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On pairs of matrices generating matrix rings and their presentations

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Abstract

Let $M_n(\mathbb{Z})$ be the ring of n-by-n matrices with integral entries, and $n \geqslant 2$. This paper studies the set $G_n(\mathbb{Z})$ of pairs $(A, B) \in M_n(\mathbb{Z})^2$ generating $M_n(\mathbb{Z})$ as a ring. We use several presentations of $M_n(\mathbb{Z})$ with generators $X = \sum_{i=1}^{n} E_{i+1,i}$ and $Y = E_{11}$ to obtain the following consequences.

(1) Let $k \geqslant 1$. The following rings have presentations with 2 generators and finitely many relations:
   (a) $\bigoplus_{j=1}^{k} M_{m_j}(\mathbb{Q})$ for any $m_1, \ldots, m_k \geqslant 2$.
   (b) $\bigoplus_{j=1}^{k} M_{n_j}(\mathbb{Z})$, where $n_1, \ldots, n_k \geqslant 2$, and the same n_i is repeated no more than three times.
(2) Let D be a commutative domain of sufficiently large characteristic over which every finitely generated projective module is free. We use 4 relations for X and Y to describe all representations of the ring $M_n(D)$ into $M_m(D)$ for $m \geqslant n$.
(3) We obtain information about the asymptotic density of $G_n(F)$ in $M_n(F)^2$ over different fields, and over the integers.

Keywords: Asymptotic density; Direct sum of matrix rings; Higman's Theorem; Magnus Embedding; Matrix rings; Noncommutative polynomials; Ring presentations; Ring representations

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1. Introduction

1.1. Terminology and notation

All rings in this paper, often denoted by R, are assumed associative with a two-sided identity element, unless stated otherwise. We denote by $\mathcal{U}(R)$ the unit group of R. We do not assume that a subring of a ring necessarily contains the identity element of the ring. All ideals in rings are assumed two-sided. The *rank* of a ring R, denoted by $\dim_{\mathbb{Z}} R$, is the rank of its additive group, that is $\dim_{\mathbb{Q}} R \otimes_{\mathbb{Z}} \mathbb{Q}$.

An algebraic closure a finite field with q elements $\mathbb{F}_q$ is denoted by $\bar{\mathbb{F}}_q$.

We denote by $M_n(R)$ the ring of n-by-n matrices with entries in R. The subscripts in matrices and in their entries will always be regarded modulo n. Let $A, B \in M_n(R)$. We define $R\langle A, B\rangle$ to be the R-subalgebra of $M_n(R)$ generated by A and B. We will study the collection of such pairs (A, B), i.e. the set

$$G_n(R) = \big\{(A, B) \in M_n(R)^2 \;\big|\; R\langle A, B\rangle = M_n(R)\big\}.$$

We also need the free noncommutative associative ring $R\{x, y\}$ whose elements we refer to as noncommutative polynomials. The ring presentations studied in this paper are quotients of $\mathbb{Z}\{x, y\}$. We do not postulate that the identity is in $R\langle A, B\rangle$, while we postulate that $1 \in R\{x, y\}$.

Many of our considerations will be based on the following two matrices:

$$X = E_{21} + E_{32} + \cdots + E_{n,n-1} + E_{1n} \quad \text{and} \quad Y = E_{11} \quad \text{for } n \geqslant 2.$$

Let $FS(x, y)$ be a free semigroup on x and y. It has the lexicographic order as well as the word length $l(w)$ counting the total number of x and y in $w \in FS(x, y)$.

The matrices $T_{m,n,R}$. Let R be a ring, and let x_{ij}, y_{ij}, where $1 \leqslant i, j \leqslant n$, be algebraically independent transcendental variables over R. We see that $\#\{w \in FS(x, y) \mid l(w) \leqslant m\} = 2^{m+1} - 2$. Below, we define the matrix

$$T_{m,n,R} \in M_{(2^{m+1}-2)\times n^2}\big(R[x_{ij}, y_{ij}]\big).$$

Let $w = w(x, y) \in FS(x, y)$. We substitute the matrices (x_{ij}) and (y_{ij}) for x and y, respectively. The result is the n-by-n matrix $(z_{ij}) = w_R((x_{ij}), (y_{ij}))$, which we write as a row vector as follows

$$(z_{11}, z_{12}, \ldots, z_{1n}, z_{21}, z_{22}, \ldots, z_{2n}, \ldots, z_{n1}, z_{n2}, \ldots, z_{nn}). \tag{1}$$

We call the operation of transforming the matrix (z_{ij}) into the vector (1) *flattening* of (z_{ij}). We define $T_{m,n,R}$ to be the matrix whose rows are the flattened matrices $w_R((x_{ij}), (y_{ij}))$ such that $l(w) \leqslant m$, the words w being ordered lexicographically.

If $A, B \in M_n(R)$, then $T_{m,n,R}(A, B)$ is the matrix obtained from $T_{m,n,R}$ by substituting the entries of A and B for (x_{ij}) and (y_{ij}), respectively.

Let $S \subseteq \mathbb{Z}^m$ and $B_k = \{(x_1, \ldots, x_m) \in \mathbb{Z}^m\colon \max_{1\leqslant i\leqslant n} |x_i| \leqslant k\}$. The *asymptotic density* of S in $\mathbb{Z}^m$ is

$$\lim_{k\to\infty} \frac{\#B_k \cap S}{\#B_k}.$$

1.2. Motivation and description of the main results

The properties of the ring $M_n(\mathbb{Z})$ are based entirely on the presentation by the elementary matrices E_{ij} subject to the relations $E_{ij}E_{kl} = \delta_{jk}E_{il}$. This set of n^2 generators may be further reduced. Moreover, the matrices X and Y generate $M_n(\mathbb{Z})$. These matrices will be used to construct several presentations of $M_n(\mathbb{Z})$ with 2 generators and finitely many relations. We investigate the interdependence between the relations in these presentations. We also use them to construct 2-generator presentations with finitely many relations of certain direct sums of matrix rings. Burnside's Theorem from [1] implies that the set $G_n(\mathbb{C})$ is infinite. This paper, in contrast, studies the set $G_n(\mathbb{Z})$. In particular, we describe $G_2(\mathbb{Z})$ in the following

Theorem 2.10. *Let $A, B \in M_2(\mathbb{Z})$. Put $I = I_2$ and $\mathcal{S} = \mathbb{Z}\langle A, B\rangle$. Then*

(1) *$I \in \mathcal{S}$ if and only if* $\gcd(\det A, \det B, \det(A + B)) = 1$.
(2) *$\mathcal{S} = M_2(\mathbb{Z})$ if and only if I, A, B, AB generate $M_2(\mathbb{Z})$ as a $\mathbb{Z}$-module.*

If I, A, B generate $M_2(\mathbb{Z})$ as a ring, then their $\mathbb{Z}$-linear combinations produce I, A_1, B_1 also generating $M_2(\mathbb{Z})$ such that

$$A_1 = \begin{pmatrix} c & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad B_1 = \begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix}$$

where $\gcd(a, b) = 1$. *Moreover, the matrices I, A_1, B_1 generate $M_2(\mathbb{Z})$ if and only if*

$$a^2 - abc - b^2 = \pm 1.$$

The set of solutions of these equations is infinite, and when $abc \neq 0$, this set is effectively described in terms of the unit group of the field $\mathbb{Q}(\sqrt{c^2+4})$.

We show that $M_n(F)^2 - G_n(F)$ is "small" for many fields. Namely, if F is a normed field having a sequence of nonzero elements whose norms tend to zero, then the set $G_n(F)$ is dense in $M_n(F)^2$. We also prove that

$$\lim_{q\to\infty} \frac{\#G_n(\mathbb{F}_q)}{\#M_n(\mathbb{F}_q)^2} = 1.$$

In contrast, the set $M_n(\mathbb{Z})^2 - G_n(\mathbb{Z})$ is not algebraic, and $G_2(\mathbb{Z})$ has zero asymptotic density in $M_2(\mathbb{Z})^2$.

The problem of minimality of presentations in ring theory admits a number of interpretations. For example, one may search for a presentation with the smallest number of both generators and relations. Unfortunately, no technique is available to solve this problem in general. More modestly, one may ask whether the removal of any of the relations in a given presentation changes the ring. We study this question and in many cases obtain information about the structure of the resulting over-rings.

We use the following noncommutative polynomials:

$$r_{1,n} = r_{1,n}(x) = x^n - 1, \qquad r_{2,n} = r_{2,n}(x,y) = \sum_{i=0}^{n-1} x^{n-i} y x^i - 1,$$

$$s_0 = s_0(y) = y^2 - y, \qquad s_j = s_j(x,y) = yx^j y \quad \text{for } j \geqslant 1.$$

Here are the presentations studied in our paper:

$$M_2(\mathbb{Z}) \cong \langle x, y \mid x^2 = y + xyx = 1,\ yxy = 0\rangle, \tag{2}$$

$$M_3(\mathbb{Z}) \cong \langle x, y \mid x^3 = y + x^2yx + xyx^2 = 1,\ yxy = 0\rangle, \tag{3}$$

$$M_4(\mathbb{Z}) \cong \langle x, y \mid r_{1,4} = r_{2,4} = s_0 = s_1 = 0\rangle, \tag{4}$$

$$M_5(\mathbb{Z}) \cong \langle x, y \mid r_{1,5} = r_{2,5} = s_0 = s_1 = 0\rangle, \tag{5}$$

$$M_n(\mathbb{Z}) \cong \langle x, y \mid r_{1,n} = r_{2,n} = s_j = 0,\ 1 \leqslant j \leqslant n-1\rangle, \tag{6}$$

$$M_n(\mathbb{Z}) \cong \langle x, y \mid r_{1,n} = r_{2,n} = s_0 = s_k = 0,\ 1 \leqslant k \leqslant \lfloor n/2 \rfloor\rangle. \tag{7}$$

While we cannot completely answer the question of minimality in the presentations above, some information is available in Theorems 3.3, 3.4, and 3.5 below. Theorems 3.3 and 3.4 investigate the effect of the removal of certain relations from (6).

Theorem 3.4.

(1) *The ring $\mathcal{R} = \langle x, y \mid r_{1,n} = s_m = 0,\ 0 \leqslant m \leqslant n-1\rangle$ is isomorphic to a direct sum of the rings $M_n(\mathbb{Z})$ and $\mathbb{Z}[x]/(x^n - 1)$.*

(2) *Let $\emptyset \neq H \subsetneqq N = \{1, 2, \ldots, n-1\}$ and $H' = N - H$. Suppose that H satisfies the following conditions modulo n:*

(a) $\{a+b \mid a,b \in -H \cup H\} \subseteq H'$.
(b) *If* $h,k,l,-h+k+l \in H$, *then* $h=k$ *or* $h=l$.
Then the ring $S(H) = \langle x, y \mid r_{1,n} = r_{2,n} = s_j = 0,\ j \in H' \rangle$ *has finite rank.*

Theorem 3.5. *The ring* $\mathbb{Z}\{x,y\}$ *has a quotient* $R = R_n$ *such that*

(1) R *is an over-ring of* $M_n(\mathbb{Z})$.
(2) *Under the natural epimorphism* $\mathbb{Z}\{x,y\} \twoheadrightarrow R$, *the images of the ideals generated by* $r_{1n}, s_1, \ldots, s_n$ *form a direct sum.*

In the proof of this theorem we introduce an analog of the Magnus Embedding (see lemma on p. 764 of Magnus [12]).

We prove the following theorem about linear representations of matrix rings.

Theorem 3.7. *Let* $\mathcal{D}$ *be a commutative domain of characteristic either zero or at least* $m+1$, *over which every finitely generated projective module is free. Let* $\mathcal{S}$ *be a subring of* $M_m(\mathcal{D})$ *generated by some nonzero* X_1 *and* Y_1 *such that*

$$X_1^{n+1} = X_1, \qquad Y_1 X_1^n = Y_1, \qquad Y_1^2 = Y_1, \qquad \sum_{i=0}^{n-1} X_1^{n-i} Y_1 X_1^i = X_1^n.$$

Then the trace k *of* Y_1 *is a positive integer, and there exist* $B \in GL_m(\mathcal{D})$ *such that, putting* $r = m - kn$, *we have*

$$B^{-1} X_1 B = \begin{pmatrix} I_k \otimes X & 0_{k\times r} \\ 0_{r\times k} & 0_{r\times r} \end{pmatrix} \quad \text{and} \quad B^{-1} Y_1 B = \begin{pmatrix} I_k \otimes Y & 0_{k\times r} \\ 0_{r\times k} & 0_{r\times r} \end{pmatrix}.$$

The rigidity of the embeddings in the above theorem also follows from more general results in Azumaya algebras (see Faith [4, pp. 481–482]).

We investigate the matrices satisfying the relations of (7). Let $x_1, \ldots, x_n$ be numbers. These numbers determine the circulant matrix $\operatorname{circ}(x_1, \ldots, x_n) = \sum_{i=1}^n x_{n-i+1} X^i$. Integral n-by-n circulant matrices are exactly the elements of the group ring $\mathbb{Z}\langle X \rangle$.

Theorem 3.10. *The set* $\mathcal{Y} = \{Y_1 \in M_n(\mathbb{Z}) \mid Y_1^2 = Y_1,\ r_{2,n}(X, Y_1) = 0\}$ *has the property that the pair* (X, Y_1) *satisfies all relations of* (7) *and all* Y_1 *have trace* 1. *If* $n = 2, 3, 4, 6$ *then* $Y_1 = E_{ii}$ *for some* i. *Otherwise,* $\mathcal{Y}$ *is infinite, and if* $Y_1 \neq E_{ii}$ *then it has both positive and negative entries.*

Any Y_1 *is of the form* $(c_i d_j)$ *for some integers* c_i, d_j *such that the matrices* $\operatorname{circ}(c_1, \ldots, c_n)$ *and* $\operatorname{circ}(d_1, \ldots, d_n)$ *are mutually inverse. Any* Y_1 *is conjugate to* Y *by an integral circulant matrix with determinant* ± 1.

This result depends on a classic theorem of G. Higman [8] about the structure of the unit group of an integral group ring of a finite Abelian group.

In the final part of this paper, we obtain some 2-generator presentations with finitely many relations for arbitrary finite direct sums $\bigoplus_{j=1}^k M_{m_j}(\mathbb{Q})$ where $m_j \geqslant 2$, and for the direct sums $\bigoplus_{j=1}^k M_{n_j}(\mathbb{Z})$ where $n_1, \ldots, n_k \geqslant 2$, and the same n_i is repeated no more than three times.

2. On the structure of $G_n(\mathbb{Z})$

The starting point of this paper is the following theorem of W. Burnside (Burnside [1]). We state it in the modern form, similar to Lam [10, p. 103].

Theorem 2.1 *(Burnside's Theorem). Let F be a field, V a finite-dimensional F-linear space, and S an F-subalgebra of the algebra $\mathrm{End}_F V$ of linear operators. Suppose that V is a simple left S-module such that $\mathrm{End}_S V$ consists exactly of scalar multiples of the identity operator on V. Then $S = \mathrm{End}_F V$.*

The condition $\mathrm{End}_S V = F\,\mathrm{id}_V$ may not always be omitted if F is not algebraically closed—counter-examples exist for any such a field. If F is algebraically closed, however, this condition is superfluous by Schur's Lemma (see Curtis and Reiner [3, 27.3]). Burnside has proved his result in a different form from first principles by linear algebra: see Burnside [1, p. 433, theorem].

In this paper, Burnside's Theorem is applied to 2-generator subalgebras of $\mathrm{End}_F V$. Therefore, below we restate the theorem for this case.

Theorem 2.2. *$F\langle A, B\rangle = \mathrm{End}_F V$ if and only if the following conditions are satisfied*:

(1) *The only subspaces of V, invariant under both A and B, are* 0 *and V.*
(2) *Only scalar multiples of* id_V *commute with both A and B.*

We need the following lemma that sometimes makes it unnecessary to verify Condition 2 of Theorem 2.2.

Lemma 2.3. *Let L/F be a field extension, then $G_n(L) \cap M_n(F)^2 = G_n(F)$.*

Proof. (1) The inclusion $G_n(L) \cap M_n(F)^2 \subseteq G_n(F)$ holds because linear independence over L implies linear independence over F.

(2) Conversely, let $(A, B) \in G_n(F)$. Then there exist n^2 words $w_1, \ldots, w_{n^2}$ in A, B that form an F-basis of $M_n(F)$. It follows that $w_1, \ldots, w_{n^2}$ form an L-basis of $M_n(L)$. Indeed, E_{ij} form an L-basis of $M_n(L)$, and the two bases are related by an invertible matrix with entries in $F \subseteq L$. □

David Saltman [14] has kindly communicated to us the following local-global principle. To state it, we need the map $\hat{p}: M_n(\mathbb{Z}) \to M_n(\mathbb{F}_p)$ that reduces modulo p every entry of a matrix.

Theorem 2.4. $G_n(\mathbb{Z}) = \bigcap_{p \text{ prime}} \hat{p}^{-1}(G_n(\mathbb{F}_p))$.

Proof. We regard $M = M_n(\mathbb{Z})$ as an additive Abelian group of rank n^2. Consider the subgroup $G = \mathbb{Z}\langle A, B\rangle$. If G is generated by t elements, then their $\hat{p}$-images generate $\hat{p}G$, so that $t \geqslant \dim_{\mathbb{F}_p} \hat{p}G = n^2$. Therefore $t = n^2$, so that the index $k = |M : G|$ is finite.

It remains to see that $k = 1$. Suppose that $k \geqslant 2$. We may choose a subgroup H of M such that $G \subseteq H$ and $h = |M : H|$ is prime. Then $hM \subseteq H$. Therefore $|\mathbb{F}_h^{n^2} : \hat{h}H| = |M/hM : H/hM| = |M : H| = h$, so that $\mathbb{F}_h^{n^2} = \hat{h}G \subseteq \hat{h}H \subsetneqq \mathbb{F}_h^{n^2}$, a contradiction. □

Combining Schur's Lemma, Lemma 2.3, Theorems 2.2 and 2.4 provides a simple method of constructing infinitely many elements (A, B) in $G_n(\mathbb{Z})$ without finding the corresponding $f_{ij} \in \mathbb{Z}\{x, y\}$ such that $E_{ij} = f_{ij}(A, B)$.

Theorem 2.5. *$(A, B) \in G_n(\mathbb{Z})$ if and only if $\bar{\mathbb{F}}_p\langle \hat{p}A, \hat{p}B\rangle x = \bar{\mathbb{F}}_p^n$ for any $0 \neq x \in \bar{\mathbb{F}}_p^n$ and any prime p.*

Example 2.6. $(X, E_{st}) \in G_n(\mathbb{Z})$ for any s and t.

Proof. We apply Theorem 2.5. Let $x = (\alpha_1, \ldots, \alpha_n) = \sum_{i=1}^n \alpha_i e_i \in \bar{\mathbb{F}}_p^n$ be a nonzero column vector. By several applications of X to x, we may assume that $\alpha_t \neq 0$. Then $y = \alpha_t^{-1} Yx = e_s$ and $\{X^i y \mid 1 \leqslant i \leqslant n\} = \{e_1, \ldots, e_n\}$. □

Example 2.7. Let $A = (a_{ij})$, $B = (b_{ij}) \in M_n(\mathbb{Z})$ be such that

(1) $a_{l-1,l} = 1$ for $2 \leqslant l \leqslant n$ and $a_{ij} = 0$ if $i \geqslant j$;
(2) $\{e_1\} \cup \{B^l e_1 \mid 2 \leqslant l \leqslant n\}$ form a $\mathbb{Z}$-basis of $\mathbb{Z}^n$.

Then $(A, B) \in G_n(\mathbb{Z})$.

Proof. Let $x \in \bar{\mathbb{F}}_p^n$ be nonzero, and k be the largest subscript corresponding to a nonzero component of x.

Case 1. If $k = 1$, then $e_1 \in \bar{\mathbb{F}}_p\langle A, B\rangle x$, so that $\{e_1\} \cup \{B^l e_1 \mid 2 \leqslant l \leqslant n\}$ form a $\bar{\mathbb{F}}_p$-basis of $\bar{\mathbb{F}}_p^n$.

Case 2. If $k \geqslant 2$, then $A^{k-1}x$ has the property that its first component is nonzero and all others are zero, so that we return to Case 1. □

These examples clearly imply that the set $G_n(\mathbb{Z})$ is infinite. This also follows from the fact that the set $\{(U^{-1}XU, U^{-1}YU) \mid U \in GL_n(\mathbb{Z})\}$ is infinite. Indeed, the centralizers of X and Y have the following properties: $C_{M_n(\mathbb{Z})}(X) = \mathbb{Z}\langle X\rangle$, and $C_{M_n(\mathbb{Z})}(Y)$ consists of the matrices (a_{ij}) such that $a_{j1} = a_{1j} = 0$ for all $2 \leqslant j \leqslant n$. Therefore, the intersection of the two centralizers with $GL_n(\mathbb{Z})$ is $\{\pm I_n\}$.

Let R be a commutative ring. Following Longstaff [11], we introduce *the minimum spanning length* msl_R for every $(A, B) \in G_n(R)$. Namely, if $(A, B) \in G_n(R)$, then $\mathrm{msl}_R(A, B)$ is the smallest integer s with the property that there exist $w_1, \ldots, w_{n^2} \in FS(x, y)$ with $\max_{1 \leqslant j \leqslant n^2} l(w_j) \leqslant s$, such that $M_n(R) = w_1(A, B)R + \cdots + w_{n^2}(A, B)R$. In the case of fields, Proposition 1 of Longstaff [11] is easily generalized to

Lemma 2.8. *Let F be a field. Then*

$$\max_{(A,B) \in G_n(F)} \mathrm{msl}_F(A, B) \leqslant n^2 - 1. \tag{8}$$

Proof. Let $\mathcal{W}_k$ be the F-linear span of all matrices that may be written as A, B-words of length $\leqslant k$. We see that $\mathcal{W}_k \subseteq \mathcal{W}_{k+1}$. Let m be the smallest value of the subscript stabilizing

this chain. Then $\dim_F \mathcal{W}_1 = 2$, and $\dim_F \mathcal{W}_{l+1} - \dim_F \mathcal{W}_l \geqslant 1$ for any $l \leqslant m-1$. Therefore $m \leqslant n^2 - 1$. □

We extend this result to $\mathbb{Z}$ below.

Theorem 2.9. *Let $A, B \in M_n(\mathbb{Z})$. Then $(A, B) \in G_n(\mathbb{Z})$ if and only if the rows of the matrix $T_{n^2-1,n^2,\mathbb{Z}}(A, B)$ span $M_n(\mathbb{Z})$.*

Proof. It suffices to prove that the condition is necessary. Let $(A, B) \in G_n(\mathbb{Z})$. Then $(A, B) \in G_n(\mathbb{F}_p)$ for every prime p. Therefore by Lemma 2.8, there exists a nonzero n^2-by-n^2 minor of $T_{n^2-1,n^2,\mathbb{F}_p}(\hat{p}A, \hat{p}B)$. Let $w_1, \ldots, w_{n^2} \in FS(x, y)$ be the words giving rise to this minor, and let $H_p = \sum_{k=1}^{n^2} w_k(A, B)\mathbb{Z}$. Then the group $H = \sum_{p \text{ prime}} H_p$ has the property that $\hat{p}H = M_n(\mathbb{F}_p)$ for every prime p. At the same time, H is a subgroup of the group generated by all row-vectors of $T_{n^2-1,n^2,\mathbb{Z}}(A, B)$. It remains to apply Theorem 2.4 and Lemma 2.8. □

The inequality (8) is not sharp, even for $n = 2$, because Proposition 2 on p. 250 of Longstaff [11] implies $\max_{(A,B)\in G_2(\mathbb{C})} \operatorname{msl}_{\mathbb{C}}(A, B) = 2$. This is true over any field: to modify the proof, in the last paragraph of Lemma 1 of Longstaff [11], we propose to replace taking adjoints with taking transposes. The paper Longstaff [11] contains an intriguing and well substantiated conjecture that $\max_{(A,B)\in G_n(\mathbb{C})} \operatorname{msl}_{\mathbb{C}}(A, B) \leqslant 2n - 2$.

2.1. Description of $G_2(\mathbb{Z})$

We relate below the elements of $G_2(\mathbb{Z})$ to the solutions of the Diophantine equation (9).

Theorem 2.10. *Let $A, B \in M_2(\mathbb{Z})$. Put $I = I_2$ and $\mathcal{S} = \mathbb{Z}\langle A, B\rangle$. Then*

(1) *$I \in \mathcal{S}$ if and only if* $\gcd(\det A, \det B, \det(A + B)) = 1$.
(2) *$\mathcal{S} = M_2(\mathbb{Z})$ if and only if I, A, B, AB generate $M_2(\mathbb{Z})$ as a $\mathbb{Z}$-module.*

If I, A, B generate $M_2(\mathbb{Z})$ as a ring, then their $\mathbb{Z}$-linear combinations produce I, A_1, B_1 also generating $M_2(\mathbb{Z})$ such that

$$A_1 = \begin{pmatrix} c & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad B_1 = \begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix}$$

where $\gcd(a, b) = 1$. *Moreover, the matrices I, A_1, B_1 generate $M_2(\mathbb{Z})$ if and only if*

$$a^2 - abc - b^2 = \pm 1. \tag{9}$$

The set of solutions of these equations is infinite, and when $abc \neq 0$, this set is effectively described in terms of the unit group of the field $\mathbb{Q}(\sqrt{c^2+4})$.

Proof. The Cayley–Hamilton Theorem successively applied to the matrices A, B, $A + B$ yields $\det(A)I, \det(B)I, \det(A + B)I \in \mathcal{S}$. Since in addition,

$$\begin{aligned} BA &= (A+B)^2 - A^2 - B^2 - AB \\ &= \operatorname{tr}(A+B)(A+B) - \det(A+B)I - \operatorname{tr}(A)A + \det(A)I - \operatorname{tr}(B)B + \det(B)I - AB, \end{aligned}$$

we conclude that

$$\mathcal{S} = g\mathbb{Z}I + \mathbb{Z}A + \mathbb{Z}B + \mathbb{Z}AB, \quad \text{where } g = \gcd\big(\det A, \det B, \det(A+B)\big). \tag{10}$$

If $g \geqslant 2$, then reducing (10) modulo g, we obtain a contradiction for reasons of cardinality. Therefore $g = 1$, and $\mathcal{S} = M_2(\mathbb{Z})$ if and only if Conditions 1 and 2 above are satisfied.

Now suppose that I, A, B generate the ring $M_2(\mathbb{Z})$. Let

$$A = (x_{ij}), \qquad B = (y_{ij}).$$

Since I, A, B generate $M_2(\mathbb{Z})$ modulo any integer m, we conclude that $\gcd(x_{12}, y_{12}) = 1$. Let a, b be integers such that $ax_{12} + by_{12} = 1$. Then

$$A' = aA + bB = \begin{pmatrix} x'_{11} & 1 \\ x'_{21} & x'_{22} \end{pmatrix}, \qquad B' = B - y_{12}A' = \begin{pmatrix} y'_{11} & 0 \\ y'_{21} & y'_{22} \end{pmatrix}$$

and therefore I, A', B' generate $M_2(\mathbb{Z})$. We use the identity matrix I to obtain

$$A'' = A' - x'_{22}I = \begin{pmatrix} x''_{11} & 1 \\ x'_{21} & 0 \end{pmatrix}, \qquad B'' = B' - y'_{22}I = -\begin{pmatrix} y''_{11} & 0 \\ y'_{21} & 0 \end{pmatrix}.$$

Again, I, A'', B'' generate $M_2(\mathbb{Z})$. We rewrite A'' and B'' as A and B, respectively; that is, we may assume

$$A = \begin{pmatrix} x_{11} & 1 \\ x_{21} & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} y_{11} & 0 \\ y_{21} & 0 \end{pmatrix}.$$

Let c, d be integers such that $cx_{21} + dy_{21} = 1$. We may replace A by

$$A' = cA + dB = \begin{pmatrix} x'_{11} & c \\ 1 & 0 \end{pmatrix}.$$

Therefore $c = \pm 1$. We will only treat the case $c = 1$. Thus we may assume

$$A = \begin{pmatrix} x_{11} & 1 \\ 1 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} y_{11} & 0 \\ y_{21} & 0 \end{pmatrix}, \qquad \gcd(y_{11}, y_{21}) = 1.$$

We want to determine when the $\mathbb{Z}$-span of I, A, B, AB is $M_2(\mathbb{Z})$. If E_{11} is a linear combination of I, A, B, AB then $E_{12} + E_{21} \in \langle I, A, B\rangle$, and therefore $\langle I, A, B\rangle = M_2(\mathbb{Z})$.

Let a, b, c, d be integers such that $aI + bA + cB + dAB = E_{11}$. As

$$aI + bA + cB + dAB = \begin{pmatrix} a + bx_{11} + cy_{11} + d(x_{11}y_{11} + y_{21}) & b \\ b + cy_{21} + dy_{11} & a \end{pmatrix},$$

the above equation has a solution if and only if

$$a = b = 0, \qquad dy_{11} = -cy_{21}, \qquad cy_{11} + d(x_{11}y_{11} + y_{21}) = 1.$$

If $y_{11} = 0$, then $dy_{21} = 1$; therefore $y_{21} = d = \pm 1$ and $c = 0$. Similarly if $y_{21} = 0$, then $y_{11} = c = \pm 1$ and $d = 0$.

We assume $y_{11}, y_{21} \neq 0$. Therefore $c, d \neq 0$, and since $\gcd(y_{11}, y_{21}) = 1$, from $dy_{11} = -cy_{21}$ we conclude that there exists an integer c' such that

$$c = c'y_{11}, \qquad d = -c'y_{21}.$$

The equation $cy_{11} + d(x_{11}y_{11} + y_{21}) = 1$ yields $c'(y_{11}^2 - x_{11}y_{21}y_{11} - y_{21}^2) = 1$ therefore $y_{11}^2 - x_{11}y_{21}y_{11} - y_{21}^2 = \pm 1$. It remains to write $a = y_{11}$, $b = y_{21}$, $c = x_{11}$, and we obtain (9). Since it is easy to analyze the solutions when one of a, b, c is zero, we will investigate the other solutions only. Equation (9) is quadratic in a; therefore, a necessary condition for (9) to have integral solutions is that the equation

$$d^2 = (bc)^2 + 4\big(b^2 \pm 1\big) \tag{11}$$

should have integral solutions too. If this is so, then

$$a = \frac{bc \pm d}{2}. \tag{12}$$

From (11) we observe that $d \equiv d^2 \equiv (bc)^2 \equiv bc \pmod 2$. In other words, (11) implies (12). Now (11) may be rewritten as

$$d^2 - \big(c^2 + 4\big)b^2 = \pm 4. \tag{13}$$

Let s be the square-free part of the number $c^2 + 4$. Then according to Fröhlich and Taylor [5, 1.3], the units of $\mathbb{Q}(\sqrt{c^2 + 4})$ uniquely, under the map $(d, b) \mapsto (1/2)(d + b\sqrt{c^2 + 4})$, correspond to the integral solutions of (13). There are infinitely many of them by the Dirichlet's Unit Theorem. Algorithm 5.7.2 in Cohen [2] computes the fundamental unit of a rational quadratic number field with positive discriminant.

Therefore, for a fixed c, we can produce units in $\mathbb{Q}(\sqrt{c^2 + 4})$, thus determining b and d; then a may found from (12). □

2.2. *Asymptotic properties of $G_n(\mathbb{Z})$*

Lemma 2.11. *Let $0 \neq f \in \mathbb{Z}[x_1, \ldots, x_n]$. Then $V(f) = \{a \in \mathbb{Z}^n \mid f(a) = 0\}$ has zero asymptotic density in $\mathbb{Z}^n$.*

Proof. Put $B_k = \{(a_1, \ldots, a_n) \in \mathbb{Z}^n \mid -k \leqslant a_i \leqslant k \text{ for all } i\}$. The case $n = 1$ is clear because $\#B_k \leqslant \deg(f)$ for all k.

Let $n = 2$, $x = x_1$, $y = x_2$ and $d = \deg(f)$. Then $f(x, y) = \sum_{j=1}^{d} f_j(x)y^j$ for some $f_j(x) \in \mathbb{Z}[x]$. Let $S = \{-k \leqslant a \leqslant k \mid f_j(a) = 0 \text{ for all } j\}$. Then $\#S \leqslant d$. We may write $V(f) = A \cup B$, where

$$A = \big\{(a, b) \in V(f) \,\big|\, a \in S\big\} \quad \text{and} \quad B = \big\{(a, b) \in V(f) \,\big|\, a \notin S\big\}.$$

If $a \in \{-k, \dots, k\} - S$, then $\#\{b \mid (a,b) \in B\} \leqslant d$. Hence,

$$\#V(f) \leqslant \#A + \#B \leqslant (\#S)\big(\#\{-k,\dots,k\}\big) + \#\big(\{-k,\dots,k\} - S\big)d = O(k).$$

Since $\#B_k = (2k+1)^2$, we conclude that the lemma is true when $n = 2$.

The case $n \geqslant 3$ is handled similarly by induction on n. □

The exponent of k in the estimate $\#(B_k \cap V(f))/\#B_k = O(k^{-1})$ in the proof of Lemma 2.11 is the best possible in general, as exemplified by the polynomial $f(x_1, \dots, x_n) = x_1$.

Corollary 2.12. *The set $M_n(\mathbb{Z})^n - G_n(\mathbb{Z})$ is not algebraic.*

Proof. Suppose that the theorem is false. Then Lemma 2.11 implies that $G_n(\mathbb{Z})$ has asymptotic density 1 in $M_n(\mathbb{Z})^2$. This is false, however, because $M_n(2\mathbb{Z})^2 \subseteq M_n(\mathbb{Z})^2 - G_n(\mathbb{Z})$, and $M_n(2\mathbb{Z})^2$ has asymptotic density 2^{-2n^2} in $M_n(\mathbb{Z})^2$, implying that $G_n(\mathbb{Z}) \cap M_n(2\mathbb{Z})^2$ is nonempty. □

In case $n = 2$, we obtain the following more precise result.

Theorem 2.13. *The set $G_2(\mathbb{Z})$ has zero asymptotic density in $M_2(\mathbb{Z})^2$.*

Proof. Put $I = I_2$. Let $A, B \in M_2(\mathbb{Z})$ such that I, A, B generate $M_2(\mathbb{Z})$ as a ring. Put $\mathcal{S} = \mathbb{Z}\langle A, B\rangle$. The Cayley–Hamilton Theorem applied to the matrices A, B, $A + B$ yields that $A^2, B^2, (A+B)^2$ are integral linear combinations of I, A, B. Since in addition, $BA = (A+B)^2 - A^2 - B^2 - AB$, we conclude that $\mathcal{S} = \mathbb{Z}I + \mathbb{Z}A + \mathbb{Z}B + \mathbb{Z}AB$. Let T be a 4-by-4 matrix whose rows are the flattened matrices I, A, B, and AB. Then $\mathcal{S} = M_2(\mathbb{Z})$ if and only if $\det T = \pm 1$. It remains to apply Lemma 2.11. □

This result sometimes clarifies the relationship between $G_2(\mathbb{Z})$ and the other subsets of $M_n(\mathbb{Z})^2$. We will give an example. Let S be set of all $(A, B) \in M_2(\mathbb{Z})^2 - G_2(\mathbb{Z})$ such that all the 8 entries are relatively prime in pairs. We will see that asymptotically, almost all elements of S lie outside of $G_2(\mathbb{Z})$. To formalize this statement, let $m_k = \prod_{p \text{ prime},\ p \leqslant k} p$ and

$$D_k = \Big\{(a_1, \dots, a_8) \in \mathbb{Z}^8 \colon \max_{1 \leqslant i \leqslant 8} |a_i| \leqslant m_k\Big\}.$$

We claim that

$$\lim_{k\to\infty} \frac{\#S \cap D_k}{\#D_k} = \prod_{p \text{ prime}} (p-1)^7(p+7)p^{-8} > 0. \tag{14}$$

We give a heuristic argument first. For a fixed prime p, we consider the Bernoulli scheme of choosing 8 integers independently and at random with the probability of success p^{-1}. Then the probability of at most 1 success is $(1-p^{-1})^8 + \binom{8}{1}p^{-1}(1-p^{-1})^7 = (p-1)^7(p+7)p^{-8}$. Taking the product over all primes gives (14).

Next we prove (14). We thank Doug Hensley [7] for communicating the following argument to us. It is convenient to decrease the sets S and D_k to retain only the 8-tuples with all positive entries. For a prime p, let S_p be the set of all 8-tuples $(a_1, \dots, a_8)$ whose entries are positive

integers, and $p \nmid \gcd(a_i, a_j)$ if $i \neq j$. Then $S = \bigcap_p S_p$. The Chinese Remainder Theorem applied to the ring $\mathbb{Z}/m_k\mathbb{Z}$ implies

$$\frac{\#S \cap D_k}{\#D_k} \leqslant \frac{\#\bigcap_{p\leqslant k} S_p \cap D_k}{\#D_k} = \prod_{p\leqslant k} (p-1)^7(p+7)p^{-8}. \tag{15}$$

For the primes $p > k$, we have $\#S_p \cap D_k \leqslant \binom{8}{1} m_k \lfloor m_k/p \rfloor^7$. Therefore

$$\frac{\#S \cap D_k}{\#D_k} \geqslant \frac{\#\bigcap_{p\leqslant k} S_p \cap D_k}{\#D_k} - \sum_{p>k} \frac{\#S_p \cap D_k}{\#D_k} = \prod_{p\leqslant k} (p-1)^7(p+7)p^{-8} + o(1). \tag{16}$$

Comparing (15) and (16) yields (14).

2.3. Asymptotic and topological properties of $G_n(F)$ for fields

Lemma 2.14. *Let F be a field. Then $M_n(F)^2 - G_n(F)$ is a nonempty algebraic set consisting of all $(A, B) \in M_n(F)^2$ such that the matrix $T_{n^2-1,n^2,F}(A, B)$ does not have full rank.*

Proof. The equality of the two sets above follows from Lemma 2.8. The set $G_n(F)$ is nonempty because $G_n(\mathbb{Z})$ is nonempty. □

Next, we will apply Lemma 2.14 to normed fields satisfying the following

Property 2.15. *F is a normed field (with the norm denoted by $|\cdot|$) such that for any $\varepsilon > 0$ there exists $0 \neq a_\varepsilon \in F$ with $|a_\varepsilon| < \varepsilon$.*

Among the fields having Property 2.15 are all the subfields of $\mathbb{C}$ or $\mathbb{C}_p$ with their respective standard Euclidean or p-adic norms.

Lemma 2.16. *Let F have Property* 2.15*, and let $Z \subsetneqq F^n$ be an algebraic set. Then $F^n - Z$ is dense in F^n in the norm topology.*

Proof. Let $z \in Z$. We show that there exists a sequence $\{z_n\}$ in $F^n - Z$ with $\lim_{n\to\infty} \|z - z_n\| = 0$. Since $Z \subsetneqq F^n$, there exists a line L_z passing though z and not contained in F^n. Substituting the parametric equations for L_z into the polynomial equations defining Z, we obtain a system of equations in one variable, which has finitely many solutions, one of them being z. We may choose $\varepsilon > 0$ sufficiently small to ensure that z is the only solution contained in the ball $B_\varepsilon(z)$ of radius ε and centered at z. Then there exists a sequence $\{z_n\}$ in $B_\varepsilon(z) \cap L_z$ such than $z_n \neq z$ and $\lim_{n\to\infty} \|z - z_n\| = 0$. In particular $z_n \in F^n - Z$. □

Theorem 2.17. *Let F have Property* 2.15*. Then $G_n(F)$ is open and dense in $M_n(F)^2$ in the norm topology.*

Proof. The result follows from Lemmas 2.14 and 2.16. □

Next we consider similar results for finite fields.

Lemma 2.18. *Let* $0 \neq f \in \mathbb{F}_q[x, y]$ *and* $V(f) = \{v \in \mathbb{F}_q^2 \mid f(v) = 0\}$. *Then* $\#V(f) \leqslant 2q \deg(f)$.

Proof. Let $d = \deg(f)$. Then $f(x, y) = \sum_{j=0}^{d} f_j(x) y^j$ for some $f_j(x) \in \mathbb{F}_q[x]$. Let $S = \{a \in \mathbb{F}_q \mid f_0(a) = \cdots = f_d(a) = 0\}$. Then $\#S \leqslant d$.

For every $a \in S$, there are at most q values of $b \in \mathbb{F}_q$ such that $(a, b) \in V(f)$. Let $A = \{(a, b) \in V(f) \mid a \in S\}$. Then $\#A \leqslant qd$.

Next let $B = \{(a, b) \in V(f) \mid a \notin S\}$. Then there are at most d values of $b \in \mathbb{F}_q$ such that $(a, b) \in V(f)$ for some $a \in \mathbb{F}_q$. Then $\#B \leqslant qd$.

Finally, $V(f) = A \cup B$, so that $\#V(f) \leqslant \#A + \#B \leqslant 2qd$. □

Theorem 2.19. *For a fixed* $n \geqslant 2$, *we have*

$$\lim_{q \to \infty} \frac{\#G_n(\mathbb{F}_q)}{\#M_n(\mathbb{F}_q)^2} = 1.$$

Proof. By Lemma 2.14, the set $M_n(\mathbb{F}_q)^2 - G_n(\mathbb{F}_q)$ is an intersection of finitely many hypersurfaces, each of them having $O(q^{2n^2-1})$ points over $\mathbb{F}_q$ by Lemma 2.18. Each such a hypersurface is defined by a polynomial equation in $2n^2$ variables with coefficients in $\mathbb{Z}$, the equations being independent of $\mathbb{F}_q$. It follows that

$$1 \geqslant \frac{\#G_n(\mathbb{F}_q)}{\#M_n(\mathbb{F}_q)^2} = 1 - \frac{\#(M_n(\mathbb{F}_q)^2 - G_n(\mathbb{F}_q))}{\#M_n(\mathbb{F}_q)^2} \geqslant 1 - \frac{O(q^{2n^2-1})}{q^{2n^2}} \xrightarrow[q \to \infty]{} 1. \qquad \square$$

However, we do not know whether the following limit exists:

$$\lim_{n,q \to \infty} \frac{\#G_n(\mathbb{F}_q)}{\#M_n(\mathbb{F}_q)^2}. \tag{17}$$

Lemma 2.14 together with Theorems 2.2, 2.17, and 2.19 imply that the set of $(A, B) \in M_n(F)^2$ having a proper common invariant subspace, is small in the appropriate sense. We note that our arguments do not involve characteristic polynomials.

3. Presentations of $M_n(\mathbb{Z})$ and their applications

We begin by recalling the definitions of the matrices $X = \sum_{i=1}^{n} E_{i+1,i}$ and $Y = E_{11}$ for some fixed $n \geqslant 2$, and the noncommutative polynomials

$$r_{1,n} = r_{1,n}(x) = x^n - 1, \qquad r_{2,n} = r_{2,n}(x, y) = \sum_{i=0}^{n-1} x^{n-i} y x^i - 1,$$
$$s_0 = s_0(y) = y^2 - y, \qquad s_j = s_j(x, y) = y x^j y \quad \text{for } j \geqslant 1.$$

Theorem 3.1. *The ring* $M_n(\mathbb{Z})$ *has the following presentations*:

$$\langle x, y \mid r_{1,n} = r_{2,n} = s_m = 0,\ 1 \leqslant m \leqslant n-1 \rangle, \tag{18}$$
$$\big\langle x, y \,\big|\, r_{1,n} = r_{2,n} = s_0 = s_k = 0,\ 1 \leqslant k \leqslant \lfloor n/2 \rfloor \big\rangle. \tag{19}$$

Both ring isomorphisms are obtained by mapping x *to* X *and* y *to* Y.

Proof. We see that X and Y satisfy all the relations of (18) and (19).

Next we prove that (18) is a presentation of $M_n(\mathbb{Z})$. To fix the notation, let $\mathcal{R}$ be the ring defined by (18). We observe that

$$1 \cdot y = \left(\sum_{i=0}^{n-1} x^{n-i} y x^i \right) y = y^2 + \sum_{i=1}^{n-1} x^{n-i} \left(y x^i y \right) = y^2.$$

Therefore, $\mathcal{R}$ is spanned as an Abelian group by the n^2 elements $x^i y x^j$ where $1 \leqslant i, j \leqslant n$; hence $\dim_{\mathbb{Z}} \mathcal{R} \leqslant n^2$. On the other hand, the map α given by $\alpha(x) = X$ and $\alpha(y) = Y$ extends to the ring epimorphism $\alpha : \mathcal{R} \twoheadrightarrow M_n(\mathbb{Z})$ because $E_{ij} = X^{i-1} Y X^{1-j}$.

It remains to show that (19) is a presentation of $M_n(\mathbb{Z})$. Since all the relations of (19) hold in (18), it remains to establish the converse. We propose to consider the cases of n even and odd separately. The arguments involved in either of them are the same; therefore, we will do only the case when $n = 2s + 1$ is odd. Multiplying the relation $1 = \sum_{i=0}^{n-1} x^{n-i} y x^i$ by y on the right yields

$$\begin{aligned} y = 1y = y^2 + x^{n-1}(yxy) + x^{n-2}\left(yx^2y\right) + \cdots + x^{n-s+1}\left(yx^s y\right) \\ + x^{n-s} y x^{s+1} y + \cdots + xyx^{n-1} y. \end{aligned} \tag{20}$$

Since $y^2 = y$ and $yxy = yx^2y = \cdots = yx^s y = 0$, and x is invertible, the formula (20) shortens:

$$yx^{n-1}y + xyx^{n-2}y + \cdots + x^s y x^{s+1} y = 0. \tag{21}$$

Multiplying (21) on the left by y, as before, yields

$$yx^{n-1}y = 0, \tag{22}$$

which is partly what we need. Now substitute (22) in (21), cancel by x on the left, and then multiply by y on the left. The result is $yx^{n-2}y = 0$. In a similar fashion, it follows that all $s_j(x, y) = 0$ for all j. □

The next theorem shows that Presentation 19 for $n = 4, 5$ may be shortened.

Theorem 3.2.

$$M_4(\mathbb{Z}) \cong \langle x, y \mid r_{1,4} = r_{2,4} = s_0 = s_1 = 0 \rangle, \tag{23}$$

$$M_5(\mathbb{Z}) \cong \langle x, y \mid r_{1,5} = r_{2,5} = s_0 = s_1 = 0 \rangle. \tag{24}$$

Proof. (1) To prove (23), observe that $0 = y r_{2,4} = s_3 x + s_2 x^2$, so that $s_3 = -s_2 x$ and $s_3 = s_3 y = -s_2 xy = -yx^2 y(xy) = 0$. Therefore $s_2 = s_3 = 0$, and the result follows from Theorem 3.1.

(2) We prove (24) in several steps.

$$\begin{aligned} 0 = y r_{1,5} = y + s_4 x + s_3 x^2 + s_2 x^3 + s_1 x^4 - y = s_4 x + s_3 x^2 + s_2 x^3 + s_1 x^4 = 0 \\ \Longrightarrow \quad s_4 + s_3 x + s_2 x^2 = 0. \end{aligned} \tag{25}$$

Similarly, by expanding $0 = r_{2,5}y$ we have

$$s_4 + xs_3 + x^2 s_2 = 0. \tag{26}$$

Multiply (26) by y on the right:

$$s_4 + s_2^2 = 0. \tag{27}$$

Equate (25) and (26): $s_3x + s_2x^2 = xs_3 + x^2s_2$, and then multiply the result by y on the right: $s_2^2 = xs_3 + x^2s_2$ implying

$$s_3 = x^4 s_2^2 - xs_2. \tag{28}$$

Multiply (28) by y on the left $s_3 = ys_3 = yx^4s_2 - yxs_2 = s_4s_2^2$ and use (27):

$$s_3 = -s_2^4. \tag{29}$$

Substitute (29) in (29):

$$-s_2^4 = x^4 s_2^2 - xs_2. \tag{30}$$

Multiply (30) by yx^2 on the left and then use (29):

$$\begin{aligned} -yx^2s_2^4 = yx^2x^4s_2^2 - yx^2xs_2 \quad &\Longrightarrow \quad -s_2^5 = -s_3s_2 = -\big(-s_2^4\big)s_2 = s_2^5 \\ &\Longrightarrow \quad 2s_2^5 = 0. \end{aligned} \tag{31}$$

Multiply (30) by yx^4 on the left: $-s_4s_2^4 = s_3s_2^2 - s_2$. Then by (27) and (29): $s_2^6 = s_3s_2^2 - s_2 = (-s_2^4)s_2^2 - s_2$. Finally, by (31): $s_2 = -2s_2^6 = -s_2(2s_2^5)$, and the claim follows from Theorem 3.1. □

Next we record some properties of Presentations (18) and (19) in connection with their minimality.

Theorem 3.3.

(1) *The ring* $\langle x, y \mid r_{2,n} = s_j = 0,\ 1 \leqslant j \leqslant n-1\rangle$ *has infinite rank.*
(2) *The ring* $\langle x, y \mid r_{1,n} = s_j = 0,\ 1 \leqslant j \leqslant n-1\rangle$ *has infinite rank.*
(3) $\langle x, y \mid r_{1,n} = r_{2,n} = 0\rangle \ncong M_n(\mathbb{Z})$.
(4) *If* $1 \leqslant k \leqslant n-1$ *and* $k \neq n/2$, *then the relation* $s_k = 0$ *follows from the other relations in* (18). *In particular, this explains why* (3) *is a presentation of* $M_3(\mathbb{Z})$.
(5) *Removing from* (18) *any two relations* $s_h = s_{n-h} = 0$ *results in a ring of an infinite rank.*
(6) *Removing from* (18) *any two relations* $s_h = s_{2h} = 0$, *provided* $1 \leqslant h < 2h \leqslant n-1$, *results in a ring of an infinite rank.*

Proof. (1) Let $\mathbb{Z}(t)$ be the ring of rational functions in t with integral coefficients. Consider the matrices $A = t\sum_{i=0}^{n-1} E_{i+1,i}$ and $B = (1/t^n)E_{11}$. Let $\mathcal{R}$ be the subring of $M_n(\mathbb{Z}(t))$ generated by A and B. These matrices satisfy all the relations of $\mathcal{R}$. At the same time, $A^n = t^n I \in \mathcal{R}$, so that $\mathcal{R}$ contains $\sum_{k=1}^{\infty} t^{kn} I$, an Abelian subgroup of infinite rank.

(2) Consider the matrices $A = \sum_{i=0}^{n-1} E_{i+1,i}$ and $B = tE_{11}$. Let $\mathcal{R}$ be the subring of $M_n(\mathbb{Z}[t])$ generated by A and B. These matrices satisfy all the relations of $\mathcal{R}$. At the same time, $\sum_{i=0}^{n-1} A^{-i} B A^i = t I_n \in \mathcal{R}$, and as above, we conclude that $\mathcal{R}$ has infinite rank.

(3) Suppose the claim is false. Then by mapping y to zero, we have $M_n(\mathbb{Z}) \cong \langle x, y \mid r_{1,n} = r_{2,n} = 0\rangle \twoheadrightarrow \mathbb{Z}[x]/(x^n - 1)$, but the ring $M_n(\mathbb{Z})$ does not a have proper ideal of infinite index.

(4) We need to show that the relation $yx^k y = 0$ follows from the other relations of (18). We have

$$0 = r_{2,n} y = y^2 + x^{-k} y x^k y - y \quad \text{and} \quad 0 = y r_{2,n} = y^2 + y x^k y x^{-k} - y.$$

Hence

$$y - y^2 = y x^k y x^{-k} = x^{-k} y x^k y. \tag{32}$$

Next, we work with the expressions $y(y - y^2)$ and $(y - y^2)y$ with the help of (32). We see that on the one hand, $y(y - y^2) = (yx^{-k}y)x^k y = 0$, and on the other hand $y(y - y^2) = y^2 x^s y x^{-k}$. Therefore $y^2 x^k y x^{-k} = 0$, and since x is invertible,

$$y^2 x^k y = 0. \tag{33}$$

Likewise, $(y - y^2)y = x^{-k} y x^k y^2 = y x^k (y x^{-k} y) = 0$, so that

$$y x^k y^2 = 0. \tag{34}$$

Applying (32), (33), and (34) yields

$$y x^k y = y x^k \big(y - y^2\big) = y x^k \big(x^{-k} y x^k y\big) = y^2 x^k y = 0.$$

(5) It suffices to give an example of the ring of infinite rank, where all the relations of (18) are satisfied except for $yx^h y = yx^{n-h} y = 0$.

Let $\mathbb{Z}[t]$ be a polynomial ring, X be the permutational matrix of oder n acting on columns, and $Y_1 = tE_{11} + (1 - t)E_{1+h,1+h}$. We denote by $\mathcal{R}$ the subring of $M_n(\mathbb{Z}[t])$ generated by X and Y_1. If $1 \leqslant i \leqslant n - 1$, then

$$X^i Y_1 X^{-i} = tE_{1+i,1+i} + (1 - t)E_{1+h+i,1+h+i} \tag{35}$$

implying $\sum_{i=0}^{n-1} X^i Y_1 X^{-i} = I$. Next, multiply (35) by Y on the left:

$$Y_1 X^i Y_1 X^{-i} = t(1 - t)(E_{11} E_{1+h+i,1+h+i} + E_{1+h,1+h} E_{1+i,1+i}). \tag{36}$$

We see that $Y_1 X^i Y_1 X^{-i} = 0$, and therefore $Y_1 X^i Y_1 = 0$, unless $i = \pm h$. In the latter cases we have that $Y_1 X^h Y_1 X^{-h} = t(1 - t)E_{11}$ and $Y_1 X^{-h} Y_1 X^h = t(1 - t)E_{1+h,1+h}$. Therefore, in $\mathcal{R}$ all the relations of (18) are satisfied except for $yx^h y = yx^{n-h} y = 0$. Another consequence of (36) is $t(t - 1)I_n \in \mathcal{R}$ because $\sum_{i=1}^{n-1} X^{-i}(Y_1 X^h Y_1 X^{-h})X^i = t(1 - t)\sum_{i=1}^{n-1} X^{-i} E_{11} X^i = t(t - 1)I_n$.

Therefore, $\mathcal{R}$ contains an Abelian subgroup of infinite rank, implying that the rank of $\mathcal{R}$ is infinite as well.

(6) As above, it suffices to give an example of the ring of infinite rank, where all the relations of (18) are satisfied except for $yx^h = yx^{2h}y = 0$, provided $1 \leqslant h < 2h \leqslant n-1$.

Let $\mathbb{Z}[t]$ be a polynomial ring, X be the permutational matrix of order n acting on columns, and $Y_1 = E_{11} + tE_{1,1+h} - tE_{1-h,1}$.

The relation $\sum_{i=0}^{n-1} X^i Y_1 X^{-i} = I_n$ is satisfied because the subscripts $(1, 1+h)$ and $(1-h, 1)$ are in the same orbit of X.

Next we investigate the monomial relations.

$$Y_1 X^i Y_1 X^{-i} = (E_{11} + tE_{1,1+h} - tE_{1-h,1})(E_{1+i,1+i} + tE_{1+i,1+h+i} - tE_{1+i-h,1+i}). \quad (37)$$

On multiplying out, we see that (37) is zero unless $i = h, 2h$. In the latter two cases, we have that

$$Y_1 X^h Y_1 X^{-h} = t^2(E_{1,1+2h} + E_{1-h,1+h}) \quad \text{and} \quad Y_1 X^{2h} Y_1 X^{-2h} = -t^2 E_{1,1+h}.$$

Finally, $-\sum_{i=0}^{n-1} X^{-i}(Y_1 X^{2h} Y_1 X^{-2h}) X^i = \sum_{i=0}^{n-1} X^{-i} t^2 E_{1,1+h} X^i = t^2 X^{1-h}$. Therefore, the ring generated by X and Y_1 has infinite rank. □

The above theorem describes some situations (with the possible exception of part (3)) where the removal of certain relations results in a ring of infinite rank. In contrast, the theorem below gives two instances in which the removal of certain relations results in a ring of finite rank.

Theorem 3.4.

(1) *The ring* $\mathcal{R} = \langle x, y \mid r_{1,n} = s_m = 0,\ 0 \leqslant m \leqslant n-1 \rangle$ *is isomorphic to a direct sum of the rings* $M_n(\mathbb{Z})$ *and* $\mathbb{Z}C_n$.

(2) *Let* $\emptyset \neq H \subsetneqq N = \{1, 2, \ldots, n-1\}$ *and* $H' = N - H$. *Suppose that* H *satisfies the following conditions modulo* n:
   (a) $\{a + b \mid a, b \in -H \cup H\} \subseteq H'$.
   (b) *If* $h, k, l, -h + k + l \in H$, *then* $h = k$ *or* $h = l$.
   Then the ring $S(H) = \langle x, y \mid r_{1,n} = r_{2,n} = s_j = 0,\ j \in H' \rangle$ *has finite rank.*

Proof. We prove the two claims of the theorem in the two respective parts below.

(1) Firstly, $r_2 \mathbf{y} = \mathbf{y} r_2 = 0$, $r_2 \mathbf{x} = \mathbf{x} r_2$, $(-r_2)^2 = -r_2$. Therefore, $r = -r_2$ is a central idempotent, and $\mathcal{R} = r\mathcal{R} \oplus (1-r)\mathcal{R} = r\mathbb{Z}\langle \mathbf{x} \rangle \oplus (1-r)\mathcal{R}$ where $(1-r)\mathcal{R} \cong M_n(\mathbb{Z})$, and $r\mathbb{Z}\langle \mathbf{x} \rangle \cong \mathbb{Z}C_n$.

(2) We construct a finite set, call it $\mathcal{S}$, such that every element of $S(H)$ may be written as an integral linear combination of the elements of $\mathcal{S}$.

Multiply the relation $r_{2,n}(x, y) = 0$ by y on the left:

$$y^2 + \sum_{i=1}^{n-1} yx^{-i} yx^i - y = 0 \quad \Longrightarrow \quad y^2 - y = -\sum_{h \in H} yx^h yx^{-h}. \quad (38)$$

Therefore, for $k \in H$, we have

$$\left(y - y^2\right) x^k y = \sum_{h \in H} yx^h yx^{-h} x^k y = yx^k y^2. \quad (39)$$

Multiply the relation $r_{2,n}(x, y) = 0$ by y on the right:

$$y^2 + \sum_{i=1}^{n-1} x^{-i} y x^i y - y = 0 \quad \Longrightarrow \quad y^2 - y = -\sum_{h \in H} x^{-h} y x^h y. \tag{40}$$

It follows that

$$y^3 - y^2 = -y \sum_{h \in H} x^{-h} y x^h y = 0. \tag{41}$$

Therefore, multiplying (39) by y on the left yields

$$y^2 x^k y^2 = -\left(y^3 - y^2\right) x^k y = 0. \tag{42}$$

Let $k \in H$, then equating the right-hand sides of (38) and (40) gives us

$$y x^k y x^{-k} = -\sum_{k \neq h \in H} y x^h y x^{-h} + \sum_{h \in H} x^{-h} y x^h y. \tag{43}$$

Next, multiplying (43) by $y x^l$ on the left and by x^k on the right yields

$$y x^l y x^k y = -\sum_{h \in H,\, h \neq k} y x^l y x^h y x^{-h+k} + \sum_{h \in H} y x^l x^{-h} y x^h y x^k = y^2 x^l y x^k. \tag{44}$$

We conclude that every word in x and y may be rewritten in such a way that the following conditions are satisfied:

(1) x occurs finitely many times with exponent between $0, \ldots, n-1$, because one of the relation in (18) is $x^n = 1$.
(2) Powers of y may occur as subwords at most twice because of (44).
(3) y occurs with exponent between 0, 1, 2 because $y^3 = y^2$ by (41).

Stated another way, every element in $S(H)$ may be written as $\mathbb{Z}$-linear combination of the words of the form $x^{\alpha_1} y^{\beta_1} x^{\alpha_2} y^{\beta_2} x^{\alpha_3}$, where $\alpha_1, \alpha_2, \alpha_3 \in \{0, \ldots, n-1\}$ and $\beta_1, \beta_2 \in \{0, 1, 2\}$. □

3.1. Magnus-type ring extension of $M_n(\mathbb{Z})$

In the proof of Theorem 3.5 below, we introduce an analog of the Magnus Embedding from Magnus [12] (see lemma on p. 764 of [12]).

Theorem 3.5. *The ring $\mathbb{Z}\{x, y\}$ has a quotient $\mathcal{R} = \mathcal{R}_n$ such that*

(1) *$\mathcal{R}$ is an over-ring of $M_n(\mathbb{Z})$.*
(2) *Under the natural epimorphism $\mathbb{Z}\{x, y\} \twoheadrightarrow \mathcal{R}$, the images of the ideals generated by $r_{1n}, s_1, \ldots, s_n$ form a direct sum.*

Proof. The proof consists of finding a ring $\mathcal{R}$ such that

(1) $\mathcal{R}$ is generated by two elements $\mathbf{x}, \mathbf{y}$ together with $1_{\mathcal{R}}$.
(2) Let $\mathcal{R}_1 = \mathcal{R}r_{1,n}(\mathbf{x})\mathcal{R}$, $\mathcal{S}_i = \mathcal{R}s_i(\mathbf{x},\mathbf{y})\mathcal{R}$ for $1 \leqslant i \leqslant n-1$, and $\mathcal{S}_0 = \mathcal{R}s_0(\mathbf{y})\mathcal{R}$. Then $\mathcal{R}_1 \cap \mathcal{S}_0 = \{0_{\mathcal{R}}\}$ and $\mathcal{S}_0 = \mathcal{S}_1 \oplus \cdots \oplus \mathcal{S}_{n-1}$.

Put $M = M_n(\mathbb{Z})$ and consider the ring $\mathcal{M} = \left(\begin{smallmatrix} M & 0 \\ \xi M \oplus \eta M & \mathbb{Z} \end{smallmatrix}\right)$ where ξ and η are independent variables commuting with each other and with every matrix from M. *Let $\mathcal{R}$ be the subring of $\mathcal{M}$ generated by the matrices*

$$\mathbf{I} = \begin{pmatrix} I & 0 \\ 0 & 1 \end{pmatrix}, \qquad \mathbf{X} = \begin{pmatrix} X & 0 \\ \xi & 1 \end{pmatrix}, \qquad \mathbf{Y} = \begin{pmatrix} Y & 0 \\ \eta & 0 \end{pmatrix}.$$

Then the projection on the top left corner is a ring epimorphism $\mathcal{R} \twoheadrightarrow M$, by Theorem 3.1. Define the polynomials $q_0(t) = 0$ and $q_i(t) = 1 + t + \cdots + t^{i-1}$, $i \geqslant 1$. Then

$$\mathbf{X}^i = \begin{pmatrix} X^i & 0 \\ \xi q_i(X) & 1 \end{pmatrix}, \qquad \mathbf{X}^{-i} = \begin{pmatrix} X^{-i} & 0 \\ -\xi q_i(X)X^{-i} & 1 \end{pmatrix},$$

$$\mathbf{X}^i\mathbf{Y} = \begin{pmatrix} X^iY & 0 \\ \xi q_i(X)Y + \eta & 0 \end{pmatrix}, \qquad \mathbf{X}^{-i}\mathbf{Y} = \begin{pmatrix} X^{-i}Y & 0 \\ -\xi q_i(X)X^{-i}Y + \eta & 0 \end{pmatrix},$$

$$\mathbf{Y}\mathbf{X}^i\mathbf{Y} = \begin{pmatrix} 0 & 0 \\ \eta X^iY & 0 \end{pmatrix},$$

$$\mathbf{X}^{-i}\mathbf{Y}\mathbf{X}^i = \begin{pmatrix} X^{-i}YX^i & 0 \\ -\xi q_i(X)X^{-i}YX^i + \eta X^i & 0 \end{pmatrix}.$$

For the remainder of the proof, let $r_1 = r_{1,n}(\mathbf{X})$, $s_j = s_j(\mathbf{X},\mathbf{Y})$, and $1 \leqslant i \leqslant n-1$. Then

$$r_1 = \begin{pmatrix} 0 & 0 \\ \xi q_n(X) & 0 \end{pmatrix}, \qquad s_0 = \begin{pmatrix} 0 & 0 \\ \eta(Y-1) & 0 \end{pmatrix}, \qquad s_i = \begin{pmatrix} 0 & 0 \\ \eta X^iY & 0 \end{pmatrix}.$$

Therefore,

$$\mathcal{R}_1 = \begin{pmatrix} 0 & 0 \\ \xi q_n(X)M & 0 \end{pmatrix}, \qquad \mathcal{S}_0 = \begin{pmatrix} 0 & 0 \\ \eta(Y-1)M & 0 \end{pmatrix}, \qquad \mathcal{S}_i = \begin{pmatrix} 0 & 0 \\ \eta X^iYM & 0 \end{pmatrix}.$$

We see that $\mathcal{R}_1 \cap \mathcal{S}_0 = \{0\}$. The significance of this fact will become apparent from the following claim that will finally prove the theorem.

Claim. *The sum $\sum_{i=1}^{n-1} \mathcal{S}_i$ is direct and equals $\mathcal{S}_0$.*

We argue as follows. An element u_0 in $\mathcal{S}_0$ has the form

$$u_0 = \begin{pmatrix} 0 & 0 \\ \eta T_0 & 0 \end{pmatrix} \quad \text{where } T_0 = (Y-1)M_0 \text{ for some } M_0 = \begin{pmatrix} M_{01} \\ \vdots \\ M_{0n} \end{pmatrix} \in M.$$

Therefore,

$$T_0 = -\begin{pmatrix} 0 \\ M_{02} \\ \vdots \\ M_{0n} \end{pmatrix}.$$

An element u_i in $\mathcal{S}_i$ $(1 \leqslant i \leqslant n-1)$ has the form

$$u_i = \begin{pmatrix} 0 & 0 \\ \eta T_i & 0 \end{pmatrix} \quad \text{where } T_i = X^i Y M_i \text{ for some } M_i = \begin{pmatrix} M_{i1} \\ \vdots \\ M_{in} \end{pmatrix} \in M.$$

Then the $(i+1)$st row of T_i is M_{i1}, the other rows being zero. Therefore,

$$\sum_{i=1}^{n-1} T_i = \begin{pmatrix} 0 \\ M_{11} \\ \vdots \\ M_{n-1,1} \end{pmatrix}$$

is of the same form as T_0, and hence $\sum_{i=1}^{n-1} T_i \in \mathcal{S}_0$. We conclude that $\sum_{i=1}^{n-1} T_i = 0$ if and only if $M_{i1} = 0$ for all $i \in \{1, \ldots, n-1\}$. □

3.2. $M_n(\mathbb{Z})$ as a quotient of rings without identity

To motivate this discussion, let $\mathcal{R} = \mathbb{Z}\{\{e_{11}, \ldots, e_{nn}\}\}$ be a free nonassociative ring without identity. Let $\mathcal{I}$ be the ideal of $\mathcal{R}$ generated by the elements $e_{ij}e_{kl} - \delta_{jk}e_{il}$. Then the quotient ring $\mathcal{R}/\mathcal{I}$ is isomorphic to $M_n(\mathbb{Z})$.

Another way to present $M_n(\mathbb{Z})$ as a quotient of a ring without identity is to modify Presentation (18) to obtain $M_n(\mathbb{Z})$ as a quotient of the integral semigroup ring $\mathbb{Z}[FS(x, y)]$. This yields the following

Theorem 3.6. *Let $X = \sum_{i=1}^{n} E_{i,i+1}$ and $Y = E_{11}$. Then the map*

$$f : \mathbb{Z}\big[FS(x, y)\big] \to M_n(\mathbb{Z}), \qquad x \mapsto X, \qquad y \mapsto Y,$$

is a ring epimorphism with kernel generated by the $n+2$ elements

$$x^{n+1} - x, \qquad yx^n - y, \qquad -x^n + \sum_{i=0}^{n-1} x^{n-i} y x^i, \qquad yx^j y, \quad 1 \leqslant j \leqslant n-1. \tag{45}$$

Proof. Put $\mathcal{R} = \mathbb{Z}[FS(x, y)]$. *All computations in this paragraph will be done modulo $\mathcal{I} = Ker(f)$.* We firstly observe that $x^{n-1}(x^{n+1} - x) = 0$ yields $x^{2n} = x^n$. Therefore $y^2 = y(\sum_{i=0}^{n-1} x^i y x^{n-i}) = yx^n = y$, so that $y = y^2 = (\sum_{i=0}^{n-1} x^i y x^{n-i})y = x^n y$. Therefore, $z = x^n$ an identity element.

It remains to show that ideal $\mathcal{I}_0$ generated by the elements (45) equals $\mathcal{I}$. Firstly, $\mathcal{I}_0 \subseteq \mathcal{I}$ because the corresponding relations are satisfied by X and Y. On the other hand, the computations in the previous paragraph show that the ring $\mathcal{R}/\mathcal{I}_0$ is generated by the n^2 elements $x^i + \mathcal{I}_0$, $x^i y x^j + \mathcal{I}_0$, $1 \leqslant i, j \leqslant n$. Since $\dim_{\mathbb{Z}} M_n(\mathbb{Z}) = n^2$, it follows that $\mathcal{I} = \mathcal{I}_0$. □

3.3. Linear representations of matrix rings

We prove below that 4 relations in X and Y are sufficient to describe $M_n(\mathbb{Z})$ in the context of matrix rings.

Theorem 3.7. *Let $\mathcal{D}$ be a commutative domain of characteristic either zero or at least $m+1$, over which every finitely generated projective module is free. Let $\mathcal{S}$ be a subring of $M_m(\mathcal{D})$ generated by some nonzero X_1 and Y_1 such that*

$$X_1^{n+1} = X_1, \qquad Y_1 X_1^n = Y_1, \qquad Y_1^2 = Y_1, \qquad \sum_{i=0}^{n-1} X_1^{n-i} Y_1 X_1^i = X_1^n. \tag{46}$$

Then the trace k of Y_1 is a positive integer, and there exist $B \in GL_m(\mathcal{D})$ such that, putting $r = m - kn$, we have

$$B^{-1} X_1 B = \begin{pmatrix} I_k \otimes X & 0_{k\times r} \\ 0_{r\times k} & 0_{r\times r} \end{pmatrix} \quad \text{and} \quad B^{-1} Y_1 B = \begin{pmatrix} I_k \otimes Y & 0_{k\times r} \\ 0_{r\times k} & 0_{r\times r} \end{pmatrix}.$$

An exposition of commutative domains over which every finitely generated projective module is free can be found in Lam [9].

Proof of Theorem 3.7. Since X_1^n is an idempotent, we decompose $\mathcal{D}^m$ as the direct sum of the image $\mathcal{P}$ and the kernel $\mathcal{N}$, i.e. $\mathcal{D}^m = \mathcal{P} \oplus \mathcal{Z}$ where

(1) $\mathcal{P}$ and $\mathcal{Z}$ have $\mathcal{D}$-ranks q and r, respectively.
(2) $X_1^n|_P = I_q$ and $X_1^n|_Z = 0_r$.

We observe from (46) that $\mathcal{P}$ and $\mathcal{Z}$ are $\mathcal{S}$-invariant and $\mathcal{S}|_{\mathcal{Z}} = 0_r$. Choose some free generating sets for $\mathcal{P}$ and $\mathcal{Z}$. Then with respect to these sets, X_1 and Y_1 are represented by the matrices $\left(\begin{smallmatrix} X_2 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} Y_2 & 0 \\ 0 & 0 \end{smallmatrix}\right)$, respectively. Furthermore, the matrices X_2 and Y_2 satisfy the following relations

$$r_{1,n}(X_2, Y_2) = r_{2,n}(X_2, Y_2) = s_0(Y_2) = 0. \tag{47}$$

Let $k = \operatorname{tr}(Y_2)$. Then (47) yield

$$q = \operatorname{tr}(I_q) = \operatorname{tr}\left(\sum_{i=0}^{n-1} X_2^i Y_2 X_2^{n-i} \right) = \sum_{i=0}^{n-1} \operatorname{tr}\left(Y_2 X_2^{n-i} X_2^i \right) = nk. \tag{48}$$

$\mathcal{P}$ decomposes with respect to the idempotent Y_2 as a direct sum of the image $\mathcal{U}$ and the kernel $\mathcal{V}$. The restriction maps $Y_2|_{\mathcal{U}}$ and $Y_2|_{\mathcal{V}}$ are the identity and zero maps, respectively. Therefore

$$k = \operatorname{tr}(Y_2) = \operatorname{tr}(Y_2|_{\mathcal{U}}) + \operatorname{tr}(Y_2|_{\mathcal{V}}) = \operatorname{tr}(Y_2|_{\mathcal{U}}) = \operatorname{tr}(\mathrm{id}_{\mathcal{U}}). \tag{49}$$

In particular, k is an integer.

Let

$$\widehat{\mathcal{U}} = \sum_{i=0}^{n-1} X_2^i(\mathcal{U}).$$

Then (46) implies that $\widehat{\mathcal{U}}$ is an $\mathcal{S}$-module. In addition, $Y_2|_{\mathcal{V}} = 0$ yields $Y_2|_{\mathcal{P}/\widehat{\mathcal{U}}} = 0$. In turn, (46) implies $X_2|_{\mathcal{P}/\widehat{\mathcal{U}}} = 0$, which amounts to the identity map acting as zero on $\mathcal{P}/\widehat{\mathcal{U}}$. Therefore $\mathcal{P} = \widehat{\mathcal{U}}$. The sum $\sum_{i=0}^{n-1} X_2^i(\mathcal{U})$ is direct because by passing to the field of fractions $\mathcal{F}$ of $\mathcal{D}$, we have $\mathcal{F}^q = \sum_{i=0}^{n-1} X_2^i(\mathcal{F} \otimes_{\mathcal{D}} \mathcal{U})$. By (53), this sum is a sum of n linear spaces of dimension k, and we know from (52) that $\dim_{\mathcal{F}} \mathcal{F}^q = nk$. Therefore

$$\mathcal{D}^q = \bigoplus_{i=0}^{n-1} X_2^i(\mathcal{U}).$$

Let $\mathcal{B} = \{s_1, \ldots, s_k\}$ be a free $\mathcal{D}$-basis of $\mathcal{U}$. Then $\widehat{\mathcal{B}} = \bigcup_{i=0}^{n-1} X_2^i(\mathcal{B})$ is a free $\mathcal{D}$-basis of $\mathcal{P}$. Hence, X_2 may be represented with respect to $\widehat{\mathcal{B}}$ by an n-by-n block matrix (X_{ij}) with k-by-k blocks, where $X_{ij} = 0$ unless $i = j + 1$, and $X_{j+1,j} = I_k$ for $1 \leqslant j \leqslant n-1$. Similarly, $Y_2 = (Y_{ij})$ where $Y_{11} = I_k$ and $Y_{ij} = 0$ for $i \neq 1$ because $Y_2|_U$ is the identity map, and $Y_2|_{\mathcal{P}/\mathcal{U}}$ is the zero map. Since $I_q = X_2^n = X_{1,n} \otimes I_n$, we arrive at $X_{1,n} = I_k$. Therefore, X_2 is represented in the basis $\widehat{\mathcal{B}}$ by the permutation matrix $X \otimes I_k$ in block form. It remains to observe that from $r_{2,n}(X_2, Y_2) = I_q$ it follows that $Y_{1j} = 0$ for $2 \leqslant j \leqslant n$. Consequently Y_2 is represented with respect to $\widehat{\mathcal{B}}$ by the matrix $Y \otimes I_k$. □

Corollary 3.8. *Let $\mathcal{D}$ be a commutative domain of characteristic either zero or at least $n+1$. Then the automorphism group of the ring $M_n(\mathcal{D})$ is generated by the automorphism group $Aut(\mathcal{D})$ of the ring $\mathcal{D}$, and by the projective general linear group $PGL_n(\mathcal{D})$, where*

(1) *$Aut(\mathcal{D})$ acts on $M_n(\mathcal{D})$ by acting on each entry of a matrix.*
(2) *$PGL_n(\mathcal{D})$ acts on $M_n(\mathcal{D})$ by conjugation.*

Proof. Any automorphism σ of the ring $M_n(\mathcal{D})$ leaves the center invariant. In other words, there exist $\alpha \in Aut(\mathcal{D})$ such that for every $a \in \mathcal{D}$, we have $\sigma(a \sum_{i=1}^n E_{ii}) = \alpha(a) \sum_{i=1}^n E_{ii}$.

Next we consider $\beta = \alpha^{-1}\sigma$, which is a $\mathcal{D}$-algebra automorphism of $M_n(\mathcal{D})$. Then the pair $(\beta X, \beta Y)$ satisfies the relations of (19). Therefore, by Theorem 3.7 there exists $U \in M_n(\mathcal{D})$ which conjugates βX to X and βY to Y. The conjugations by U and $-U$ produce identical results, and there are no further such identifications. Therefore the automorphism group of the $\mathcal{D}$-algebra $M_n(\mathcal{D})$ is isomorphic to $PGL_n(\mathcal{D})$. □

The result of Corollary 3.8 is not new. More general results are contained Rosenberg and Zelinsky [13]. In particular, that paper shows that Corollary 3.8 is false, for example, for Dedekind domains with class number at least 2.

We will need the following theorem of G. Higman [8].

Theorem 3.9 *(G. Higman's Theorem). The unit group $\mathcal{U}$ of the integral group ring of a finite Abelian group $\mathcal{G}$ is given by $\mathcal{U} = \pm\mathcal{G} \times \mathcal{F}$, where $\mathcal{F}$ is a free Abelian group of rank*

$$\frac{1}{2}(\#\mathcal{G} + t_2 - 2l + 1). \tag{50}$$

Here t_2 is the number of elements of $\mathcal{G}$ of order 2, and l is the number of cyclic subgroups of $\mathcal{G}$.

By analyzing some elementary inequalities, it follows that $\mathcal{F} = \{0\}$ if and only if $n = 2, 3, 4, 6$.

Theorem 3.10. *The set $\mathcal{Y} = \{Y_1 \in M_n(\mathbb{Z}) \mid Y_1^2 = Y_1,\ r_{2,n}(X, Y_1) = 0\}$ has the property that the pair (X, Y_1) satisfies all relations of* (19)*, and all Y_1 have trace* 1*. If $n = 2, 3, 4, 6$ then $Y_1 = E_{ii}$ for some i. Otherwise, $\mathcal{Y}$ is infinite, and if $Y_1 \neq E_{ii}$ then it has both positive and negative entries.*

Any Y_1 is of the form $(c_i d_j)$ for some integers c_i, d_j such that the matrices $\operatorname{circ}(c_1, \ldots, c_n)$ *and* $\operatorname{circ}(d_1, \ldots, d_n)$ *are mutually inverse. Any Y_1 is conjugate to Y by an integral circulant matrix with determinant* ± 1.

Proof. Let $Y_1 = (y_{ij})$. Then $r_{2,n}(X, Y_1) = 0$ implies

$$\sum_{k=0}^{n} y_{i+k,\,j+k} = \delta_{ij}. \tag{51}$$

These formulas prove the claim about the possible signs of entries of Y_1.

Applying the trace to $r_{2,n}(X, Y_1) = 0$ implies

$$n = \operatorname{tr}(I_n) = \operatorname{tr}\left(\sum_{i=0}^{n-1} X^i Y_1 X^{n-i}\right) = \sum_{i=0}^{n-1} \operatorname{tr}\left(Y_1 X^{n-i} X^i\right) = n \operatorname{tr}(Y_1). \tag{52}$$

$\mathbb{Z}^n$ decomposes with respect to the idempotent Y_1 as a direct sum of the image $\mathcal{I}$ and kernel $\mathcal{K}$. Therefore

$$1 = \operatorname{tr}(Y_1) = \operatorname{tr}(Y_1|_{\mathcal{I}}) + \operatorname{tr}(Y_1|_{\mathcal{K}}) = \operatorname{tr}(Y_1|_{\mathcal{I}}) = \operatorname{tr}(\mathrm{id}_{\mathcal{I}}). \tag{53}$$

Therefore, Y_1 is a rank 1 projection. The image of Y_1 is an Abelian group is generated by some $(d_1, \ldots, d_n) \in \mathbb{Z}^n$. It follows that on the standard basis $e_1 = (1, 0, \ldots, 0), \ldots, e_n = (0, \ldots, 0, 1)$ the action of Y_1 is described by $Y_1 e_i = c_i d_1 + \cdots + c_i d_n$ for some integer c_i. Therefore $Y_1 = (c_i d_j)$. Next, from $r_{2,n}(X, Y_1) = 0$ we conclude that $\sum_{k=0}^{n-1} c_{i+k} d_{j+k} = \delta_{ij}$, which is the same as saying that the matrices $\operatorname{circ}(c_1, \ldots, c_n)$ and $\operatorname{circ}(d_1, \ldots, d_n)$ are mutually inverse.

Now, going back to (51), we see that the relations $Y_1 X^k Y_1 = 0$ follow from the relations $r_{1,n}(X) = r_{2,n}(X, Y_1) = 0$. Indeed, $(X^k Y_1)_{ij} = c_{i+k} d_j$. Therefore $(Y_1 X^k Y_1)_{ij} = c_i (\sum_{u=1}^{n} d_u c_{u+k}) d_j = c_i \delta_{kn} d_j = 0$. It follows that $(X, Y_1) \in G_n(\mathbb{Z})$ by Theorem 3.1 and because all proper quotients of the ring $M_n(\mathbb{Z})$ are finite.

In the cases of $n = 2, 3, 4, 6$ the group $\mathcal{U}(\mathbb{Z}\langle X\rangle)$ consists precisely of $2n$ matrices $\pm E_{ii}$. □

Theorem 3.10 may be strengthened as follows. If all entries of $X_1 \in M_n(\mathbb{Z})$ are nonnegative, and $X_1^n = I_n$, then in each row of X_1 there exactly one positive entry, and it equals 1. We will prove this assertion in 2 steps.

(1) Suppose that in each row of X_1 there is exactly one nonzero entry. Then from $\det X_1 = \pm 1$ it follows that X_1 is of the required form.

(2) Suppose that $X_1 = (x_{ij})$ has a row with at least 2 positive entries x_{ij} and $x_{ij'}$. The ith column of X_1 contains a nonzero entry x_{mi}. We conclude that the matrix $X_1^2 = (t_{kl})$ has the property that $t_{mj}, t_{mj'} > 0$. Similarly, any positive power of X_1 has at least two positive entries in some row. We obtain a contradiction, however, by considering $X_1^n = I_n$.

We remark that G. Higman's Theorem 3.9, when applied to a cyclic group of order n, may be restated in terms of solutions of the following Diophantine equations:

$$\det \operatorname{circ}(x_1, \ldots, x_n) = \pm 1. \tag{54}$$

Unfortunately, there appears to be no efficient algorithm to find solutions of (54). Computer experiments with (54) eventually led us to Theorem 3.10.

3.4. Presentations of direct sums of matrix rings over $\mathbb{Q}$ and $\mathbb{Z}$

Our next result gives infinitely many 2-generator presentations for the ring $M_n(\mathbb{Z})$. We obtain, as a consequence, presentations for several types of direct sums of matrix rings. We do not write down these presentations explicitly based on the following reason. If $\mathcal{I}$ and $\mathcal{J}$ are ideals of a ring $\mathcal{R}$ such that $\mathcal{I} + \mathcal{J} = \mathcal{R}$, then $\mathcal{I} \cap \mathcal{J} = \mathcal{I}\mathcal{J} + \mathcal{J}\mathcal{I}$. Therefore, if the ideals $\mathcal{I}$ and $\mathcal{J}$ are generated by explicitly given i and j elements, respectively, then $\mathcal{I} \cap \mathcal{J}$ is generated by at most $2ij$ explicitly given elements.

Theorem 3.11. *The ring $\mathbb{Z}\{x, y\}$ has an infinite family of ideals $\{\mathcal{I}_n(m)\}_{m\in\mathbb{Z}}$ defined by*

$$\mathcal{I}_n(m) = \big(r_{1,n}(x, mx + y), r_{2,n}(x, mx + y), s_j, 1 \leqslant j \leqslant n-1\big), \qquad \mathcal{I}_n = \mathcal{I}_n(0).$$

This family of ideals has the following properties:

(1) $\mathbb{Z}\{x, y\}/\mathcal{I}_n(m) \cong M_n(\mathbb{Z})$ *for any integer m.*

(2) *If $m_1, \ldots, m_k \geqslant 2$ are integers, and the sets $S_1, \ldots, S_k \subseteq \mathbb{Z}$ are finite, then*

$$\frac{\mathbb{Z}\{x, y\}}{\bigcap_{i=1}^k \bigcap_{s_i \in S_i} \mathcal{I}_{m_i}(s_i)} \otimes_{\mathbb{Z}} \mathbb{Q} \cong \bigoplus_{i=1}^k M_{m_i}(\mathbb{Q})^{\#S_i}.$$

(3) *If $|k - l| \geqslant 2$, then even though $\mathbb{Z}\{x, y\}/\mathcal{I}_n(k) \cap \mathcal{I}_n(l) \ncong M_n(\mathbb{Z})^2$, it embeds as a subring of finite index.*

(4) *Define the map $^t : \mathbb{Z}\{x, y\} \to \mathbb{Z}\{x, y\}$ by $f(x, y)^t = f(y, x)$, then for any pairwise different integers $n_1, \ldots, n_k \geqslant 2$, we have*

$$\frac{\mathbb{Z}\{x, y\}}{\bigcap_{j=1}^k \mathcal{I}_{n_j}^t \cap \mathcal{I}_{n_j} \cap \mathcal{I}_{n_j}(1)} \cong \bigoplus_{j=1}^k M_{n_j}(\mathbb{Z})^3.$$

(5) $\mathbb{Z}\{x, y\}/\mathcal{I}_2 \cap \mathcal{I}_2^t \cap \mathcal{I}_2(1) \cap \mathcal{I}_2(1)^t \cong M_2(\mathbb{Z})^4$.

Proof. We find it convenient to introduce a family of ring automorphisms $\{\varphi_m\}_{m\in\mathbb{Z}}$ of $\mathbb{Z}\{x, y\}$ given by $\varphi_m(x) = x$ and $\varphi_m(y) = mx + y$. Then $\mathcal{I}_n(m) = \varphi_m(\mathcal{I}_n)$. Theorem 3.1 implies $\mathbb{Z}\{x, y\}/\mathcal{I}_n(m) \cong M_n(\mathbb{Z})$.

We will show that all these ideals are different, i.e. $\mathcal{I}_n(k) = \mathcal{I}_m(l)$ if and only if $m = n$ and $k = l$.

Let us consider the case $m = n$. Suppose that our claim is false, so that $\mathcal{I}_n(k) = \mathcal{I}_n(l)$ for some $k \neq l$. Then

$$\mathcal{I}_n = \varphi_{-k}\big(\mathcal{I}_n(k)\big) = \mathcal{I}_n(l-k) = \mathcal{I}_n(a), \quad \text{where } 0 \neq a = l-k.$$

Therefore,

- $r_{2,n}(x, ax + y) = a\sum_{i=0}^{n-1} x^{n+1} + r_{2,n}(x, y) \equiv nax \pmod{\mathcal{I}_n}$,
- $r_{2,n}(x, ax + y) \in \mathcal{I}_n$,
- x is invertible modulo $\mathcal{I}_n$

imply that $na \in \mathcal{I}_n$. Therefore, $\{0\} = na(\mathbb{Z}\{x, y\}/\mathcal{I}_n) \cong M_n(\mathbb{Z})$, a contradiction.

Now suppose that $m < n$. Then $\mathcal{I}_n = \mathcal{I}_m(l-k)$. Therefore modulo either of these ideals, $0 = s_m(x, y) = yx^m y = y^2 = y$, yielding $0 = r_{2,n}(x, y) = -1$.

Since the ring $M_n(\mathbb{Q})$ is simple, the arguments above together with Chinese Remainder Theorem prove part (2).

Next we prove part (3). We observe that the restriction of the maps φ_m to $\mathbb{Z}$ is the identity map. Therefore,

$$\big(\mathcal{I}_n(k) + \mathcal{I}_n(l)\big) \cap \mathbb{Z} \subseteq \big(\mathcal{I}_n + \mathcal{I}_n(k-l)\big) \cap \mathbb{Z} \equiv \mathcal{I}_n \cap \mathbb{Z} \equiv \{0\} \pmod{k-l},$$

so that $(\mathcal{I}_n(k) + \mathcal{I}_n(l)) \cap \mathbb{Z} \neq \mathbb{Z}$ and therefore $\mathcal{I}_n(k) + \mathcal{I}_n(l) \neq \mathbb{Z}\{x, y\}$.

We prove part (4) by showing that the sum of any two of the three ideals $\mathcal{I}_k^t$, $\mathcal{I}_m$, $\mathcal{I}_n(1)$ is $\mathbb{Z}\{x, y\}$.

(1) We claim that $\mathcal{J} = \mathcal{I}_m + \mathcal{I}_n(1) = \mathbb{Z}\{x, y\}$. If $m > n$, then modulo $\mathcal{J}$ we have $0 = s_n(x, y) = yx^n y = y^2 = y$, so that $0 = r_{2,m}(x, y) = -1$. Suppose that $m < n$, then consider the ideal $\mathcal{J}' = \varphi_{-1}(\mathcal{J}) = \mathcal{I}_m(-1) + \mathcal{I}_n$. Then modulo $\mathcal{J}'$ we have $0 = s_m(x, y) = yx^m y = y^2 = y$, so that $0 = r_{2,n}(x, y) = -1$.

(2) We claim that $\mathcal{K} = \mathcal{I}_m^t + \mathcal{I}_n = \mathbb{Z}\{x, y\}$. All computations here are done modulo $\mathcal{K}$. From $x^2 = x$ and $x^n = 1$ we conclude that $x = 1$, and therefore $0 = xyx = y$, and consequently $0 = 0^m = y^m = 1$.

(3) The proof that $\mathcal{L} = \mathcal{I}_m^t + \mathcal{I}_n(1) = \mathbb{Z}\{x, y\}$ is exactly as above.

It remains to prove part (5) of the theorem.

(4) We claim that $\mathcal{N} = \mathcal{I}_2(1) + \mathcal{I}_2(1)^t = \mathbb{Z}\{x, y\}$. The computations will be done modulo $\mathcal{N}$.

$$x + y = (x + y)^2 = x^2 + xy + yx + y^2 = 2 + xy + yx, \tag{55}$$

$$0 = (x + y)x(x + y)y = \big(x^3 + x^2y + yx^2 + yxy\big)y = xy + 2 + yx. \tag{56}$$

The right-hand sides of (55) and (56) are equal, hence $x + y = 0$. Therefore, $0 = r_{2,2}(x, x + y) = r_{2,2}(x, 0) = -1$.

(5) We claim that $\mathcal{O} = \mathcal{I}_2 + \mathcal{I}_2(1)^t = \mathbb{Z}\{x, y\}$. The computations will be done modulo $\mathcal{O}$. Multiplying both sides of (55) by y on the right yields $y = 0$, so that $0 = r_{2,2}(x, y) = -1$.

(6) The equality $\mathcal{I}_2^t + \mathcal{I}_2(1)^t = \mathbb{Z}\{x, y\}$ is proved similarly, by multiplying (55) by x on the right. □

For all sufficiently large k, the ring $M_n(\mathbb{Z})^k$ does not admit 2 generators, because the same holds modulo any prime by a simple counting argument. Therefore, it should be of interest to investigate the minimum number of generators for finite direct sums of integral matrix rings. The situation is philosophically similar to the result of Philip Hall [6, p. 137], that "the direct product of the nineteen icosahedrals can be generated by two elements, but not the direct product of twenty."

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