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Determining cusp forms by central values of Rankin–Selberg L-functions ☆

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ABSTRACT

Let g be a fixed normalized Hecke–Maass cusp form for $SL(2,\mathbb{Z})$ associated to the Laplace eigenvalue $\frac{1}{4}+\nu^2$. We show that g is uniquely determined by the central values of the family $\{L(s, f \otimes g)\colon\ g \in H_k(1)\}$ for k sufficiently large, where $H_k(1)$ denotes a Hecke basis of the space of holomorphic cusp forms for $SL(2,\mathbb{Z})$.

1. Introduction

Determining modular forms by central values of L-functions has been an interesting subject. In [9], Luo and Ramakrishnan showed that if two cuspidal normalized newforms f and f' have the property that $L(1/2, f, \chi_d) = L(1/2, f', \chi_d)$ for all quadratic characters χ_d, then f must equal f'. Chinta and Diaconu [2] further generalized this result to forms on $GL(3)$. Replacing twisting $GL(1)$ objects χ_d by twisting $GL(2)$ holomorphic cusp forms, Luo [8] proved the following. Let f and f' be two normalized new forms of weight $2k$ (resp. $2k'$) and level N (resp. N'). Suppose there exists a positive integer l and infinitely many primes p, such that for all forms h in the Hecke basis $H_{2l}(\Gamma_0(p))$ of new forms of weight $2l$ and level p,

$$L\left(\frac{1}{2}, f \otimes h\right) = L\left(\frac{1}{2}, f' \otimes h\right).$$

Then, $k = k'$, $N = N'$, and $f = f'$. This is to determine modular forms by the central values of its twisting families of L-functions varying in level. Recently, Ganguly, Hoffstein and Sengupta [3] studied

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the case of determining modular forms by the central values of its twisting families of L-functions varying in weight. Precisely, let $H_k(1)$ denote a Hecke basis of the space of holomorphic cusp forms of weight k for $SL(2,\mathbb{Z})$. Suppose that $g \in H_l(1)$ and $g' \in H_{l'}(1)$. If

$$L\left(\frac{1}{2}, f \otimes g\right) = L\left(\frac{1}{2}, f \otimes g'\right)$$

for all $f \in H_k(1)$ for infinitely many k with k sufficiently large, then $l = l'$ and $g = g'$.

In this paper, we are concerned with Maass cusp forms. It is proved that they can be uniquely determined by the central values of its twisting families of L-functions varying in weight.

Theorem 1.1. *Suppose g and g' are fixed Hecke–Maass cusp forms for $SL(2,\mathbb{Z})$ with Laplace eigenvalues $\frac{1}{4} + \nu^2$ and $\frac{1}{4} + \nu'^2$ respectively. Assume that g and g' are normalized such that the first Fourier coefficients are 1. If*

$$L\left(\frac{1}{2}, f \otimes g\right) = L\left(\frac{1}{2}, f \otimes g'\right) \tag{1.1}$$

for all $f \in H_k(1)$ for infinitely many k with k sufficiently large, then $\nu = \nu'$ and $g = g'$.

To prove Theorem 1.1, we use the ideas of Ganguly, Hoffstein and Sengupta [3]. By the multiplicity one theorem (see [10]), it is enough to prove that for all but finitely many primes p,

$$\lambda_g(p) = \lambda_{g'}(p).$$

Define

$$\omega_f = \frac{(4\pi)^{k-1}}{\Gamma(k-1)} \langle f, f\rangle,$$

where $\langle , \rangle$ denotes the Petersson norm. Then Theorem 1.1 follows from the following result.

Theorem 1.2. *Let f and g be as in Theorem 1.1. Let h be a real valued function, which is smooth, compactly supported on $[1,2]$, and satisfies $h^{(j)} \ll_j 1$. Then for K sufficiently large,*

$$\sum_{k \equiv 0 \ (\mathrm{mod}\ 2)} h\left(\frac{k-1}{K}\right) \sum_{f \in H_k(1)} \omega_f^{-1} L\left(\frac{1}{2}, f \otimes g\right) \lambda_f(p) = \frac{\lambda_g(p)}{\sqrt{p}} K\left(\hat{h}(0)\log K + C\right) + O_{\nu,p}(1), \tag{1.2}$$

with

$$C = \int_{\mathbb{R}} h(x) \log x \, dx + c_0 \hat{h}(0),$$

where $c_0 = \gamma_0 - \log 2 - \frac{1}{2}\log 4\pi^2 p$, and $\hat{h}$ denotes the Fourier transform of h. Here γ_0 is Euler's constant.

The proof of Theorem 1.2 starts from the approximate functional equation and Petersson trace formula. Subsequently, the diagonal term from the Petersson formula gives the main term and the error term in (1.2). For the non-diagonal term, differently from [3], by dealing with an averaging of J-Bessel functions, we can show that it is negligible.

2. Preliminaries

Let $H_k(1)$ denote a Hecke basis of the space of holomorphic cusp forms of weight k for $SL(2,\mathbb{Z})$. Any $f \in H_k(1)$ has a Fourier expansion

$$f(z) = \sum_{n \geqslant 1} \lambda_f(n) n^{\frac{k-1}{2}} e(nz), \tag{2.1}$$

normalized such that $\lambda_f(1) = 1$. For $\mathrm{Re}(s) > 1$, the L-function associated to f is defined by

$$L(s, f) = \sum_{n \geqslant 1} \frac{\lambda_f(n)}{n^s}$$

which satisfies the functional equation

$$\Lambda(s, f) = (2\pi)^{-s} \Gamma\left(s + \frac{k-1}{2}\right) L(s, f) = i^{-k} \Lambda(1-s, f).$$

Let g be a Hecke–Maass cusp form for $SL(2,\mathbb{Z})$ with Laplace eigenvalue $\frac{1}{4} + \nu^2$, either even or odd. g has a Fourier expansion

$$g(z) = \sum_{n \neq 0} \lambda_g(n) \sqrt{y} K_{i\nu}\big(2\pi |n| y\big) e(nx), \tag{2.2}$$

normalized such that $\lambda_g(1) = 1$. The L-function associated to g is given by

$$L(s, g) = \sum_{n \geqslant 1} \frac{\lambda_g(n)}{n^s},$$

which converges absolutely for $\mathrm{Re}(s) > 1$. It satisfies the functional equation

$$\Lambda(s, g) = \pi^{-s} \Gamma\left(\frac{s + \delta + i\nu}{2}\right) \Gamma\left(\frac{s + \delta - i\nu}{2}\right) = (-1)^{\delta} \Lambda(1-s, g),$$

where $\delta = 0$ if g is even, and $\delta = 1$ if g is odd.

For $\mathrm{Re}(s)$ large, the Rankin–Selberg L-function $L(s, f \otimes g)$ for f in (2.1) and g in (2.2) is defined by

$$L(s, f \otimes g) = \zeta(2s) \sum_{m \geqslant 1} \frac{\lambda_f(m)\lambda_g(m)}{m^s} = \sum_{n \geqslant 1} \frac{b_{f\otimes g}(n)}{n^s},$$

with

$$b_{f\otimes g}(n) = \sum_{n = mj^2} \lambda_f(m)\lambda_g(m). \tag{2.3}$$

It is known that $L(s, f \otimes g)$ extends to an entire function on $\mathbb{C}$ and satisfies the functional equation (see Section 4 in Kowalski, Michel and Vanderkam [7])

$$\Lambda(s, f\otimes g) = (4\pi^2)^{-s}\Gamma\left(s+\frac{k-1}{2}+i\nu\right)\Gamma\left(s+\frac{k-1}{2}-i\nu\right) = \Lambda(1-s, f\otimes g). \tag{2.4}$$

Let $G(u) = e^{u^2}$. We have the following approximate functional equation for $L(s, f\otimes g)$ (see p. 98, Theorem 5.3 in Iwaniec and Kowalski [5]).

Lemma 2.1. *We have*

$$L\left(\frac{1}{2}, f\otimes g\right) = 2\sum_{n\geqslant 1}\frac{b_{f\otimes g}(n)}{\sqrt{n}}V\left(k, 4\pi^2 n\right), \tag{2.5}$$

where

$$V(k,y) = \frac{1}{2\pi i}\int_{(3)} y^{-u}\frac{\Gamma(u+\frac{k}{2}+i\nu)\Gamma(u+\frac{k}{2}-i\nu)}{\Gamma(\frac{k}{2}+i\nu)\Gamma(\frac{k}{2}-i\nu)}\frac{G(u)}{u}\,du. \tag{2.6}$$

$V(k,y)$ has the following properties (see p. 100, Proposition 5.4 in Iwaniec and Kowalski [5]).

Lemma 2.2. *For $y>0$, we have*

$$V(k,y) = 1 + O_\nu\left(\frac{y}{k^2}\right),$$

$$V(k,y) \ll_{\nu,A} \left(1+\frac{y}{k^2}\right)^{-A}.$$

The following lemma appears in Baker [1] (see Propositions 6 and 8).

Lemma 2.3.

(i) *For fixed c, and $\operatorname{Re}(z+c)\geqslant\delta$, we have*

$$\log\Gamma(z+c) = \left(z+c-\frac{1}{2}\right)\log z - z + O_{c,\delta}(1). \tag{2.7}$$

(ii) *In $|\arg z|\leqslant \pi-\delta$,*

$$\frac{\Gamma'(z)}{\Gamma(z)} = \log z - \frac{1}{2z} + O_\delta\left(\frac{1}{|z|^2}\right). \tag{2.8}$$

We need the following Petersson trace formula (See Theorem 3.6 in Iwaniec [4]).

Lemma 2.4. *We have*

$$\sum_{f\in H_k(1)}\omega_f^{-1}\lambda_f(m)\lambda_f(n) = \delta(m,n) + 2\pi i^{-k}\sum_{c\geqslant 1}c^{-1}S(m,n;c)J_{k-1}\left(\frac{4\pi\sqrt{mn}}{c}\right), \tag{2.9}$$

where $J_{k-1}(x)$ is the standard J-Bessel function and $S(m,n;c)$ is the classical Kloosterman sum defined by

$$S(m,n;c)=\sum_{d\bar{d}\equiv 1 \pmod{c}} e\left(\frac{md+n\bar{d}}{c}\right).$$

We also need the Poisson summation formula in arithmetic progressions (see p. 70, (4.24) in Iwaniec and Kowalski [5]).

Lemma 2.5. *Suppose that both f and its Fourier transform $\hat{f}$ are in $L^1(\mathbb{R})$ and have bounded variation. Then*

$$\sum_{m\equiv a \pmod{q}} f(m)=\frac{1}{q}\sum_{n\in\mathbb{Z}}\hat{f}\left(\frac{n}{q}\right)e\left(-\frac{an}{q}\right).$$

3. Proof of Theorem 1.2

Applying the approximate functional equation in (2.5) and Petersson trace formula in (2.9), we have

$$\begin{aligned}
&\sum_{k\equiv 0 \pmod 2} h\left(\frac{k-1}{K}\right)\sum_{f\in H_k(1)}\omega_f^{-1}L\left(\frac{1}{2},f\otimes g\right)\lambda_f(p)\\
&=2\sum_{k\equiv 0 \pmod 2} h\left(\frac{k-1}{K}\right)\sum_{f\in H_k(1)}\omega_f^{-1}\sum_{n\geqslant 1}\frac{b_{f\otimes g}(n)}{\sqrt{n}}V\left(k,4\pi^2 n\right)\lambda_f(p)\\
&=2\sum_{k\equiv 0 \pmod 2} h\left(\frac{k-1}{K}\right)\sum_{f\in H_k(1)}\omega_f^{-1}\sum_{m\geqslant 1}\sum_{j\geqslant 1}\frac{\lambda_f(m)\lambda_g(m)}{j\sqrt{m}}V\left(k,4\pi^2 mj^2\right)\lambda_f(p)\\
&=2\sum_{k\equiv 0 \pmod 2} h\left(\frac{k-1}{K}\right)\sum_{m\geqslant 1}\frac{\lambda_g(m)}{\sqrt{m}}\sum_{j\geqslant 1}\frac{V(k,4\pi^2 mj^2)}{j}\sum_{f\in H_k(1)}\omega_f^{-1}\lambda_f(m)\lambda_f(p)\\
&=\mathcal{D}+\mathcal{ND},
\end{aligned}$$

where $\mathcal{D}$ is the contribution from the diagonal term

$$\mathcal{D}=2\frac{\lambda_g(p)}{\sqrt{p}}\sum_{k\equiv 0 \pmod 2} h\left(\frac{k-1}{K}\right)\sum_{j\geqslant 1}\frac{V(k,4\pi^2 pj^2)}{j}, \tag{3.1}$$

and $\mathcal{ND}$ is the contribution from the non-diagonal term

$$\mathcal{ND}=4\pi\sum_{k\equiv 0 \pmod 2} i^k h\left(\frac{k-1}{K}\right)\sum_{m\geqslant 1}\frac{\lambda_g(m)}{\sqrt{m}}\sum_{j\geqslant 1}\frac{V(k,4\pi^2 mj^2)}{j}\sum_{c\geqslant 1}c^{-1}S(m,p;c)J_{k-1}\left(\frac{4\pi\sqrt{mp}}{c}\right). \tag{3.2}$$

Then Theorem 1.2 follows from the following two estimates:

$$\mathcal{D}=\frac{\lambda_g(p)}{\sqrt{p}}K\left(\hat{h}(0)\log K+C\right)+O_{\nu,p}(1), \tag{3.3}$$

where C is defined in Theorem 1.2, and

$$\mathcal{N}\mathcal{D} = O_{\nu,p}\big(K^{-2+\epsilon}\big). \tag{3.4}$$

We will establish (3.3) and (3.4) in Sections 4 and 5, respectively.

4. Estimation of the diagonal term $\mathcal{D}$

Recall that

$$\mathcal{D} = 2\frac{\lambda_g(p)}{\sqrt{p}} \sum_{k\equiv 0 \ (\mathrm{mod}\ 2)} h\left(\frac{k-1}{K}\right) M_p(k,\nu), \tag{4.1}$$

where

$$M_p(k,\nu) = \sum_{j\geqslant 1} \frac{V(k, 4\pi^2 pj^2)}{j}.$$

By the definition of $V(k,y)$ in (2.6), we have

$$\begin{aligned} M_p(k,\nu) &= \sum_{j\geqslant 1} \frac{1}{j}\frac{1}{2\pi i}\int_{(3)} \big(4\pi^2 pj^2\big)^{-u} \frac{\Gamma(u+\frac{k}{2}+i\nu)\Gamma(u+\frac{k}{2}-i\nu)}{\Gamma(\frac{k}{2}+i\nu)\Gamma(\frac{k}{2}-i\nu)} \frac{G(u)}{u}\,\mathrm{d}u \\ &= \frac{1}{2\pi i}\int_{(3)} \zeta(2u+1)\big(4\pi^2 p\big)^{-u} \frac{\Gamma(u+\frac{k}{2}+i\nu)\Gamma(u+\frac{k}{2}-i\nu)}{\Gamma(\frac{k}{2}+i\nu)\Gamma(\frac{k}{2}-i\nu)} \frac{G(u)}{u}\,\mathrm{d}u. \end{aligned} \tag{4.2}$$

Moving the line of integration in (4.2) to $\mathrm{Re}(u) = -\frac{1}{2}$, passing a double pole at $u = 0$ with residue

$$\frac{1}{2}\left(\frac{\Gamma'}{\Gamma}\left(\frac{k}{2}+i\nu\right) + \frac{\Gamma'}{\Gamma}\left(\frac{k}{2}-i\nu\right) + 2\gamma_0 - \log\big(4\pi^2 p\big)\right)$$

by [3] (see p. 852), we have

$$M_p(k,\nu) = \frac{1}{2}\left(\frac{\Gamma'}{\Gamma}\left(\frac{k}{2}+i\nu\right) + \frac{\Gamma'}{\Gamma}\left(\frac{k}{2}-i\nu\right) + 2\gamma_0 - \log\big(4\pi^2 p\big)\right) + \mathcal{R}, \tag{4.3}$$

where

$$\mathcal{R} = \frac{1}{2\pi i}\int_{(-\frac{1}{2})} \zeta(2u+1)\big(4\pi^2 p\big)^{-u} \frac{\Gamma(u+\frac{k}{2}+i\nu)\Gamma(u+\frac{k}{2}-i\nu)}{\Gamma(\frac{k}{2}+i\nu)\Gamma(\frac{k}{2}-i\nu)} \frac{G(u)}{u}\,\mathrm{d}u. \tag{4.4}$$

The remainder term $\mathcal{R}$ in (4.4) is

$$\begin{aligned} \mathcal{R} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \zeta(2it)\big(4\pi^2 p\big)^{\frac{1}{2}-it} \frac{\Gamma(\frac{k-1}{2}+i(t+\nu))}{\Gamma(\frac{k-1}{2})} \frac{\Gamma(\frac{k-1}{2})}{\Gamma(\frac{k}{2}+i\nu)} \\ \times \frac{\Gamma(\frac{k-1}{2}+i(t-\nu))}{\Gamma(\frac{k-1}{2})} \frac{\Gamma(\frac{k-1}{2})}{\Gamma(\frac{k}{2}-i\nu)} \frac{G(-\frac{1}{2}+it)}{-\frac{1}{2}+it}\,\mathrm{d}t. \end{aligned} \tag{4.5}$$

By (2.7), we have

$$\frac{\Gamma(\frac{k}{2}-\frac{1}{2})}{\Gamma(\frac{k}{2}+i\nu)} = \exp\left(\log\Gamma\left(\frac{k}{2}-\frac{1}{2}\right) - \log\Gamma\left(\frac{k}{2}+i\nu\right)\right)$$
$$= \exp\left(\left(-\frac{1}{2}-i\nu\right)\log\frac{k}{2} + O_\nu(1)\right)$$
$$= O_\nu\big(k^{-\frac{1}{2}}\big). \tag{4.6}$$

Similarly,

$$\frac{\Gamma(\frac{k}{2}-\frac{1}{2})}{\Gamma(\frac{k}{2}-i\nu)} = O_\nu\big(k^{-\frac{1}{2}}\big). \tag{4.7}$$

Note that $|\Gamma(x+iy)| \leqslant |\Gamma(x)|$. By (4.5), (4.6) and (4.7), we obtain

$$\mathcal{R} \ll_\nu \frac{1}{k}\int_{-\infty}^{\infty}\big(1+|t|\big)^{\frac{1}{2}}\frac{e^{-t^2}}{1+|t|}\,\mathrm{d}t \ll_\nu \frac{1}{k}. \tag{4.8}$$

Thus by (4.3) and (4.8),

$$M_p(k,\nu) = \frac{1}{2}\left(\frac{\Gamma'}{\Gamma}\left(\frac{k}{2}+i\nu\right) + \frac{\Gamma'}{\Gamma}\left(\frac{k}{2}-i\nu\right) + 2\gamma_0 - \log\big(4\pi^2 p\big)\right) + O_\nu\left(\frac{1}{k}\right)$$
$$= \log k + \gamma_0 - \log 2 - \frac{1}{2}\log 4\pi^2 p + O_\nu\left(\frac{1}{k}\right). \tag{4.9}$$

Here we have used (2.8). Denote $c_0 = \gamma_0 - \log 2 - \frac{1}{2}\log 4\pi^2 p$. Then by (4.1) and (4.9),

$$\mathcal{D} = 2\frac{\lambda_g(p)}{\sqrt{p}}\sum_{k\equiv 0 \ (\mathrm{mod}\ 2)} h\left(\frac{k-1}{K}\right)(\log k + c_0) + O_\nu(1).$$

By Lemma 2.5,

$$\mathcal{D} = \frac{\lambda_g(p)}{\sqrt{p}}\sum_{n\in\mathbb{Z}}\int_{\mathbb{R}} h\left(\frac{x-1}{K}\right)(\log x + c_0)e\left(\frac{nx}{2}\right)\mathrm{d}x + O_\nu(1)$$
$$= \frac{\lambda_g(p)}{\sqrt{p}}\int_{\mathbb{R}} h\left(\frac{x-1}{K}\right)(\log x + c_0)\,\mathrm{d}x + \frac{\lambda_g(p)}{\sqrt{p}}\sum_{n\neq 0}\int_{\mathbb{R}} h\left(\frac{x-1}{K}\right)(\log x + c_0)e\left(\frac{nx}{2}\right)\mathrm{d}x + O_\nu(1)$$
$$= \frac{\lambda_g(p)}{\sqrt{p}}K\int_{\mathbb{R}} h(x)\left(\log K + \log x + c_0 + \log\left(1+\frac{1}{Kx}\right)\right)\mathrm{d}x$$
$$+ \frac{\lambda_g(p)}{\sqrt{p}}K\sum_{n\neq 0} e\left(\frac{n}{2}\right)\int_{\mathbb{R}} h(x)\big(\log(Kx+1) + c_0\big)e\left(\frac{nKx}{2}\right)\mathrm{d}x + O_\nu(1)$$
$$= \frac{\lambda_g(p)}{\sqrt{p}}K\big(\hat{h}(0)\log K + C\big) + \mathcal{R}^* + O_{\nu,p}(1), \tag{4.10}$$

with

$$C=\int_{\mathbb{R}} h(x)\log x\,\mathrm{d}x + c_0\hat{h}(0), \tag{4.11}$$

where

$$\mathcal{R}^* = \frac{\lambda_g(p)}{\sqrt{p}} K \sum_{n\neq 0} e\left(\frac{n}{2}\right)\int_{\mathbb{R}} h(x)\big(\log(Kx+1)+c_0\big)e\left(\frac{nKx}{2}\right)\mathrm{d}x. \tag{4.12}$$

By partial integration twice,

$$\begin{aligned}
&\int_{\mathbb{R}} h(x)\big(\log(Kx+1)+c_0\big)e\left(\frac{nKx}{2}\right)\mathrm{d}x\\
&\quad=\frac{1}{\pi inK}\int_{\mathbb{R}} h(x)\big(\log(Kx+1)+c_0\big)\,\mathrm{d}e\left(\frac{nKx}{2}\right)\\
&\quad=\frac{-1}{\pi inK}\int_{\mathbb{R}} e\left(\frac{nKx}{2}\right)\left(h'(x)\big(\log(Kx+1)+c_0\big)+h(x)\frac{K}{Kx+1}\right)\mathrm{d}x\\
&\quad=\left(\frac{-1}{\pi inK}\right)^2\int_{\mathbb{R}} e\left(\frac{nKx}{2}\right)\left(h''(x)\big(\log(Kx+1)+c_0\big)+2h'(x)\frac{K}{Kx+1}-h(x)\frac{K^2}{(Kx+1)^2}\right)\mathrm{d}x\\
&\quad=O\left(\frac{\log K}{n^2K^2}\right).
\end{aligned} \tag{4.13}$$

By (4.12) and (4.13), $\mathcal{R}^* = O_p(K^{-1}\log K)$ and thus by (4.10),

$$\mathcal{D}=\frac{\lambda_g(p)}{\sqrt{p}}K\big(\hat{h}(0)\log K + C\big)+O_{\nu,p}(1),$$

where C is defined in (4.11). This proves (3.3).

5. Estimation the non-diagonal term $\mathcal{ND}$

In this section, we will estimate $\mathcal{ND}$ in (3.2). We have

$$\mathcal{ND}=4\pi\sum_{m\geqslant 1}\frac{\lambda_g(m)}{\sqrt{m}}\sum_{j\geqslant 1}\frac{1}{j}\sum_{c\geqslant 1}\frac{S(m,p;c)}{c}\sum_{k\equiv 0\ (\mathrm{mod}\ 2)} i^k h\left(\frac{k-1}{K}\right)V\big(k,4\pi^2mj^2\big)J_{k-1}\left(\frac{4\pi\sqrt{mp}}{c}\right).$$

The J-Bessel functions satisfy (see p. 73, (5.16) in Iwaniec [4])

$$J_{k-1}(x)\ll\min\big\{x^{k-1},x^{-\frac{1}{2}}\big\}.$$

Thus by Lemma 2.2, the contribution from $4\pi^2mj^2\gg K^{2+\varepsilon}$ to $\mathcal{ND}$ is

$$\ll_{p,A} \sum_{j\geqslant 1} j^{-1} \sum_{m\gg K^{2+\varepsilon}j^{-2}} \sum_{c\geqslant 1} c^{-1/2+\varepsilon} \sum_{K\leqslant k-1\leqslant 2K} \left(\frac{k^2}{mj^2}\right)^A \min\left\{\left(\frac{4\pi\sqrt{mp}}{c}\right)^{k-1}, \left(\frac{4\pi\sqrt{mp}}{c}\right)^{-1/2}\right\}$$

$$\ll_{p,A} K^{2A} \sum_{j\geqslant 1} j^{-1-2A} \sum_{m\gg K^{2+\varepsilon}j^{-2}} m^{-A} \sum_{K\leqslant k-1\leqslant 2K} \left(\sum_{c\ll\sqrt{m}} c^{\varepsilon} m^{-1/4} + \sum_{c\gg\sqrt{m}} c^{-1/2+\varepsilon}\left(\frac{\sqrt{m}}{c}\right)^{k-1}\right)$$

$$\ll_{p,A} K^{1+2A} \sum_{j\geqslant 1} j^{-1-2A} \sum_{m\gg K^{2+\varepsilon}j^{-2}} m^{-A+1/4+\varepsilon}$$

$$\ll_{p,A,\varepsilon} K^{7/2+\varepsilon-\varepsilon A}$$

which is negligible by choosing A sufficiently large. Therefore, we only need to estimate

$$\mathcal{ND}^* = \sum_{m\ll K^{2+\varepsilon}} \frac{\lambda_g(m)}{\sqrt{m}} \sum_{mj^2\ll K^{2+\varepsilon}} \frac{1}{j} \sum_{c\geqslant 1} \frac{S(m,p;c)}{c} \mathcal{W}(m,j,c), \tag{5.1}$$

where

$$\mathcal{W}(m,j,c) = 2\sum_{k\equiv 0 \ (\mathrm{mod}\ 2)} i^k h\left(\frac{k-1}{K}\right) V\left(k, 4\pi^2 mj^2\right) J_{k-1}\left(\frac{4\pi\sqrt{mp}}{c}\right). \tag{5.2}$$

This kind of averaging of J-Bessel function has been studied in Iwaniec, Luo and Sarnak [6] (see p. 102, Corollary 8.2).

Lemma 5.1. *Fixed a real valued function $\omega \in \mathcal{C}_0^\infty(\mathbb{R}^+)$ and $K \geqslant 1$. Then*

$$2\sum_{k\equiv 0 \ (\mathrm{mod}\ 2)} i^k \omega\left(\frac{k-1}{K}\right) J_{k-1}(x) = -\frac{K}{\sqrt{x}} \operatorname{Im}\left(e^{ix-\frac{2\pi i}{8}} \Omega\left(\frac{K^2}{2x}\right)\right) + O\left(\frac{x}{K^4}\right),$$

where

$$\Omega(v) = \int_0^\infty \frac{\omega(\sqrt{u})}{\sqrt{2\pi u}} e^{iuv}\, du$$

satisfying

$$\Omega(v) \ll_A \left(1+|v|\right)^{-A}.$$

Applying Lemma 5.1 with $\omega(u) = h(u)V(uK+1, 4\pi^2 mj^2)$, we have that $\mathcal{W}(m,j,c)$ in (5.2) is

$$\mathcal{W}(m,j,c) = -\frac{K\sqrt{c}}{2\sqrt{\pi}(mp)^{\frac{1}{4}}} \operatorname{Im}\left(e^{i\frac{4\pi\sqrt{mp}}{c}-\frac{2\pi i}{8}} \Omega\left(\frac{K^2 c}{8\pi\sqrt{mp}}\right)\right) + O_{\nu,p}\left(\frac{\sqrt{m}}{cK^4}\right), \tag{5.3}$$

where

$$\Omega(v) = \int_0^\infty \frac{h(\sqrt{u})V(\sqrt{u}K+1, 4\pi^2 mj^2)}{\sqrt{2\pi u}} e^{iuv}\, du$$

satisfying

$$\Omega(\nu) \ll_{A,\nu} \big(1+|\nu|\big)^{-A}. \tag{5.4}$$

Note that in (5.3), $\frac{K^2c}{8\pi\sqrt{mp}} \gg_p \frac{K^2c}{\sqrt{m}} \gg cK^{1-\varepsilon}$ for any $\varepsilon > 0$. By (5.4), the contribution to $\mathcal{ND}^*$ from the first term in (5.3) is negligible. By Weil's bound for Kloosterman sum,

$$\big|S(m,n;c)\big| \leqslant c^{\frac{1}{2}}(m,n,c)^{\frac{1}{2}}\tau(c),$$

the contribution from second term $O_\nu(\frac{\sqrt{m}}{cK^4})$ to $\mathcal{ND}^*$ in (5.1) is at most

$$\begin{aligned}
&\ll_{\nu,p} \sum_{m\ll K^{2+\epsilon}} \frac{|\lambda_g(m)|}{\sqrt{m}} \sum_{j\ll \frac{K^{1+\varepsilon}}{\sqrt{m}}} \frac{1}{j} \sum_{c\geqslant 1} c^{-1}c^{\frac{1}{2}+\epsilon}\cdot\frac{\sqrt{m}}{cK^4}\\
&\ll_{\nu,p} K^{-4+\epsilon} \sum_{m\ll K^{2+\epsilon}} \big|\lambda_g(m)\big|\\
&\ll_{\nu,p} K^{-4+\epsilon}\Bigg(\sum_{m\ll K^{2+\epsilon}} 1\Bigg)^{\frac{1}{2}}\Bigg(\sum_{m\ll K^{2+\epsilon}} \big|\lambda_g(m)\big|^2\Bigg)^{\frac{1}{2}}\\
&\ll_{\nu,p} K^{-2+\epsilon}.
\end{aligned}$$

This finishes the proof of (3.4).

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