Quantum black hole wave packet: Average area entropy and temperature dependent width

CrossMark

Aharon Davidson *, Ben Yellin

Physics Department, Ben-Gurion University of the Negev, Beer-Sheva 84105, Israel

ABSTRACT

A quantum Schwarzschild black hole is described, at the mini super spacetime level, by a non-singular wave packet composed of plane wave eigenstates of the momentum Dirac-conjugate to the mass operator. The entropy of the mass spectrum acquires then independent contributions from the average mass and the width. Hence, Bekenstein's area entropy is formulated using the $\langle mass^2 \rangle$ average, leaving the $\langle mass \rangle$ average to set the Hawking temperature. The width function peaks at the Planck scale for an elementary (zero entropy, zero free energy) micro black hole of finite rms size, and decreases Doppler-like towards the classical limit.

© 2014 The Authors. Published by Elsevier B.V. This is an open access article under the CC BY license (http://creativecommons.org/licenses/by/3.0/). Funded by SCOAP³.

Bekenstein–Hawking black hole area entropy [1] constitutes a triple point in the phase of physical theories, connecting gravity, quantum mechanics, and statistical mechanics. However, despite several illuminating derivations [2], the statistical roots of black hole entropy have not been fully revealed, not even at the level of discrete models [3]. There exist a few extreme black hole solutions [4], notably beyond general relativity, where one can apparently count micro states. But as far as the prototype Schwarzschild black hole is concerned, we still do not have the finest idea where these micro states are hiding, and how to enumerate them. A classical black hole is characterized by its event horizon, but once $\hbar$ is switched on (to allow for a finite Hawking temperature and non-zero Bekenstein entropy), even the innocent looking question 'where is this horizon located' lacks a decisive answer in the quantum or even in the semi-classical level.

The quantum-mechanical Schwarzschild black hole is hereby described by a non-singular minimal uncertainty wave packet composed of plane wave eigenstates. We carry out our analysis at the mini super spacetime level [5] without relying on theories beyond general relativity such as string theory [6], the fuzzball proposal [7], or loop quantum gravity [8] (see [9] for a different approach). Treating the black hole as a subsystem (a field theory defined on a black hole background is expected to be in a thermal state), its Gaussian mass spectrum becomes temperature dependent. We invoke Fowler prescription [10] for dealing with such

subsystems, and show that the associated statistical entropy acquires independent contributions from the average mass as well as from the width, and consistently formulate Bekenstein's area ansatz by means of the $\langle mass^2 \rangle$ average. While, as expected, the $\langle mass \rangle$ average turns out to be inversely proportional to Hawking temperature, a novel temperature dependent width function makes its appearance. The width function is maximal at the reduced Planck mass for an elementary quantum-mechanical black hole of finite rms size, for which both the entropy and free energy vanish and are minimal, and decreases Doppler-like towards the classical limit.

Let our starting point be the most general static radially symmetric line element, expressed in the form

$$ds^2 = -\frac{y(r)}{2r}dt^2 + \frac{2r}{x(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta \, d\phi^2). \qquad (1)$$

The unfamiliar x, y representation has been carefully designed to avoid the appearance of explicit r-dependence in the constrained Hamiltonian formalism (see Ref. [11] for the canonically transformed r-dependent Berry–Keating type [12] Hamiltonian). A tenable gauge pre-fixing option, namely defining a radial marker r whose geometrical interpretation is x, y-independent, has been harmlessly exercised. This has to be contrasted with the forbidden gauge prefixing of the 'lapse' function (the coefficient of dr^2 in this case), which kills the Hamiltonian constraint and introduces an unphysical degree of freedom (no gauge pre-fixing in Kuchar's midi superspace approach [13]). The more so at the mini superspacetime level, where the general relativistic action $\int \mathcal{R}\sqrt{-g}\,d^4x$ is integrated out over time and solid angle into the mini action $\int \mathcal{L}(x, x', y, y', r)dr$.

---

* Corresponding author.
E-mail addresses: davidson@bgu.ac.il (A. Davidson), yellinb@bgu.ac.il (B. Yellin).
URL: http://www.bgu.ac.il/~davidson (A. Davidson).

A word of caution is in order: Throughout this paper we treat $\int \mathcal{L}(q, q', r)dr$ in full mathematical analogy with $\int \mathcal{L}(q, \dot{q}, t)dt$. Technically, the t-evolution is traded for the r-evolution, both classically and quantum-mechanically, with the notions of Lagrangian and Hamiltonian being adapted accordingly. A similar technique has been introduced by York and Schmekel [14]. To sharpen the point, we clarify that our 'Hamiltonian' (to be identified with the momentum Dirac-conjugate [15] to the mass operator) has nothing to do with the physical mass of the black hole.

Up to a total derivative and an overall absorbable factor, the mini super-spacetime Lagrangian takes the form

$$\mathcal{L}(x, x', y, y') = \left(\frac{3x'}{4} - 2\right)\sqrt{\frac{y}{x}} - \frac{y'}{4}\sqrt{\frac{x}{y}}. \tag{2}$$

Being linear in the 'velocities', it gives rise to two primary second class constraints, namely

$$\phi_y = p_y + \frac{1}{4}\sqrt{\frac{x}{y}} \approx 0, \qquad \phi_x = p_x - \frac{3}{4}\sqrt{\frac{y}{x}} \approx 0, \tag{3}$$

whose Poisson brackets do not vanish $\{\phi_y, \phi_x\} = \frac{1}{2\sqrt{xy}}$. Following Dirac prescription [15], we are then driven from the naive Hamiltonian $\mathcal{H} = p_x x' + p_y y' - \mathcal{L} = 2\sqrt{y/x}$ to the total Hamiltonian

$$\mathcal{H}_T = 2\sqrt{\frac{y}{x}} + 2\frac{y}{x}\phi_y + 2\phi_x. \tag{4}$$

One can verify that the corresponding classical solution is (and is nothing but) the Schwarzschild solution

$$\frac{y(r)}{2\omega^2 r} = \frac{x(r)}{2r} = 1 - \frac{2m}{r}, \tag{5}$$

with no restrictions on the sign of the integration parameters m and ω. Along the classical trajectories the Hamiltonian takes the value $\mathcal{H} = 2\omega$, telling us that the $\mathcal{H}$ is not the total physical mass of the system.

To quantize the system it becomes crucial to calculate the Dirac brackets, and here one finds first of all

$$\boxed{\{x, y\}_D = 2\sqrt{xy} \neq 0} \tag{6}$$

Counter-intuitively, and potentially with far reaching consequences, two metric components do not Dirac-commute. Moreover, the relation $\{x, \frac{1}{2}\mathcal{H}\}_D = 1$ paves then the way for the quantum-mechanical commutation relations $[x, \frac{1}{2}\mathcal{H}] = i\hbar$. $\mathcal{H}$ is then faithfully represented by

$$\mathcal{H} = -2i\hbar\frac{\partial}{\partial x}. \tag{7}$$

By the same token, in accord with Eq. (6), the other metric component y is represented by $y = \frac{1}{4}\mathcal{H}x\mathcal{H}$.

$\phi_{x,y}$ are second class constraints, so $\phi_x\psi = \phi_y\psi = 0$ are automatically fulfilled. Denoting by 2ω the eigenvalues of $\mathcal{H}$, the corresponding eigenstates are simple plane waves. Their full r-'evolution' is given by

$$\psi_\omega(x, r) = \frac{1}{\sqrt{4\pi}}e^{\frac{i}{\hbar}\omega(x-2r)}. \tag{8}$$

They are not localized and form a δ-normalizable set. The most general solution is of the form $\psi(x - 2r)$. We are however after the 'most classical' wave packet defined by the minimal uncertainty relation $\Delta x \Delta \mathcal{H} = \hbar$, namely

$$\psi(x, r) = \frac{e^{-\frac{(x-2r+4m)^2}{64\sigma^2}}}{2(2\pi)^{\frac{1}{4}}\sqrt{\sigma}}, \tag{9}$$

for which the classical Schwarzschild solution (5) is both the average as well as the most probable configuration. We thus expect the wave packet Eq. (9) to capture all semi-classical essence of black hole thermodynamics. We have limited ourselves in this paper to the 'most classical' black hole wave packet simply because Bekenstein–Hawking thermodynamics is formulated in the background of a classical event horizon. One can even construct an orthonormal tower of non-minimal uncertainty wave packets [11], to be regarded a prediction of the mini super-spacetime approach (to be discuss elsewhere), none of which sharing the Schwarzschild configuration as the most probable. Eq. (9) is a superposition of plane waves. Its Fourier transform is given by

$$\tilde{\psi}(\mathcal{H}) = \frac{2\sqrt{\sigma}}{(2\pi)^{\frac{1}{4}}}e^{-4\sigma^2\mathcal{H}^2}e^{2im\mathcal{H}}. \tag{10}$$

We identify the mass operator as $M = \frac{1}{4}(2r - x)$ (in the $\mathcal{H}$-language it reads $M = -\frac{1}{2}i\hbar\frac{\partial}{\partial\mathcal{H}}$). For the Gaussian wave packet (9) it means

$$\langle M \rangle = m, \qquad \langle M^2 \rangle = m^2 + \sigma^2. \tag{11}$$

The black wave packet probability density $\psi^\dagger\psi$ can be directly translated into a statistical mechanics normalized mass spectrum

$$\rho(M; m, \sigma) = \frac{e^{-\frac{(M-m)^2}{2\sigma^2}}}{\sqrt{2\pi}\,\sigma}. \tag{12}$$

While a non-negative average mass $m \geq 0$ (the classical choice) is soon to be dictated on thermodynamical grounds, the mass distribution must cover, for the sake of quantum completeness, the entire range $-\infty < M < \infty$. However, the probability to have negative masses drops like $\sim \exp(-m^2/2\sigma^2)$ towards the classical limit.

At this stage, one may wonder where is the black hole horizon actually located? As far as our wave packet is concerned, there is nothing special going on in the neighbourhood of $r = 2Gm/c^2$ (and actually also not near the origin). Supported by Eq. (6), this suggests that a sharp horizon is merely a classical gravitational concept. However, one may still effectively interpret Eq. (12) as the quantum-mechanical profile of the horizon, with a probability density $\rho(M; m, \sigma)$ to find it at radius $2GM/c^2$. In some sense, this reminds us of the fuzzball proposal [7] where the black hole arises from coarse graining over horizon-free non-singular geometries. See [16] for horizon wave packets, and [14,17] for horizon fluctuations.

Treating the quantum-mechanical black hole as a subsystem (a field theory defined on a black hole background is expected to be in a thermal state whose temperature at infinity is the Hawking temperature), its Gaussian mass spectrum is temperature dependent. Following Fowler prescription for dealing with such a case, the naive partition function must be modified according to

$$Z(\beta) = \sum_n \rho_n e^{-\beta F_n(\beta)}, \tag{13}$$

with the Boltzmann factor being traded for the Gibbs–Helmholtz factor. The Helmholtz free energy function F obeys the Gibbs–Helmholtz equation

$$F + \beta\frac{\partial F}{\partial\beta} = E(\beta) \quad \Rightarrow \quad \beta F(\beta) = \int_{\beta_0}^{\beta} E(b)db, \tag{14}$$

with β_0 to be fixed on physical grounds. To proceed, it is convenient to discretize the problem by dividing the normal mass distribution $\rho(M; m, \sigma)$ into N equal probability and temperature
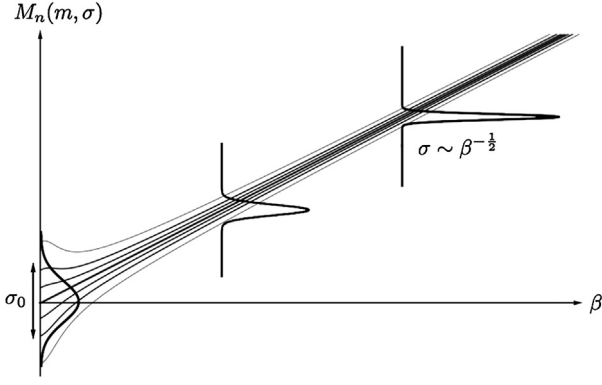
Fig. 1. The black hole wave packet mass spectrum is plotted as a function of the inverse Hawking temperature β. The average mass m is linear in β. The width σ peaks at σ_0 and decreases Doppler-like towards the classical limit (where the probability of the negative masses is practically negligible).

independent sections, each of which representing a wide energy level, such that

$$\int_{M_n}^{M_{n+1}} \rho(M; m, \sigma)\,dM = \frac{1}{N}. \tag{15}$$

This equation is formally solved by invoking the inverse error function $\mathrm{erf}^{-1} x$, that is

$$M_n(\beta) = m(\beta) - \sqrt{2}\sigma(\beta)\,\mathrm{erf}^{-1}\left(1 - \frac{2n}{N}\right), \tag{16}$$

for $n = 1, ..., N-1$. The normal mass spectrum is depicted in Fig. 1. A straightforward solution of the differential Gibbs–Helmholtz equation (14), with $M_n(\beta)$ serving as the source term, reveals the Helmholtz free energy associated with the n-th mass level

$$\beta F_n = \int_{\beta_0}^{\beta} m(b)\,db - \sqrt{2}\,\mathrm{erf}^{-1}\left(1 - \frac{2n}{N}\right)\int_{\beta_0}^{\beta} \sigma(b)\,db, \tag{17}$$

which we can now substitute into the partition function $Z(\beta) = \frac{1}{N}\sum_{n=1}^{N} e^{-\beta F_n(\beta)}$. Let $N \to \infty$, define a continuous integration variable $\xi = \frac{n}{N}$, and use the integration formula $\int_0^1 e^{\sqrt{2}\lambda\,\mathrm{erf}^{-1}(1-2\xi)}\,d\xi = e^{\frac{1}{2}\lambda^2}$ to arrive at

$$Z(\beta) = \exp\left(-\int_{\beta_0}^{\beta} m(b)\,db + \frac{1}{2}\left(\int_{\beta_0}^{\beta} \sigma(b)\,db\right)^2\right). \tag{18}$$

The entropy $S = \left(1 - \beta\frac{\partial}{\partial\beta}\right)\log Z$ can now be calculated, and turns out to be the sum of two separate contributions

$$\boxed{S(\beta) = S_m(\beta) + S_\sigma(\beta)} \tag{19}$$

$$S_m(\beta) = \beta m(\beta) - \int_{\beta_0}^{\beta} m(b)\,db, \tag{20a}$$

$$S_\sigma(\beta) = -\beta\sigma(\beta)\int_{\beta_0}^{\beta} \sigma(b)\,db + \frac{1}{2}\left(\int_{\beta_0}^{\beta} \sigma(b)\,db\right)^2. \tag{20b}$$

The associated internal energy exhibits a similar structure, and is given by

$$U(\beta) = m(\beta) - \sigma(\beta)\int_{\beta_0}^{\beta} \sigma(b)\,db, \tag{21}$$

closing on the first law $S'(\beta) = \beta U'(\beta)$.

At this stage, $m(\beta)$ and $\sigma(\beta)$ are two yet unspecified independent functions of β. The connection with black hole physics requires some input beyond the mini super-spacetime model. This is hereby established by invoking the Bekenstein area entropy ansatz

$$\boxed{S = \frac{\langle M^2\rangle}{2\eta^2} + c_S} \tag{22}$$

quantum-mechanically adjusted however by trading the classical $\langle M\rangle^2 = m^2$ for $\langle M^2\rangle = m^2 + \sigma^2$. The constant η will be recognized as the reduced Planck mass

$$\eta = \sqrt{\frac{\hbar c}{8\pi G}} \tag{23}$$

as soon as the contact with Hawking temperature is analytically established, and c_S is a constant to be determined. Having the first law for a Gaussian mass distribution at our disposal, with its compelling $m \leftrightarrow \sigma$ split Eqs. (19), (21), we can now proceed to calculate the independent functions $m(\beta)$ and $\sigma(\beta)$. The corresponding non-linear integral–differential equations to solve are

$$S_m(\beta) = \frac{m^2(\beta)}{2\eta^2} + c_m, \qquad S_\sigma(\beta) = \frac{\sigma^2(\beta)}{2\eta^2} + c_\sigma, \tag{24}$$

with $c_m + c_\sigma = c_S$.

The exact solution of the first equation is straightforward, and is noticeably β_0, c_S-independent, namely

$$m(\beta) = \eta^2\beta, \tag{25}$$

reassuring us that the reciprocal Hawking temperature β is proportional, as expected (but non-trivial in the absence of a sharp horizon), to the necessarily positive average mass. The recovery of $m(\beta)$ is a necessary stage preceding the $\sigma(\beta)$ calculation. Fixing β_0 will then determine $c_m = \frac{1}{2}\eta^2\beta_0^2$. The solution of the second equation is somewhat more complicated. Define $f(\beta) \equiv \int_{\beta_0}^{\beta} \sigma(b)\,db$, so that $\sigma(\beta) = f'(\beta)$, and attempt to solve numerically

$$f'(\beta) = -\eta^2\beta f(\beta) + \eta\sqrt{(1+\eta^2\beta^2)f^2(\beta) - 2c_\sigma}, \tag{26}$$

subject to $f(\beta_0) = 0$. Before doing so, however, it is crucial to first fix the β_0 parameter on physical grounds.

Fowler and Rushbrooke could not give a general rule for fixing β_0. They say "The ambiguity has its counterpart in the use of the Gibbs Helmholtz equation to derive free energy from true energy. One needs to know, for instance, the entropy of the substance at some one particular temperature". Under $\beta_0 \to \beta_0 + \delta\beta_0$, the entropy $S(\beta)$ gets shifted by a β-dependent amount. In other words, the choice of β_0 is a physical choice. And since it cannot be sensitive to $S \to S + const$, its roots must be at the $S'(\beta_0) = \beta_0 U'(\beta_0)$ level. The only tenable choice is $\beta_0 = 0$; it is universal in the sense that

$$S(0) = S'(0) = U(0) = 0, \tag{27}$$

suggesting (to be implied later) that $U'(0) = \eta^2 - \sigma_0^2$ should vanish as well. In fact, there is even a simpler argument to support the $\beta_0 = 0$ choice. Hawking temperature (25) tells us that choosing β_0 means choosing a special average mass m_0, but there is no such a special mass. The accompanying constants take then the values

$$c_m = 0, \qquad c_\sigma = -\frac{\sigma_0^2}{2\eta^2} = c_S. \tag{28}$$

Unfortunately, Eq. (26) does not admit an exact analytic solution. It tells us, however, that $\sigma(\beta) = f'(\beta)$ is a monotonically decreasing function of β, solely parameterized by the maximal width σ_0. As far as the small-$\eta\beta$ region is concerned, we derive the asymptotic expansion

$$\sigma(\beta) = \sigma_0 \left(1 - \frac{1}{2}\eta^2\beta^2 + \frac{3}{8}\eta^4\beta^4 + \dots \right). \tag{29}$$

Even the special case $m = 0$, which classically leads to a flat space-time, is quantum-mechanically accompanied by a wave packet of non-vanishing width. Similarly for large-$\eta\beta$, we face

$$\sigma(\beta) = \frac{s\sigma_0}{2\sqrt{\eta\beta}} \left(1 + \frac{1}{2s^2\eta\beta} + \dots \right), \tag{30}$$

where $s \simeq 0.6185$ has been fixed numerically. No log-terms [18] at this stage. The Hawking temperature dependent width of macro black hole wave packets highly reminds us (but apparently without any physics in common) of the Doppler broadening of spectral lines.

m and σ have been gradually elevated from being two independent parameters to two explicit functions of the Hawking temperature. Treating β as a parameter, one can express $\sigma(m)$, and proceed to discuss the entropy and the internal energy. At the classical limit $m \gg \eta$ there are no surprises, with the leading Bekenstein–Hawking formulas acquire only tiny corrections

$$S(m) = \frac{m^2}{2\eta^2} - \frac{\sigma_0^2}{2\eta^2} + \frac{s^2\sigma_0^2}{8\eta m} + \dots$$
$$U(m) = m - \frac{s^2\sigma_0^2}{2\eta} + \dots \tag{31}$$

At the quantum regime $m \leq \eta$, on the other hand, we find ourselves in an unfamiliar territory governed by

$$S(m) = \left(1 - \frac{\sigma_0^2}{\eta^2} \right)\frac{m^2}{2\eta^2} + \frac{\sigma_0^2 m^4}{2\eta^6} + \dots$$
$$U(m) = \left(1 - \frac{\sigma_0^2}{\eta^2} \right)m + \frac{2\sigma_0^2 m^3}{3\eta^4} + \dots \tag{32}$$

Regarding the value of σ_0, several possibilities arise:

(i) If $\sigma_0 = 0$, we recover the Bekenstein–Hawking black hole thermodynamics of Schwarzschild spacetime.
(ii) If $\sigma_0 > \eta$, the entropy function develops a local maximum at $m = 0$. This in turn causes the small-m section of $S(m)$ to be negative, and hence must be rejected on entropy positivity grounds.
(iii) If $\sigma_0 < \eta$, the entropy $S(m)$ exhibits an absolute minimum at $m = 0$. The minimal entropy is still proportional to $S_{BH} = m^2/2\eta^2$, but is suppressed now by a factor of $1 - \sigma_0^2/\eta^2$.
(iv) If $\sigma_0 = \eta$ (accompanied by $c_S = -\frac{1}{2}$), the black hole entropy barely keeps its minimum at $m = 0$, and the internal energy gives up its linear small-m behavior.

The smallest size quantum-mechanical black hole wave packet comes with $m = 0$ and $\sigma = \sigma_0$. We insist on attaching to it a minimal entropy, but do we have a physical reason which can single out one particular value for $\sigma_0 \leq \eta$? In fact, we do. Recall that $U'(0) = \eta^2 - \sigma_0^2$, so $\sigma_0 = \eta$ can now complete the partial set of initial conditions (27) by supplementing the missing piece $U'(0) = 0$.

Carrying zero entropy, this micro black hole represents a single degree of freedom, and in this respect can be regarded elementary. It is characterized by a finite root mean square mass $m_{RMS} = \eta$ (consistent with the fact that Compton wavelength puts a limit on the minimum size of the region in which a mass can be localized), yet it is divergently hot, a feature which is supposed to play a crucial role at the final stage of black hole evaporation.

While a classical event horizon is apparently mandatory for formulating black hole thermodynamics, Bekenstein entropy will explode and Hawking temperature vanish if $\hbar$ is switched off. It is thus relieving to learn that black hole thermodynamics can be consistently resumed when $\hbar$ is switched on, causing inevitable horizon smoothening. The mass spectrum, which on quantum-mechanical consistency grounds must contain negative masses, plays here a central role. While the average mass sets the recovered Hawking temperature, a novel temperature dependent width function contributes to Bekenstein entropy. It peaks at the Planck mass for an elementary quantum black hole of finite rms size, and decreases Doppler like towards the classical limit.

Acknowledgement

References

[1] S.W. Hawking, Phys. Rev. Lett. 26 (1971) 1344;
J.D. Bekenstein, Lett. Nuovo Cimento 4 (1972) 737;
J.D. Bekenstein, Phys. Rev. D 7 (1973) 2333;
S.W. Hawking, Nature 248 (1974) 30;
J.D. Bekenstein, Phys. Rev. D 9 (1974) 3292;
S.W. Hawking, Commun. Math. Phys. 43 (1975) 199.
[2] G.W. Gibbons, S.W. Hawking, Phys. Rev. D 15 (1977) 2752;
G. t' Hooft, Nucl. Phys. B 256 (1985) 727;
R.M. Wald, Phys. Rev. D 48 (1993) 3427;
V. Iyer, R.M. Wald, Phys. Rev. D 50 (1994) 846;
S. Carlip, Class. Quantum Gravity 16 (1999) 3327.
[3] J.D. Bekenstein, V.F. Mukhanov, Phys. Lett. B 360 (1995) 7.
[4] A. Strominger, C. Vafa, Phys. Lett. B 379 (1996) 99;
C. Rovelli, Phys. Rev. Lett. 77 (1996) 3288;
R. Emparan, G.T. Horowitz, Phys. Rev. Lett. 97 (2006) 141601.
[5] B.S. DeWitt, Phys. Rev. 160 (1967) 1113;
J.A. Wheeler, in: Battelle Rencontres, Benjamin, NY, 1968, p. 242;
C.W. Misner, Phys. Rev. Lett. 22 (1969) 1071;
W.E. Blyth, C. Isham, Phys. Rev. D 11 (1975) 768;
S.W. Hawking, I.G. Moss, Phys. Lett. B 110 (1982) 35;
J.B. Hartle, S.W. Hawking, Phys. Rev. D 28 (1983) 2960;
J.J. Halliwell, S.W. Hawking, Phys. Rev. D 31 (1985) 1777.
[6] D. Garfinkle, G.T. Horowitz, A. Strominger, Phys. Rev. D 43 (1991) 3140;
A. Sen, Nucl. Phys. B 440 (1995) 421;
J.M. Maldacena, Ph.D. thesis, arXiv:hep-th/9607235;
L. Susskind, J. Lindseday, in: Black Holes, Information and the String Theory Revolution, World Scientific, 2005.
[7] O. Lunin, S.D. Mathur, Nucl. Phys. B 623 (2002) 342;
K. Skenderis, M. Taylor, Phys. Rep. 467 (2008) 117.
[8] T. Thiemann, H.A. Kastrup, Nucl. Phys. B 399 (1993) 211;
R. Gambini, J. Pullin, Phys. Rev. Lett. 110 (2013) 211301.
[9] V. Frolov, I. Novikov, Phys. Rev. D 48 (1993) 4545.
[10] R.H. Fowler, in: Statistical Mechanics, Cambridge, 1936;
R.H. Fowler, E.A. Guggenheim, in: Statistical Thermodynamics, Cambridge Univ. Press, 1939;
G.S. Rushbrooke, Trans. Faraday Soc. 36 (1940) 1055;
E.W. Elcock, P.T. Landsberg, Proc. Phys. Soc. B 70 (1957) 161.
[11] A. Davidson, B. Yellin, Gen. Relativ. Gravit. 46 (2014) 1662.
[12] A. Connes, Sel. Math. 5 (1999) 29;
M.V. Berry, in: I.V. Lerner, et al. (Eds.), Supersymmetry and Trace Formulae, Kluver Academic, NY, 1999;
M.V. Berry, J.P. Keating, SIAM Rev. 41 (1999) 236;
G. Sierra, P.K. Townsend, Phys. Rev. Lett. 101 (2008) 110201.
[13] K.V. Kuchar, Phys. Rev. D 50 (1994) 3961;

T. Brotz, C. Kiefer, Phys. Rev. D 55 (1997) 2186;
C. Kiefer, J. Louko, Ann. Phys. (Leipzig) 8 Ser. 67 (1999);
T. Christodoulakis, N. Dimakis, P.A. Terzis, G. Doulis, Th. Grammenosc, E. Melasd, A. Spanou, J. Geom. Phys. 71 (2013) 127.
[14] J.W. York Jr., B.S. Schmekel, Phys. Rev. D 72 (2005) 024022.
[15] P.A.M. Dirac, in: Lectures on Quantum Mechanics, Dover Publications, 1964.
[16] R. Casadio, F. Scardigli, Eur. Phys. J. C 74 (2014) 2685.
[17] R.D. Sorkin, in: Proc. 1st Australian Conf. on GRG, University of Adelaide, 1996, p. 163;

A. Casher, F. Englert, N. Itzhaki, S. Massar, R. Parentani, Nucl. Phys. B 484 (1997) 419;
D. Marolf, Springer Proc. Phys. 98 (2005) 99.
[18] J.L. Cardy, Nucl. Phys. B 270 (1986) 186;
R.K. Kaul, P. Majumdar, Phys. Rev. Lett. 84 (2000) 5255;
S. Carlip, Class. Quantum Gravity 17 (2000) 4175;
S. Carlip, Phys. Rev. Lett. 88 (2002) 241301;
A. Chatterjee, P. Majumdar, Phys. Rev. Lett. 92 (2004) 141301;
S.N. Solodukhin, Living Rev. Relativ. 14 (2011) 8.