

In the case of one independent variable x and one dependent variable y, we will use the common notation and write $z=(x,y,y',y'',\ldots,y^{(s)},\ldots)$. Then the total differentiation (1.8) is written as follows:

$$D_x=\frac{\partial}{\partial x}+y'\frac{\partial}{\partial y}+y''\frac{\partial}{\partial y'}+\cdots \tag{2.4}$$

and the Euler–Lagrange operator (2.3) becomes

$$\frac{\delta}{\delta y}=\frac{\partial}{\partial y}-D_x\frac{\partial}{\partial y'}+D_x^2\frac{\partial}{\partial y''}-D_x^3\frac{\partial}{\partial y'''}+\cdots. \tag{2.5}$$

The main constructions of this section are based on the concept of multipliers and the following lemmas (for the proofs, see [6, Section 8.4]).

Lemma 2.1. *Let $f(x,y,y',\ldots,y^{(s)})\in\mathcal{A}$. If $D_x(f)=0$ identically in all variables $x,y,y',\ldots,y^{(s)}$, and $y^{(s+1)}$, then $f=C=\text{const}$. Likewise, if $f(x,u,u_{(1)},\ldots,u_{(s)})$ is a differential function with one independent variable x and several dependent variables $u=(u^1,\ldots,u^m)$, the equation $D_x(f)=0$ implies that $f=C$.*

Lemma 2.2. *A differential function $f(x,u,\ldots,u_{(s)})\in\mathcal{A}$ with one independent variable x is a total derivative*:

$$f=D_x(g),\quad g(x,u,\ldots,u_{(s-1)})\in\mathcal{A}, \tag{2.6}$$

if and only if the following equations hold identically in $x,u,u_{(1)},\ldots$:

$$\frac{\delta f}{\delta u^\alpha}=0,\quad \alpha=1,\ldots,m. \tag{2.7}$$

Lemma 2.3. *A function $f(x,u,\ldots,u_{(s)})\in\mathcal{A}$ with several independent variables $x=(x^1,\ldots,x^n)$ and several dependent variables $u=(u^1,\ldots,u^m)$ is a divergence of a vector field $H=(h^1,\ldots,h^n)$, $h^i\in\mathcal{A}$*:

$$f=\operatorname{div}H\equiv D_i\big(h^i\big), \tag{2.8}$$

if and only if the following equations hold identically in $x,u,u_{(1)},\ldots$:

$$\frac{\delta f}{\delta u^\alpha}=0,\quad \alpha=1,\ldots,m. \tag{2.9}$$

2.2. *Integrating factor for higher-order equations*

Definition 2.1. Consider sth-order ordinary differential equations of the form

$$a\big(x,y,y',\ldots,y^{(s-1)}\big)y^{(s)}+b\big(x,y,y',\ldots,y^{(s-1)}\big)=0. \tag{2.10}$$

A differential function $\mu(x,y,y',\ldots,y^{(s-1)})$ is called an integrating factor for Eq. (2.10) if the multiplication by μ converts the left-hand side of Eq. (2.10) into a total derivative of some function $\Phi(x,y,y',\ldots,y^{(s-1)})\in\mathcal{A}$:

$$\mu a y^{(s)}+\mu b=D_x(\Phi). \tag{2.11}$$

Knowledge of an integrating factor allows one to reduce the order of Eq. (2.10). Indeed, Eqs. (2.10)–(2.11) are written $D_x(\Phi)=0$, and Lemma 2.1 yields the $(s-1)$-order equation

$$\Phi\big(x,y,y',\ldots,y^{(s-1)}\big)=C. \tag{2.12}$$

Definition 2.1 can be readily extended to systems of ordinary differential equations of any order.

Theorem 2.1. *The integrating factors for Eq.* (2.10) *are determined by the following equation*:

$$\frac{\delta}{\delta y}\big(\mu a y^{(s)} + \mu b\big) = 0, \tag{2.13}$$

where $\delta/\delta y$ is the variational derivative (2.5). *Equation* (2.13) *involves the variables $x, y, y', \ldots, y^{(2s-2)}$ and should be satisfied identically in all these variables.*

Proof. Equation (2.13) is obtained from Lemma 2.2. The highest derivative that may appear after the variational differentiation (2.5) has the order $2s-1$. It occurs in the terms

$$(-1)^s D_x^s(\mu a) \quad \text{and} \quad (-1)^{s-1} D_x^{s-1}\left[y^{(s)} \frac{\partial(\mu a)}{\partial y^{(s-1)}}\right].$$

We have, dropping the terms that certainly do not involve $y^{(2s-1)}$:

$$(-1)^s D_x^s(\mu a) = -(-1)^{s-1} D_x^{s-1}\left[y^{(s)} \frac{\partial(\mu a)}{\partial y^{(s-1)}}\right] + \cdots.$$

Thus, the terms containing $y^{(2s-1)}$ annihilate each other, and hence Eq. (2.13) involves only the variables $x, y, y', \ldots, y^{(2s-2)}$. This completes the proof. □

For the first-order equation (1.1), $a(x,y)y' + b(x,y) = 0$, Eq. (2.13) is written:

$$\frac{\delta}{\delta y}(\mu a y' + \mu b) = y'(\mu a)_y + (\mu b)_y - D_x(\mu a) = 0.$$

Since $D_x(\mu a) = (\mu a)_x + y'(\mu a)_y$, we arrive at Eq. (1.6), $(\mu b)_y - (\mu a)_x = 0$.

Consider the second-order equation

$$a(x, y, y')y'' + b(x, y, y') = 0. \tag{2.14}$$

The integrating factors μ depend on x, y, y', and Eq. (2.13) for determining $\mu(x, y, y')$ is written:

$$\frac{\delta}{\delta y}(\mu a y'' + \mu b) = y''(\mu a)_y + (\mu b)_y - D_x\big[y''(\mu a)_{y'} + (\mu b)_{y'}\big] + D_x^2(\mu a) = 0.$$

We have

$$\begin{aligned}
D_x(\mu a) &= y''(\mu a)_{y'} + y'(\mu a)_y + (\mu a)_x,\\
D_x^2(\mu a) &= y'''(\mu a)_{y'} + y''^2(\mu a)_{y'y'} + 2y'y''(\mu a)_{yy'} + 2y''(\mu a)_{xy'}\\
&\quad + y''(\mu a)_y + y'^2(\mu a)_{yy} + 2y'(\mu a)_{xy} + (\mu a)_{xx},\\
D_x\big(y''(\mu a)_{y'}\big) &= y'''(\mu a)_{y'} + y''^2(\mu a)_{y'y'} + y'y''(\mu a)_{yy'} + y''(\mu a)_{xy'},\\
D_x\big((\mu b)_{y'}\big) &= y''(\mu b)_{y'y'} + y'(\mu b)_{yy'} + (\mu b)_{xy'},
\end{aligned}$$

and hence

$$\begin{aligned}
\frac{\delta}{\delta y}(\mu a y'' + \mu b) &= y''\big[y'(\mu a)_{yy'} + (\mu a)_{xy'} + 2(\mu a)_y - (\mu b)_{y'y'}\big]\\
&\quad + y'^2(\mu a)_{yy} + 2y'(\mu a)_{xy} + (\mu a)_{xx} - y'(\mu b)_{yy'} - (\mu b)_{xy'} + (\mu b)_y.
\end{aligned}$$

Since this expression should vanish identically in x, y, y' and y'', we arrive at the following statement.

Theorem 2.2. *The integrating factors* $\mu(x,y,y')$ *for the second-order equation* (2.14) *are determined by the following system of two equations*:

$$y'(\mu a)_{yy'} + (\mu a)_{xy'} + 2(\mu a)_y - (\mu b)_{y'y'} = 0, \tag{2.15}$$

$$y'^2(\mu a)_{yy} + 2y'(\mu a)_{xy} + (\mu a)_{xx} - y'(\mu b)_{yy'} - (\mu b)_{xy'} + (\mu b)_y = 0. \tag{2.16}$$

Theorem 2.2 shows that the second-order equations, unlike the first-order ones, may have no integrating factors. Indeed, the integrating factor $\mu(x,y)$ for any first-order equation is determined by the single first-order linear partial differential equation (1.6) which always has infinite number of solutions. In the case of second-order equations (2.14), one unknown function $\mu(x,y,y')$ should satisfy two second-order linear partial differential equations (2.15)–(2.16). An integrating factor exists only if the over-determined system (2.15)–(2.16) is compatible.

Remark 2.1. If a second-order equation (2.14) has two integrating factors, its general solution can be found without additional integration.

Example 2.1. Let us calculate integrating factors for the following equation:

$$y'' + \frac{y'^2}{y} + 3\frac{y'}{x} = 0. \tag{2.17}$$

Equation (2.17) has the form (2.14) with

$$a = 1, \qquad b = \frac{y'^2}{y} + 3\frac{y'}{x}\cdot$$

For the sake of simplicity, we will look for the integrating factors of the particular form $\mu = \mu(x,y)$. Then Eq. (2.15) reduces to $2\mu_y - (\mu b)_{y'y'} = 0$. Since $(\mu b)_{y'y'} = 2\mu/y$, we obtain the equation

$$\frac{\partial\mu}{\partial y} - \frac{\mu}{y} = 0,$$

whence $\mu = \phi(x)y$. Thus, we have

$$\mu = \phi(x)y, \qquad \mu_{yy} = 0, \qquad \mu_{xy} = \phi', \qquad \mu_{xx} = \phi'' y, \qquad \mu b = \phi y'^2 + 3\frac{\phi}{x}yy',$$

$$(\mu b)_y = 3\frac{\phi}{x}y', \qquad (\mu b)_{yy'} = 3\frac{\phi}{x}, \qquad (\mu b)_{xy'} = 2\phi' y' + 3\left(\frac{\phi'}{x} - \frac{\phi}{x^2}\right)y.$$

Substitution in Eq. (2.16) leads to the following Euler's equation:

$$x^2\phi'' - 3x\phi' + 3\phi = 0.$$

Integrating it by the standard change of the independent variable, $t = \ln x$, we obtain two independent solutions, $\phi = x$ and $\phi = x^3$. Thus, Eq. (2.17) has two integrating factors:

$$\mu_1 = xy, \qquad \mu_2 = x^3 y, \tag{2.18}$$

and can be solved without an additional integration (see Remark 2.1).

Indeed, multiplying Eq. (2.17) by the first integrating factor, we have

$$xy\left(y'' + \frac{y'^2}{y} + 3\frac{y'}{x}\right) = xyy'' + xy'^2 + 3yy' = 0.$$

Substituting $xyy'' = D_x(xyy') - yy' - xy'^2$, we reduce it to

$$D_x(xyy') + 2yy' = D_x\big(xyy' + y^2\big) = 0,$$

whence

$$xyy' + y^2 = C_1. \tag{2.19}$$

The similar calculations by using the second integrating factor (2.18) yields

$$x^3 yy' = C_2. \tag{2.20}$$

Eliminating y' from Eqs. (2.19)–(2.20), we obtain the following general solution to Eq. (2.17):

$$y = \pm\sqrt{C_1 - \frac{C_2}{x^2}}. \tag{2.21}$$

2.3. Adjoint equations

Definition 2.2. Consider a system of sth-order partial differential equations

$$F_\alpha(x, u, \ldots, u_{(s)}) = 0, \quad \alpha = 1, \ldots, m, \tag{2.22}$$

where $F_\alpha(x, u, \ldots, u_{(s)}) \in \mathcal{A}$ are differential functions with n independent variables $x = (x^1, \ldots, x^n)$ and m dependent variables $u = (u^1, \ldots, u^m)$, $u = u(x)$. The system of *adjoint equations* to Eqs. (2.22) is defined by

$$F_\alpha^*(x, u, v, \ldots, u_{(s)}, v_{(s)}) \equiv \frac{\delta(v^\beta F_\beta)}{\delta u^\alpha} = 0, \quad \alpha = 1, \ldots, m, \tag{2.23}$$

where $v = (v^1, \ldots, v^m)$ are new dependent variables, $v = v(x)$.

Remark 2.2. In the case of linear equations, adjoint equations given by Definition 2.2 are identical with the classical adjoint equations discussed in Section 1.2. Therefore, the adjoint equation to a linear equation (or a system) $F(x, u, \ldots, u_{(s)}) = 0$ for $u(x)$ is a linear equation (a system) $F^*(x, v, \ldots, v_{(s)}) = 0$ for $v(x)$, and the relation to be adjoint is symmetric, i.e., $F^{**} = F$. More specifically, if the adjoint equation to $F^*(x, v, \ldots, v_{(s)}) = 0$ is $F^{**}(x, w, \ldots, w_{(s)}) = 0$, then setting $w = u$ in the latter equation we obtain the original equation.

Definition 2.3. A system of equations (2.22) is said to be *self-adjoint* if the system obtained from the adjoint equations (2.23) by the substitution $v = u$:

$$F_\alpha^*(x, u, u, \ldots, u_{(s)}, u_{(s)}) = 0, \quad \alpha = 1, \ldots, m,$$

is identical with the original system (2.22).[1]

Example 2.2. Let us take $n = 1, m = 1$, set $u = y, v = z$, and consider the first-order linear ordinary differential equation (1.15):

$$F(x, y, y') \equiv a_0 y' + a_1 y - f(x) = 0.$$

[1] In general, it does not mean that $F_\alpha^*(x, u, u, \ldots, u_{(s)}, u_{(s)}) = F_\alpha(x, u, \ldots, u_{(s)})$, see, e.g., Example 2.6.

Equation (2.23) defining the adjoint equation is written:

$$\frac{\delta(zF)}{\delta y} = \left(\frac{\partial}{\partial y} - D_x \frac{\partial}{\partial y'}\right)\left(z\left[a_0 y' + a_1 y - f(x)\right]\right) = 0.$$

Since

$$\frac{\partial}{\partial y}\left(z\left[a_0 y' + a_1 y - f(x)\right]\right) = a_1 z, \qquad \frac{\partial}{\partial y'}\left(z\left[a_0 y' + a_1 y - f(x)\right]\right) = a_0 z,$$

Eq. (2.23) yields the adjoint equation $a_1 z - D_x(a_0 z) = 0$, or

$$a_1 z - (a_0 z)' = 0$$

the left-hand side of which is identical with the adjoint operator (1.16).

Example 2.3. For the second-order equation (1.17),

$$a_0 y'' + a_1 y' + a_2 y = f(x),$$

Definition 2.2 yields the adjoint equation

$$\left(\frac{\partial}{\partial y} - D_x \frac{\partial}{\partial y'} + D_x^2 \frac{\partial}{\partial y''}\right)\left(z\left[a_0 y'' + a_1 y' + a_2 y - f(x)\right]\right) = 0.$$

Proceeding as in the previous example, one obtains the adjoint equation (1.18):

$$(a_0 z)'' - (a_1 z)' + a_2 z = 0.$$

Example 2.4. Consider the second-order linear partial differential equation (1.9):

$$L[u] \equiv a^{ij}(x) u_{ij} + b^i(x) u_i + cu = f(x).$$

The definition (2.23) of the adjoint equation is written

$$\left(\frac{\partial}{\partial u} - D_i \frac{\partial}{\partial u_i} + D_i D_j \frac{\partial}{\partial u_{ij}}\right)\left(v\left[a^{ij}(x) u_{ij} + b^i(x) u_i + cu - f(x)\right]\right) = 0$$

and yields the adjoint equation (1.14):

$$L^*[u] \equiv D_i D_j\left(a^{ij} v\right) - D_i\left(b^i v\right) + cv = 0.$$

Example 2.5. Consider the heat equation

$$u_t - c(x) u_{xx} = 0,$$

where $c(x)$ is a variable or constant coefficient. Equation (2.23) is written (see (2.2)):

$$\frac{\delta}{\delta u}\left(v\left[c(x) u_{xx} - u_t\right]\right) = \left(-D_t \frac{\partial}{\partial u_t} + D_x^2 \frac{\partial}{\partial u_{xx}}\right)\left(v\left[c(x) u_{xx} - u_t\right]\right) = 0$$

and yields the adjoint equation $D_x^2(c(x)v) + D_t(v) = 0$, or

$$v_t + (cv)_{xx} = 0.$$

Let us calculate by Definition 2.2 the adjoint equations to several well-known non-linear equations from mathematical physics.

Example 2.6. Consider the Korteweg–de Vries equation

$$u_t = u_{xxx} + uu_x. \tag{2.24}$$

We take $F(t, x, u, \ldots, u_{(3)}) = u_t - u_{xxx} - uu_x$ and write the left-hand side of Eq. (2.23) in the form

$$\frac{\delta}{\delta u}\big(v[u_t - u_{xxx} - uu_x]\big) = -v_t + v_{xxx} - vu_x + D_x(uv) = -v_t + v_{xxx} + uv_x.$$

Hence, $F^*(t, x, u, v, \ldots, u_{(3)}, v_{(3)}) = -(v_t - v_{xxx} - uv_x)$, and the adjoint equation to the Korteweg–de Vries equation (2.24) is

$$v_t = v_{xxx} + uv_x. \tag{2.25}$$

We have

$$F^*(t, x, u, u, \ldots, u_{(3)}, u_{(3)}) = -(u_t - u_{xxx} - uu_x) \equiv -F(t, x, u, \ldots, u_{(3)}).$$

Thus, Eq. (2.24) is self-adjoint (see Definition 2.3).

Let us find the adjoint equation to Eq. (2.25). We have

$$\frac{\delta}{\delta v}\big(w[v_t - v_{xxx} - uu_x]\big) = -w_t + w_{xxx} + D_x(uw) = -w_t + w_{xxx} + uw_x + wu_x.$$

Hence, the adjoint to Eq. (2.25) is $w_t = w_{xxx} + uw_x + wu_x$. Setting here $w = u$, we obtain the equation

$$u_t = u_{xxx} + 2uu_x$$

different from the original Korteweg–de Vries equation (2.24) (cf. Remark 2.2).

Example 2.7. Consider the Burgers equation

$$u_t = uu_x + u_{xx}. \tag{2.26}$$

The left-hand side of Eq. (2.23) is written:

$$\frac{\delta}{\delta u}\big(v[u_t - uu_x - u_{xx}]\big) = -v_t - vu_x + D_x(uv) - v_{xx} = -v_t + uv_x - v_{xx}.$$

Hence, adjoint equation to the Burgers equation (2.26) is (see also [7])

$$v_t = uv_x - v_{xx}. \tag{2.27}$$

Example 2.8. Consider the non-linear heat equation:

$$u_t = \big[k(u)u_x\big]_x. \tag{2.28}$$

The left-hand side of Eq. (2.23) is written:

$$\begin{aligned}&\frac{\delta}{\delta u}\big(v\big[u_t - k(u)u_{xx} - k'(u)u_x^2\big]\big)\\ &\quad = -v_t - k'(u)vu_{xx} - k''(u)vu_x^2 - D_x^2\big(k(u)v\big) + 2D_x\big(k'(u)vu_x\big).\end{aligned} \tag{2.29}$$

We have $D_x(k(u)v) = kv_x + k'vu_x$ and therefore

$$-D_x^2\big(k(u)v\big) + 2D_x\big(k'(u)vu_x\big) = -D_x(kv_x) + D_x\big(k'vu_x\big).$$

Inserting this in Eq. (2.29) and making simple calculations we arrive at the following adjoint equation to the non-linear heat equation (2.28) (see also [7]):

$$v_t + k(u)v_{xx} = 0. \tag{2.30}$$

Let us find the adjoint equation to (2.30). We have

$$\frac{\delta}{\delta v}\big(w\big[v_t + k(u)v_{xx}\big]\big) = -w_t + D_x^2\big[k(u)w\big].$$

Hence, the adjoint equation to (2.30) is $w_t = [k(u)w]_{xx}$ and does not coincide with Eq. (2.28) upon setting $w = u$.

2.4. Lagrangians

Theorem 2.3. *Any system of sth-order differential equations* (2.22)*,*

$$F_\alpha(x, u, \ldots, u_{(s)}) = 0, \quad \alpha = 1, \ldots, m, \tag{2.22}$$

considered together with its adjoint equation (2.23)*,*

$$F_\alpha^*(x, u, v, \ldots, u_{(s)}, v_{(s)}) \equiv \frac{\delta(v^\beta F_\beta)}{\delta u^\alpha} = 0, \quad \alpha = 1, \ldots, m, \tag{2.23}$$

has a Lagrangian. Namely, the simultaneous system (2.22)–(2.23) *with* $2m$ *dependent variables* $u = (u^1, \ldots, u^m)$ *and* $v = (v^1, \ldots, v^m)$ *is the system of Euler–Lagrange equations* (1.21) *with the Lagrangian* $\mathcal{L}$ *defined by*[2]

$$\mathcal{L} = v^\beta F_\beta. \tag{2.31}$$

Proof. Indeed, we have

$$\frac{\delta \mathcal{L}}{\delta v^\alpha} = F_\alpha(x, u, \ldots, u_{(s)}) \tag{2.32}$$

and

$$\frac{\delta \mathcal{L}}{\delta u^\alpha} = F_\alpha^*(x, u, v, \ldots, u_{(s)}, v_{(s)}). \qquad \square \tag{2.33}$$

Let us turn to examples. Consider linear equations, e.g. the homogeneous linear second-order partial differential equation (1.9):

$$L[u] \equiv a^{ij}(x)u_{ij} + b^i(x)u_i + c(x)u = 0. \tag{2.34}$$

The Lagrangian (2.31) is written:

$$\mathcal{L} = vL[u] = v\big(a^{ij}(x)u_{ij} + b^i(x)u_i + c(x)u\big). \tag{2.35}$$

We have

$$\frac{\delta \mathcal{L}}{\delta v} = \frac{\partial \mathcal{L}}{\partial v} = L[u] \tag{2.36}$$

[2] See also the concept of a weak Lagrangian introduced in [3].

and

$$\frac{\delta\mathcal{L}}{\delta u^\alpha} = D_i D_j\left(\frac{\partial\mathcal{L}}{\partial u_{ij}}\right) - D_i\left(\frac{\partial\mathcal{L}}{\partial u_i}\right) + \frac{\partial\mathcal{L}}{\partial u}$$
$$= D_i D_j\big(a^{ij}v\big) - D_i\big(b^i v\big) + cv = L^*[v]. \tag{2.37}$$

Theorem 2.4. *Let the linear operator $L[u]$ be self-adjoint, $L^*[u] = L[u]$. Then Eq.* (2.34) *is obtained from the Lagrangian*

$$\mathcal{L} = \frac{1}{2}\big[c(x)u^2 - a^{ij}(x)u_i u_j\big]. \tag{2.38}$$

Proof. We rewrite the Lagrangian (2.35) in the form

$$\mathcal{L} = v\big(a^{ij}u_{ij} + b^i u_i + cu\big) = D_j\big(va^{ij}u_i\big) - vu_i D_j\big(a^{ij}\big) + vb^i u_i - a^{ij}u_i v_j + cuv.$$

The first term at the right-hand side can be dropped by Lemma 2.3, while the second and the third terms annihilate each other by the condition (1.13). Finally, we set $v = u$, divide by two and arrive at the Lagrangian (2.38). □

Example 2.9. For the Helmholtz equation $\Delta u + k^2 u = 0$, (2.38) gives the well-known Lagrangian $\mathcal{L} = (k^2u^2 - |\nabla u|^2)/2$.

If one deals with linear equations that are not self-adjoint or with non-linear equations, one obtains a Lagrangian formulation by considering the equation in question together with its adjoint equation.

Example 2.10. The linear heat equation is not self-adjoint. Therefore, we consider it together with its adjoint equation and obtain the system of two equations:

$$u_t - c(x)u_{xx} = 0, \qquad v_t + (cv)_{xx} = 0 \tag{2.39}$$

which is derived from the Lagrangian

$$\mathcal{L} = vu_t - c(x)vu_{xx}. \tag{2.40}$$

Example 2.11. According to Example 2.6, the Lagrangian

$$\mathcal{L} = v[u_t - u_{xxx} - uu_x] \tag{2.41}$$

leads to the Korteweg–de Vries equation (2.24) and its conjugate (2.25) combined in the following system:

$$u_t = u_{xxx} + uu_x, \qquad v_t = v_{xxx} + uv_x. \tag{2.42}$$

Example 2.12. Likewise, we obtain from Example 2.8 the Lagrangian

$$\mathcal{L} = v\big[u_t - k(u)u_{xx} - k'(u)u_x^2\big] \tag{2.43}$$

that leads to the non-linear heat equation (2.28) and its conjugate (2.30) combined in the following system:

$$u_t = \big[k(u)u_x\big]_x, \qquad v_t + k(u)v_{xx} = 0. \tag{2.44}$$

Example 2.13. One of fundamental equations in quantum mechanics is the Dirac equation

$$\gamma^k \frac{\partial \psi}{\partial x^k} + m\psi = 0, \quad m = \text{const.} \tag{2.45}$$

The dependent variable ψ is a 4-dimensional column vector with complex valued components $\psi^1, \psi^2, \psi^3, \psi^4$. The independent variables compose the four-dimensional vector $x = (x^1, x^2, x^3, x^4)$, where x^1, x^2, x^3 are the real valued spatial variables and x^4 is the complex variable defined by $x^4 = ict$ with t being time and c the light velocity. Furthermore, γ^k are the following 4×4 complex matrices called the Dirac matrices:

$$\gamma^1 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}, \qquad \gamma^2 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix},$$
$$\gamma^3 = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \\ i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}, \qquad \gamma^4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

Equation (2.45) does not have a Lagrangian. Therefore, it is considered together with the conjugate equation

$$\frac{\partial \tilde{\psi}}{\partial x^k}\gamma^k - m\tilde{\psi} = 0. \tag{2.46}$$

Here $\tilde{\psi} = \overline{\psi}^T \gamma^4$ is the row vector, where $\overline{\psi}$ denotes the complex-conjugate to ψ and T the transposition. The system (2.45)–(2.46) has the Lagrangian

$$\mathcal{L} = \frac{1}{2}\left[\tilde{\psi}\left(\gamma^k \frac{\partial \psi}{\partial x^k} + m\psi\right) - \left(\frac{\partial \tilde{\psi}}{\partial x^k}\gamma^k - m\tilde{\psi}\right)\psi\right].$$

Indeed, we have

$$\frac{\delta \mathcal{L}}{\delta \psi} = -\left(\frac{\partial \tilde{\psi}}{\partial x^k}\gamma^k - m\tilde{\psi}\right), \qquad \frac{\delta \mathcal{L}}{\delta \tilde{\psi}} = \gamma^k \frac{\partial \psi}{\partial x^k} + m\psi.$$

3. Application to the Maxwell equations

This section is dedicated to illustration of the method by applying Noether's theorem to the Maxwell equations. Consider the Maxwell equations in vacuum:

$$\frac{1}{c}\frac{\partial \boldsymbol{E}}{\partial t} = \operatorname{curl} \boldsymbol{H}, \quad \operatorname{div} \boldsymbol{E} = 0,$$
$$\frac{1}{c}\frac{\partial \boldsymbol{H}}{\partial t} = -\operatorname{curl} \boldsymbol{E}, \quad \operatorname{div} \boldsymbol{H} = 0. \tag{3.1}$$

The system (3.1) contains six dependent variables, namely, the components of the electric field $\boldsymbol{E} = (E^1, E^2, E^3)$ and the magnetic field $\boldsymbol{H} = (H^1, H^2, H^3)$, and eight equations, i.e. it is *over-determined*. On the other hand, the number of equations in the Euler–Lagrange equations (1.21) is equal to the number of dependent variables. Consequently, the system (3.1) cannot have a

Lagrangian. What is considered in the literature as a variational problem in electrodynamics (see, e.g. [1,8]) provides a Lagrangian for the wave equation

$$\Delta \boldsymbol{A} - \frac{1}{c^2}\frac{\partial^2 \boldsymbol{A}}{\partial t^2} = 0$$

for the vector potential $\boldsymbol{A}$ of the electromagnetic field, but not for the Maxwell equations (3.1).

Let us find a Lagrangian for the electromagnetic field by using Theorem 2.3. First we note that the equations $\operatorname{div}\boldsymbol{E} = 0$, $\operatorname{div}\boldsymbol{H} = 0$ hold at any time provided that they are satisfied at the initial time $t = 0$. Hence, they are merely initial conditions (see, e.g. [2] or [6]). Therefore, we will consider the following *determined system* of differential equations (we set $t' = ct$ and take t' as new t):

$$\operatorname{curl}\boldsymbol{E} + \frac{\partial \boldsymbol{H}}{\partial t} = 0, \qquad \operatorname{curl}\boldsymbol{H} - \frac{\partial \boldsymbol{E}}{\partial t} = 0. \tag{3.2}$$

We introduce six new dependent variables, namely the components of the vectors $\boldsymbol{V} = (V^1, V^2, V^3)$ and $\boldsymbol{W} = (W^1, W^2, W^3)$, and introduce the Lagrangian

$$\mathcal{L} = \boldsymbol{V}\cdot\left(\operatorname{curl}\boldsymbol{E} + \frac{\partial \boldsymbol{H}}{\partial t}\right) + \boldsymbol{W}\cdot\left(\operatorname{curl}\boldsymbol{H} - \frac{\partial \boldsymbol{E}}{\partial t}\right) \tag{3.3}$$

in accordance with the definition (2.31).

One can readily verify that the Lagrangian (3.3) yields the system (3.2) together with its adjoint, namely:

$$\frac{\delta\mathcal{L}}{\delta\boldsymbol{V}} \equiv \operatorname{curl}\boldsymbol{E} + \frac{\partial \boldsymbol{H}}{\partial t} = 0, \qquad \frac{\delta\mathcal{L}}{\delta\boldsymbol{W}} \equiv \operatorname{curl}\boldsymbol{H} - \frac{\partial \boldsymbol{E}}{\partial t} = 0, \tag{3.4}$$

$$\frac{\delta\mathcal{L}}{\delta\boldsymbol{E}} \equiv \operatorname{curl}\boldsymbol{V} + \frac{\partial \boldsymbol{W}}{\partial t} = 0, \qquad \frac{\delta\mathcal{L}}{\delta\boldsymbol{H}} \equiv \operatorname{curl}\boldsymbol{W} - \frac{\partial \boldsymbol{V}}{\partial t} = 0. \tag{3.5}$$

If we set $\boldsymbol{V} = \boldsymbol{E}$, $\boldsymbol{W} = \boldsymbol{H}$, Eqs. (3.5) coincide with (3.4). Hence, the operator in (3.2) is self-adjoint. Therefore we set $\boldsymbol{V} = \boldsymbol{E}$, $\boldsymbol{W} = \boldsymbol{H}$ in (3.3), divide by two and obtain the Lagrangian for the Maxwell equations (3.2) (cf. Theorem 2.4):

$$\mathcal{L} = \frac{1}{2}\left[\boldsymbol{E}\cdot\left(\operatorname{curl}\boldsymbol{E} + \frac{\partial \boldsymbol{H}}{\partial t}\right) + \boldsymbol{H}\cdot\left(\operatorname{curl}\boldsymbol{H} - \frac{\partial \boldsymbol{E}}{\partial t}\right)\right]. \tag{3.6}$$

In coordinates, the Lagrangian (3.6) is written:

$$\begin{aligned}\mathcal{L} = {} & E^1\left(E^3_y - E^2_z + H^1_t\right) + E^2\left(E^1_z - E^3_x + H^2_t\right) + E^3\left(E^2_x - E^1_y + H^3_t\right)\\ & + H^1\left(H^3_y - H^2_z - E^1_t\right) + H^2\left(H^1_z - H^3_x - E^2_t\right) + H^3\left(H^2_x - H^1_y - E^3_t\right).\end{aligned} \tag{3.7}$$

The symmetries of the Maxwell equations are well known, and one can apply Noether's theorem by using the Lagrangian (3.6). We will employ, as an example, the invariance of Eqs. (3.2) with respect to the group of transformations

$$\boldsymbol{H}' = \boldsymbol{H}\cos\theta + \boldsymbol{E}\sin\theta, \qquad \boldsymbol{E}' = \boldsymbol{E}\cos\theta - \boldsymbol{H}\sin\theta \tag{3.8}$$

with the generator

$$X = \boldsymbol{E}\frac{\partial}{\partial \boldsymbol{H}} - \boldsymbol{H}\frac{\partial}{\partial \boldsymbol{E}} \equiv \sum_{i=1}^{3}\left(E^i\frac{\partial}{\partial H^i} - H^i\frac{\partial}{\partial E^i}\right). \tag{3.9}$$

The prolongation (1.25) of this generator is written

$$X = \boldsymbol{E}\frac{\partial}{\partial \boldsymbol{H}} - \boldsymbol{H}\frac{\partial}{\partial \boldsymbol{E}} + \boldsymbol{E}_t\frac{\partial}{\partial \boldsymbol{H}_t} - \boldsymbol{H}_t\frac{\partial}{\partial \boldsymbol{E}_t} + \boldsymbol{E}_x\frac{\partial}{\partial \boldsymbol{H}_x} - \boldsymbol{H}_x\frac{\partial}{\partial \boldsymbol{E}_x} + \cdots. \tag{3.10}$$

Acting by the operator (3.10) on the Lagrangian (3.6), we have

$$\begin{aligned} X(\mathcal{L}) = \frac{1}{2}\big[&-\boldsymbol{H}\cdot(\operatorname{curl}\boldsymbol{E} + \boldsymbol{H}_t) + \boldsymbol{E}\cdot(\operatorname{curl}\boldsymbol{H} - \boldsymbol{E}_t) \\ &+ \boldsymbol{E}\cdot(-\operatorname{curl}\boldsymbol{H} + \boldsymbol{E}_t) + \boldsymbol{H}\cdot(\operatorname{curl}\boldsymbol{E} + \boldsymbol{H}_t)\big] = 0. \end{aligned}$$

Hence, the condition (1.24) is satisfied and one can obtain a conservation law by the formula (1.23). We will write the conservation law in the form

$$D_t(\tau) + \operatorname{div}\boldsymbol{\chi} = 0, \tag{3.11}$$

where $\boldsymbol{\chi} = (\chi^1, \chi^2, \chi^3)$, $\operatorname{div}\boldsymbol{\chi} = D_x(\chi^1) + D_y(\chi^2) + D_z(\chi^3)$. Equation (1.23) yields

$$\tau = \boldsymbol{E}\cdot\frac{\partial\mathcal{L}}{\partial \boldsymbol{H}_t} - \boldsymbol{H}\cdot\frac{\partial\mathcal{L}}{\partial \boldsymbol{E}_t} = \frac{1}{2}\big[\boldsymbol{E}\cdot\boldsymbol{E} - \boldsymbol{H}\cdot(-\boldsymbol{H})\big] = \frac{1}{2}\big[E^2 + H^2\big].$$

Hence, τ is the energy density. Likewise, calculating the spatial coordinates of the conserved vector (1.23), one can verify that $\boldsymbol{\chi}$ is the Poynting vector, $\boldsymbol{\chi} = \boldsymbol{E}\times\boldsymbol{H}$. Thus, we have obtained the conservation of energy (see, e.g. [8]):

$$D_t\left(\frac{E^2 + H^2}{2}\right) + \operatorname{div}(\boldsymbol{E}\times\boldsymbol{H}) = 0. \tag{3.12}$$

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