Matrix Evolution Equations and Special Functions

G. DATTOLI
Unità Tecnica Scientifica Tecnologie Fisiche Avanzate
ENEA, Centro Ricerche Frascati, C.P. 65
Via E. Fermi, 45 00044, Frascati, Roma, Italia
dattoli@frascati.enea.it

B. GERMANO
Dipartimento di Metodi e Modelli Matematici per le Scienze Applicate
Università degli Studi di Roma "La Sapienza"
Via A Scarpa, 14, 00161 Roma, Italia
germano@dmmm.uniroma1.it

P. E. RICCI
Dipartimento di Matematica "Guido Castelnuovo"
Università degli Studi di Roma "La Sapienza"
P.le A. Moro, 2, 00185 Roma, Italia
ricci@uniroma1.it

(Received and accepted March 2004)

Abstract—We extend the technique of the evolution operator to matrix differential equations. It is shown that the combined use of the Cayley-Hamilton theorem and of nonstandard special functions may provide a new form of solutions for a wide family of this type of equation. © 2004 Elsevier Ltd. All rights reserved.

Keywords—Evolution matrix equations, Cayley-Hamilton theorem, Multivariable Hermite polynomials, Tricomi functions.

1. INTRODUCTION

Operational methods associated with the evolution operator technique are powerful tools to deal with families of evolution type differential equations. The method is very flexible and can be exploited in different contexts. An apparently unorthodox application is the solution of the following vector type equation

\[
\frac{d}{dt} \vec{R} = \vec{\Omega} \times \vec{R},
\]

\[
\vec{R}(0) = \vec{R}_0,
\]

which is often encountered in physical problems ranging from rigid body rotation, to atomic and nuclear physics [1].
If the torque vector $\tilde{\Omega}$ is independent of time, we can cast the formal solution of equation (1) in the form \[1\),
\[\tilde{R}(t) = \exp\left(t \left[\tilde{\Omega}\right]\right) \tilde{R}_0,\]
\[\left[\tilde{\Omega}\right] = \tilde{\Omega} \times,\] with the exponential on the r.h.s. of equation (2) playing the role of the evolution operator, associated with equation (1).

The explicit solution of equation (1) can be obtained by expanding the exponential and thus finding,
\[\tilde{R}(t) = \sum_{n=0}^{\infty} \frac{t^n}{n!} \tilde{\Omega}^n \tilde{R}_0,\] which should be interpreted properly by noting that,
\[[\tilde{a}] \tilde{b} = \tilde{a} \times \tilde{b},\]
\[[\tilde{a}]^2 \tilde{b} = \tilde{a} \times \left(\tilde{a} \times \tilde{b}\right), \ldots.\] By combining equation (3) with the cyclical properties of the vector product, we eventually end up with \[1\),
\[\tilde{R}(t) = \cos \left(\tilde{\Omega} t\right) \tilde{R}_0 + \sin \left(\tilde{\Omega} t\right) \left(\tilde{n} \times \tilde{R}_0\right) + \left[1 - \cos \left(\tilde{\Omega} t\right)\right] \left(\tilde{n} \cdot \tilde{R}_0\right) \tilde{n},\] where $\tilde{n}$ is a unit vector in the direction of the torque vector $\tilde{\Omega}$.

Now, we can give an example of the type of generalization we will consider in this paper. Therefore, let us consider the following equation,
\[
\frac{d}{dt} \tilde{R} = \left(a \left[\tilde{\Omega}\right] + b \left[\tilde{\Omega}^2\right]\right) \tilde{R},
\] having the same initial condition as in equation (1) and with $a$, $b$ constants.

Therefore, the formal solution of equation (6) can be written as,
\[\tilde{R}(t) = \exp \left(a \left[\tilde{\Omega}\right] + b \left[\tilde{\Omega}^2\right]\right) \tilde{R}_0, \quad \tilde{a} = at, \quad \tilde{b} = bt.\] To make a proper expansion of the exponential, we remind that the generating function of the Hermite polynomials reads,
\[\exp(\chi t + yt^2) = \sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(x, y),\]
\[H_n(x, y) = n! \sum_{r=0}^{n} \frac{x^{n-2r} y^r}{(n-2r)!}.\] By using the vector operator $[\tilde{\Omega}]$ as expansion parameter, we can write equation (7) as
\[\tilde{R}(t) = \sum_{n=0}^{\infty} \frac{1}{n!} H_n(\tilde{a}, \tilde{b}) \left[\tilde{\Omega}\right]^n \tilde{R}_0.\]
The above solution can be written in a form very similar to equation (5), by just replacing the circular functions according to the prescription,

\[
\cos (|\vec{\Omega}| t) \rightarrow \exp \left(-\frac{b}{|\vec{\Omega}|^2}\right) \cos \left(\frac{|\vec{\Omega}|}{a} \right), \\
\sin (|\vec{\Omega}| t) \rightarrow \exp \left(-\frac{b}{|\vec{\Omega}|^2}\right) \sin \left(\frac{|\vec{\Omega}|}{a} \right). 
\]

The same result can also be obtained in a more formal way, by recalling that [2],

\[
H_n(x,y) = \exp \left(\frac{\partial^2}{\partial x^2}\right) x^n. 
\]

Accordingly, we can recast equation (9) in the form,

\[
\dot{\vec{R}}(t) = \exp \left(\frac{b}{|\vec{\Omega}|} \frac{\partial^2}{\partial a^2}\right) \left[\vec{\Omega}\right]^n \vec{R}_0, 
\]

so that equation (10) can be easily recovered, by ordinary algebraic means.

Now, let us consider the more general case,

\[
\frac{d}{dt} \vec{R} = \sum_{s=1}^{m} a_s \left[\vec{\Omega}\right]^s \vec{R}, 
\]

which admits a solution analogous to that reported in equation (9), provided that the Hermite polynomials, \(H_n(x,y)\), be replaced by the Bell-type polynomials [3],

\[
H^{(m)}_n (\{a_s\}^m_1) ; \quad \{a_s\}^m_1 = a_1, \ldots, a_m \\
H^{(m)}_n (\{a_s\}^m_1) = n! \sum_{r=0}^{\infty} \frac{a_m H^{(m-1)}_{n-mr} (\{a_s\}^{m-1}_1)}{(n-mr)!r!}. 
\]

Furthermore, the generalization of equation (10) is obtained using the operational definition [2],

\[
H^{(m)}_n (\{a_s\}^m_1) = \exp \left(\sum_{s=2}^{m} a_s \frac{\partial^2}{\partial a^2_s}\right) \left[a_1^n\right]. 
\]

This last result opens the interesting perspective of obtaining solutions for vector equations of the type,

\[
\frac{d}{dt} \vec{R} = F \left(\left[\vec{\Omega}\right]\right) \vec{R}. 
\]

If \(F(x)\) is a function admitting a series expansion of the type,

\[
F(x) = \sum_{s=1}^{\infty} a_s x^s, 
\]

(the 0th-order term of the series is nonessential to the present discussion) the solution of equation (17) can be written in terms of infinite variable Bell-type polynomials, thus, getting, e.g.,

\[
\vec{R} (t) = \sum_{n=0}^{\infty} \frac{1}{n!} H^{(n)}_n (\{a_s\}_1^n) \left[\vec{\Omega}\right]^n \vec{R}_0. 
\]
This last example and the previous discussion show that the use of operational methods and of special polynomials allow a fairly direct solution of vector type equations. In the forthcoming sections, we will show that the same method can be extended to matrix equations.

2. OPERATIONAL METHODS AND MATRIX EQUATIONS

The vector equation we have considered in (1) can be written also in the form of a matrix equation of the type,

$$\frac{d}{dt} y = \dot{A} y,$$

where $\dot{A}$ is a $3 \times 3$ matrix and $y$ is a three column vector. This restriction, in general, can be relaxed and in the following, we will consider $n \times n$ matrices.

The evolution operator,

$$U(t) = \exp (\dot{A} t),$$

can be written in a finite form, using the Cayley-Hamilton characteristic polynomial,

$$\exp (\dot{A} t) = \sum_{s=0}^{n-1} c_s t^s \dot{A}^s,$$  \hspace{1cm} (21)

with the $c_s$ coefficients specified by the equations,

$$\exp (\lambda_i t) = \sum_{s=0}^{n-1} c_s t^s \lambda_i^s$$  \hspace{1cm} (22)

if the eigenvalues $\lambda_i$ of matrix $\dot{A}$ are all distinct, if $\lambda_i$ is an $r$th-order multiple root of the matrix characteristic equation, then equation (22) should be replaced by,

$$\exp (\lambda_i t) = \left( \frac{d}{d\lambda} \right)^s r(\lambda)|_{\lambda_i t},$$

$$s = 1, \ldots, k-1, \quad r(\lambda) = \sum_{m=0}^{n-1} c_m \lambda^m.$$  \hspace{1cm} (23)

Now, it is evident that what we have discussed in the previous section can be extended, mutatis mutandis, to the present case. In particular, if the r.h.s. of equation (1) is replaced by a combination of the same type of equation (6), we expect that the characteristic polynomial can be rewritten as

$$\exp (\bar{a} \dot{A} + \bar{b} \dot{A}^2) = \exp \left( \bar{b} \frac{\partial^2}{\partial t^2} \sum_{s=0}^{n-1} c_s \dot{A}^s \bar{a}^s \right).$$  \hspace{1cm} (24)

Since the coefficients $c_s$ may be functions of $\bar{a}$, we find

$$\exp (\bar{a} \dot{A} + \bar{b} \dot{A}^2) = \sum_{s=0}^{n-1} \dot{C}_s H_s (\bar{a}, \bar{b}),$$  \hspace{1cm} (25)

where $\dot{C}_s$ is an operator, specified by

$$\dot{C}_s = \exp \left( \bar{b} \frac{\partial^2}{\partial \bar{a}^2} \right) c_s \exp \left( -\bar{b} \frac{\partial^2}{\partial \bar{a}^2} \right).$$  \hspace{1cm} (26)

If, e.g.,

$$c_s = \exp (\bar{a} \lambda_s),$$  \hspace{1cm} (27)
we find
\[ \hat{C}_s = \exp \left[ \left( \tilde{a} + 2\lambda \frac{\partial}{\partial \tilde{a}} \right) \lambda_s \right] = \exp \left( \tilde{a} \lambda_s + \tilde{b} \lambda_s^2 \right) \exp \left( 2\lambda \lambda_s \frac{\partial}{\partial \tilde{a}} \right), \]
whose derivation follows by using Weyl identity.

Now, let us discuss an example which may better illustrate the procedure. We consider, therefore, the matrix
\[ \hat{A} = \begin{pmatrix} 3 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{pmatrix}. \]
The eigenvalues of this matrix are all identical \( \lambda_1 = \lambda_2 = \lambda_3 \), so that
\[ \exp \left( \tilde{a} \hat{A} \right) = \exp (3\tilde{a}) \begin{pmatrix} 1 & \frac{\tilde{a}^2}{2} \\ 0 & 1 \\ 0 & 0 \end{pmatrix}, \]
and by applying the previously quoted rules, we end up with
\[ \exp \left( \tilde{a} \hat{A} + \tilde{b} \hat{A}^2 \right) = \exp (3\tilde{a} + 9\tilde{b}) \begin{pmatrix} 1 & H_1 (\tilde{a} + 6\tilde{b}, \tilde{b}) & \frac{1}{2}H_2 (\tilde{a} + 6\tilde{b}, \tilde{b}) \\ 0 & 1 \\ 0 & 0 \end{pmatrix}. \]

In the forthcoming section, we will extend the above results to more general cases.

### 3 CONCLUDING REMARKS

In the situation where we are dealing with more complicated cases, let us say \( \exp (\tilde{a} \hat{A} + \tilde{b} \hat{A}^m) \) with \( m < n \) (with \( n \) being the dimensionality of the matrix), we can apply the same procedure developed in the previous section. The method is only hampered by more complicated operational identities which should be exploited, along with higher order Hermite polynomials.

We find, indeed,
\[ \exp \left( \tilde{a} \hat{A} + \tilde{b} \hat{A}^m \right) = \exp \left( \tilde{b} \frac{\partial^m}{\partial \tilde{b}^m} \right) \sum_{s=0}^{n-1} c_s \hat{A}^s \tilde{a}^s \]
\[ = \sum_{s=0}^{n-1} \hat{C}^{(m)}_s \left( \tilde{a}, \tilde{b} \right), \]
\[ \hat{C}^{(m)}_s = \exp \left( \tilde{b} \frac{\partial^m}{\partial \tilde{b}^m} \right) c_s \exp \left( -\tilde{b} \frac{\partial^m}{\partial \tilde{b}^m} \right). \]

Assuming for \( c_s \) the same form given in equation (27), we find
\[ \hat{C}^{(m)}_s = \exp \left( \tilde{a} \lambda_s \right) \exp \left[ \sum_{r=1}^{m} \left( \begin{array}{c} m \\ r \end{array} \right) \lambda_s^r \frac{\partial}{\partial \tilde{a}} \right] \]
\[ = \exp \left[ \sum_{s=2}^{m} \pi_s \frac{\partial}{\partial x_s} \right] H^{(m)}_n (x, y) = H^{(m)}_n (\{ \phi \}^m_1) \]
\[ \{ \phi \}^m_1 = x, \pi_2, \pi_3, \ldots, y + \pi_m. \]
In these concluding remarks, we will touch on a further example which illustrates the usefulness of the combined use of special functions, and of operational methods.

We consider, indeed, the equation,

$$\frac{d}{dt} \frac{d}{dt} y = \dot{A} y,$$  \hspace{1cm} (35)

which can be treated by exploiting the methods developed in reference [5]. By recalling, indeed, that [4],

$$\frac{d}{dt} \frac{d}{dt} = \frac{d}{d\dot{D}_t^{-1}},$$  \hspace{1cm} (36)

where $\dot{D}_t^{-1}$ is a negative derivative operator, whose properties have been defined in [5], we find that the relevant evolution operator can be cast in the form,

$$\dot{U}(t) = \exp \left( \dot{D}_t^{-1} \dot{A} \right).$$  \hspace{1cm} (37)

We can now safely apply the Cayley-Hamilton theorem by replacing $t$ with $\dot{D}_t^{-1}$, and by recalling that,

$$\exp \left( \alpha \dot{D}_t^{-1} \right) = C_0(\alpha t),$$

$$\dot{D}_t^{-m} \exp \left( \alpha \dot{D}_t^{-1} \right) = t^m C_m(\alpha t),$$  \hspace{1cm} (38)

$$C_m(x) = \sum_{r=0}^{\infty} \frac{x^r}{r! (m+r)!},$$

where $C_m(x)$ is the $m$th-order Tricomi function [2].

According to the above identities, in the case of matrix (31), we find

$$\exp \left( \dot{D}_t^{-1} \dot{A} \right) = \begin{pmatrix} C_0(3t) & t C_1(3t) & \frac{t^2}{2} C_2(3t) \\ 0 & C_0(3t) & \frac{t}{2} C_1(3t) \\ 0 & 0 & C_0(3t) \end{pmatrix}.$$  \hspace{1cm} (39)

Before closing the paper, let us note that, according to equations (37),(38), the evolution operator can be written as

$$\dot{U}(t) = C_0(\dot{A}t).$$  \hspace{1cm} (40)

In the case of equations of the type,

$$\frac{d}{dt} \frac{d}{dt} y + m \frac{d}{dt} y = \dot{A} y,$$  \hspace{1cm} (41)

the evolution operator can be written as

$$\dot{U}(t) = m! C_m(\dot{A}t)$$  \hspace{1cm} (42)

and the method can still be applied without any significant change. In the case of example (39), it will be sufficient to replace $C_r(x)$ with $C_{m+r}(x)$.

The method we have discussed can be extended to higher-order equations and to more complicated operational forms. In a forthcoming investigation, we will discuss specific applications and further extension of the technique developed in this paper.
REFERENCES