Note on 2-rainbow domination and Roman domination in graphs

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\textbf{Abstract}

A Roman dominating function of a graph \( G \) is a function \( f : V \rightarrow \{0,1,2\} \) such that every vertex with 0 has a neighbor with 2. The minimum of \( f(V(G)) = \sum_{v \in V} f(v) \) over all such functions is called the Roman domination number \( \gamma_R(G) \). A 2-rainbow dominating function of a graph \( G \) is a function \( g \) that assigns to each vertex a set of colors chosen from the set \{1, 2\}. The 2-rainbow domination number \( \gamma_2(G) \) is the minimum of \( w(g) = \sum_{v \in V} |g(v)| \) over all such functions. We prove \( \gamma_2(G) \leq \gamma_R(G) \) and obtain sharp lower and upper bounds for \( \gamma_2(G) + \frac{\gamma_R(G)}{2} \). Finally, we give a proof of the characterization of graphs with \( \gamma_2(G) = \gamma(R) + k \) for \( 2 \leq k \leq \gamma(G) \). 

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1. Introduction

In this paper, we follow the notation of [1]. Specifically, let \( G = (V, E) \) be a graph with vertex set \( V \) and edge set \( E \). \( \delta(G) \) and \( \Delta(G) \) denote the minimum degree and maximum degree of \( G \), respectively. For any vertex \( v \in V \), the open neighborhood of \( v \) is the set \( N(v) = \{u \in V \mid uv \in E\} \) and the closed neighborhood is the set \( N[v] = N(v) \cup \{v\} \). For a set \( S \subseteq V \), the open neighborhood \( N(S) = \bigcup_{v \in S} N(v) \) and the closed neighborhood is \( N(S) = N(S) \cup S \). A set \( S \subseteq V \) is a dominating set of \( G \) if \( N[S] = V \). The domination number of \( G \), denoted by \( \gamma(G) \), is the minimum cardinality of a dominating set. A dominating set of cardinality \( \gamma(G) \) is called a \( \gamma \)-set of \( G \). The concept of domination in graphs, with its many variations, is now well studied in graph theory. A thorough study of domination appears in [2].

There is a variant of the domination number—Roman domination number, which is suggested by Stewart [3]. A Roman dominating function (RDF) on a graph \( G \) is a function \( f : V \rightarrow \{0,1,2\} \) satisfying the condition that every vertex \( u \) for which \( f(u) = 0 \) is adjacent to at least one vertex \( v \) for which \( f(v) = 2 \). The weight of \( f \) is \( f(V(G)) = \sum_{v \in V} f(v) \). The Roman domination number, denoted by \( \gamma_R(G) \), equals the minimum weight of an RDF of \( G \), and we say that a function \( f \) is a \( \gamma_R(G) \)-function if it is an RDF and \( f(V(G)) = \gamma_R(G) \). For a graph \( G \), let \( f : V \rightarrow \{0,1,2\} \), and let \( (V_0, V_1, V_2) \) be the order partition of \( V \) induced by \( f \), where \( V_i = \{v \in V(G) \mid f(v) = i\} \) for \( i = 0, 1, 2 \). Note that there exists a 1–1 correspondence between the functions \( f : V \rightarrow \{0,1,2\} \) and the ordered partitions \( (V_0, V_1, V_2) \) of \( V(G) \). Thus we will write \( f = (V_0, V_1, V_2) \).

Let \( g \) be a function that assigns to each vertex a set of colors chosen from the set \{1, \ldots, \( k \)}; that is, \( g : V(G) \rightarrow 2^V \). If for each vertex \( v \in V(G) \) such that \( g(v) = \emptyset \). We have

\[ \bigcup_{u \in N(v)} g(u) = \{1, \ldots, k\}. \]
Then $g$ is called a $k$-rainbow dominating function of $G$. The weight, $w(g)$, of a function $g$ is defined as $w(g) = \sum_{v \in V} |g(v)|$. Given a graph $G$, the minimum weight of a $k$-rainbow dominating function is called the $k$-rainbow domination number of $G$, which we denote by $\gamma_k(G)$.

For a pair of graphs $G$ and $H$, the Cartesian product $G \square H$ of $G$ and $H$ is the graph with vertex set $V(G) \times V(H)$ and where two vertices are adjacent if and only if they are equal in one coordinate and adjacent in the other. Rainbow domination of a graph $G$ coincides with the ordinary domination of the Cartesian product of $G$ with the complete graph, in particular $\gamma_k(G) = \gamma(G \square K_k)$ for any graph $G$ [4]. In the language of domination of Cartesian products, Hartnell and Rall obtained several observations about rainbow domination, for instance, $\min\{|V(G)|, \gamma(G) + k - 2\} \leq \gamma_k(G) \leq k\gamma(G)$, for any $k \geq 2$ and any graph $G$ [5]. The attempt in [5] to characterize graphs with $\gamma(G) = \gamma_2(G)$ was inspired by the following famous open problem [6].

Vizing’s conjecture. For any graphs $G$ and $H$, $\gamma(G)\gamma(H) \leq \gamma(G \square H)$.

Brešar and Šumenjak [7] showed that the problem of deciding if a graph has a 2-rainbow dominating function of a given weight is NP-complete even when restricted to bipartite graphs or chordal graphs. Some exact values of 2-rainbow domination number of several classes of graphs are found in [7,8]. Wu [9] presents some general bounds on the 2-rainbow domination number of a graph that are expressed in terms of the order and domination number of a graph.

2. Main results

For Roman domination, Cockayne et al. [10] showed the following inequality.

**Proposition 1 (Cockayne et al. [10]).** For any graph $G$, $\gamma(G) \leq \gamma_k(G) \leq 2\gamma(G)$.

In fact, we can insert the parameter $\gamma_2(G)$ into the above inequality.

**Proposition 2.** Let $G$ be a graph. Then

$$\gamma(G) \leq \gamma_2(G) \leq \gamma_k(G) \leq 2\gamma(G).$$

**Proof.** We only need to show $\gamma_2(G) \leq \gamma_k(G)$. Suppose $f = (V_0, V_1, V_2)$ is an RDF of $G$. Then $V_0 \subseteq N(V_2)$. Now we set

$$g(v) = \begin{cases} \emptyset & v \in V_0, \\ \{1\} \text{ or } \{2\} & v \in V_1, \\ \{1, 2\} & v \in V_2. \end{cases}$$

It is clear that this is a 2-rainbow dominating function of $G$. Then $\gamma_2(G) \leq w(g) = |V_1| + 2|V_2| = \gamma_k(G)$. □

The corona $HoK_1$ of a graph $H$ is obtained by attaching one pendent edge at each vertex of $H$. Let $\mathcal{F}$ be the family of graphs obtained from a connected graph $H$ by identifying each vertex of $H$ with the central vertex of a path $P_3$ or with an internal vertex of a path $P_4$ where the $V(H)$ paths are vertex-disjoint. $\mathcal{F}$ is the family of graphs of $\mathcal{F}$ such that each vertex of $H$ is identified with an internal vertex of a path $P_4$. Favaron, Karami, Khoeilar and Sheikholeslami [11] obtained the following result:

**Theorem 1 (Favaron et al. [11]).** For any connected graph $G$ of order $n \geq 3$, then $\gamma_k(G) + \frac{\gamma(G)}{2} \leq n$ with equality if and only if $G$ is $C_4, C_5, C_4\circ K_1$ or $G$ belongs to $\mathcal{F}$.

Let now $G$ be a graph of $\mathcal{F}$ composed of $k_1$ paths $P_3$ and $k_2$ paths $P_5$. Then $\gamma_2(G) = 2k_1 + 2k_2, \gamma_2(G) = 3k_1 + 3k_2$. With Proposition 2, the following corollary is obtained.

**Corollary 1.** For any connected graph $G$ of order $n \geq 3$, then $\gamma_2(G) + \frac{\gamma(G)}{2} \leq n$ with equality if and only if $G$ is $C_4\circ K_1$ or $G$ belongs to $\mathcal{F}$. If $\gamma_2(G) = 1$, then $G$ is a trivial graph, i.e., all graphs $G$ of order at least two with $\gamma_2(G) \geq 2$.

**Proposition 3.** Let $G$ be a graph of order $|V(G)| = n \geq 2$. Then $\gamma_2(G) = 2$ if and only if $K_{1,n-1}$ or $K_{2,n-2}$ is a spanning subgraph of $G$.

**Proof.** If $K_{1,n-1}$ or $K_{2,n-2}$ is a spanning subgraph of $G$, then it holds. Conversely, $\gamma(G) \leq 2$ since $\gamma(G) \leq \gamma_2(G)$. If $G$ is an edge $uv$, then it holds. So assume $|V(G)| \geq 3$. Suppose $f$ is a 2-rainbow dominating function with weight 2. If there is only one vertex with color $\{1, 2\}$ and all the other vertices with empty set, then $\gamma(G) = 1$, i.e., $K_{1,n-1}$ is a spanning subgraph of $G$. Otherwise, there exist two vertices $u$ and $v$ with colors 1 and 2, respectively. Since $f$ is a 2-rainbow dominating function, then for each vertex $t \in V(G) - \{u, v\}, \{u, v\} \subseteq N_G(t)$. Hence $K_{2,n-2}$ is a spanning subgraph of $G$. □

Let $v$ be a vertex in $G$ with maximum degree $\Delta(G)$. If we set

$$f(v) = \begin{cases} \emptyset & u \in N(v), \\ \{1\} \text{ or } \{2\} & u \in V(G) - N[v], \\ \{1, 2\} & u = v. \end{cases}$$

Then $f$ is 2-rainbow dominating function of $G$. So the following proposition holds.
**Proposition 4.** If $G$ is a graph of order $n$, then $\gamma_2(G) \leq n - \Delta(G) + 1$.

Let $\overline{G}$ be the complement of a graph $G$. We show the following result.

**Theorem 2.** If $G$ is a graph with order $n \geq 3$, then

$$5 \leq \gamma_2(G) + \gamma_2(\overline{G}) \leq n + 2.$$  

Moreover, the equalities can be obtained.

**Proof.** When $G$ has at least three vertices, $\gamma_2(G) \geq 2$. By **Proposition 3**, the equality holds only when $\gamma(G) = 1$ or $\gamma(G) = 2$ and $G$ has an independent dominating set $\{u, v\}$ and $\Delta(G) \leq |V(G)| - 2$. A graph and its complement can not both have dominating vertices, so if $\gamma(G) = 1$, then $\gamma_2(G) \geq 3$. Otherwise $\gamma(G) = 2$ and $|V(G)| \geq 4$, then $\overline{G}$ contains at least two components and one of them is an edge. So $\gamma_2(G) \geq 4$. Thus the left equality holds if and only if $G$ (resp. $\overline{G}$) has a dominating vertex and $\overline{G}$ (resp. $G$) contains an isolated vertex $x$ such that $\gamma_2(\overline{G} - x) = 2$ (resp. $\gamma_2(G - x) = 2$).

By **Proposition 4**, $\gamma_2(G) + \gamma_2(\overline{G}) \leq (n - \Delta(G) + 1) + (n - \Delta(\overline{G}) + 1) = n - \Delta(G) + \Delta(G) + 3 \leq n + 3$.

If $\gamma_2(G) + \gamma_2(\overline{G}) = n + 3$, then equality holds throughout the above calculation, and $\Delta(G) = \delta(G)$. Hence $G$ is $k$-regular for some $k$. Without loss of generality assume that $k \leq (n - 1)/2$, since our argument is symmetric in $G$ and $\overline{G}$. Since equality holds, $\gamma_2(G) = n - k + 1$ and $\gamma_2(\overline{G}) = k + 2$.

Let $v \in V(G)$. If some vertex $u$ outside $N[v]$ in $G$ has at least two neighbors outside $N[u]$, then set $f(v) = f(u) = \{1, 2\}$, for $s \in V(G) - N[v] \cup N[u]$, let $f(s) = \{1\}$ and other vertices with empty set. Then $f$ is a 2-rainbow dominating function of $G$ with weight at most $n - k$, a contradiction. Hence every vertex not in $N[v]$ has at least $k - 1$ neighbors in $N(v)$. A similar argument shows that each vertex in $N(v)$ has at most two neighbors outside $N[v]$.

Suppose $m$ is the number of edges joining $N(v)$ and $V(G) - N[v]$, we thus have $(k - 1)(n - k - 1) \leq m \leq 2k$. For $k \geq 2$, then $n \leq k + 1 + 2k/(k - 1)$. Since $n \geq 2k + 1$, we have $k \leq 2k/(k - 1)$, which requires $2 \leq k \leq 3$. If $k = 2$, we have $n \leq k + 1 + 2k/(k - 1) = 7$, and also $n \geq 2k + 1 = 5$. However $\gamma_2(C_3) = 3$, $\gamma_2(C_6) = 2$, $\gamma_2(C_4) = 4$, and $\gamma_2(C_5) = \gamma_2(C_7) + \gamma_2(C_8) = 4$, which is a contradiction to $\gamma_2(G) = n - k + 1$. If $k = 3$, then $n = 7$. It is a contradiction to that $G$ is 3-regular with even order. For $k = 1$, the only example is $(n/2)K_2, \gamma_2(G) + \gamma_2(\overline{G}) = n + 2$. For $k = 0$, the only example is $G = K_3$, where $\gamma_2(G) + \gamma_2(\overline{G}) = n + 2$. Then it implies equality does not hold. Hence $\gamma_2(G) + \gamma_2(\overline{G}) \leq n + 2$. □

Cockayne et al. [10] characterized the connected graphs $G$ with $\gamma_k$-functions of weight $\gamma(G) + 1$ and $\gamma(G) + 2$.

**Proposition 5 (Cockayne et al. [10]).** If $G$ is a connected graph of order $n$, then $\gamma_k(G) = \gamma(G) + 1$ if and only if there is a vertex $v \in V(G)$ of degree $n - \gamma(G)$.

**Proposition 6 (Cockayne et al. [10]).** If $G$ is a connected graph of order $n$, then $\gamma_k(G) = \gamma(G) + 2$ if and only if:

(a) $G$ does not have a vertex $v \in V(G)$ of degree $n - \gamma(G)$;

(b) either $G$ has a vertex of degree $n - \gamma(G) - 1$ or $G$ has two vertices $v$ and $w$ such that $|N[v] \cup N[w]| = n - \gamma(G) + 2$.

Subsequently, Xing, Chen and Chen [12] gave a characterization of graphs for which $\gamma_k(G) = \gamma(G) + k$ for $2 \leq k \leq \gamma(G)$. However, their proof has a logical mistake. Here we give a correct proof.

**Theorem 3 (Xing, Chen and Chen [12]).** Let $G$ be a connected graph of order $n$ and the domination number $\gamma(G) \geq 2$. If $k$ is an integer such that $2 \leq k \leq \gamma(G)$, then $\gamma_k(G) = \gamma(G) + k$ if and only if:

(a) for any integer $s$ with $1 \leq s \leq k - 1$, $G$ does not have a set $U_t$ of $(1 \leq t \leq s)$ vertices such that $|\bigcup_{t \in U_t} N[v]| = n - \gamma(G) - s + 2t$;

(b) there exists an integer $l$ with $1 \leq l \leq k$, and $G$ has a set $W_l$ of $l$ vertices such that $|\bigcup_{v \in W_l} N[v]| = n - \gamma(G) - k + 2l$.

**Proof.** By induction. If $k = 2$ then it holds by **Proposition 6**. So we assume $\gamma(G) \geq k \geq 3$ and the theorem holds for all values less than $k$. Let $f = (V_0, V_1, V_2)$ be a $\gamma_k(G)$-function of weight $\gamma_k(G) = \gamma(G) + k$.

First we prove that condition (a) holds. Suppose the contrary, that is, there exist two integers $s_0$ and $t_0$ with $1 \leq t_0 \leq s_0 \leq k - 1$, and $G$ has a set $U_{t_0}$ of $t_0$ vertices such that $|\bigcup_{v \in U_{t_0}} N[v]| = n - \gamma(G) - s_0 + 2t_0$. By **Proposition 5**, $s_0 \geq 2$ and without loss of generality assume that for any integer $s$ with $1 \leq s \leq s_0 - 1$, $G$ does not have a set $U_t$ of $(1 \leq t \leq s)$ vertices such that $|\bigcup_{v \in U_t} N[v]| = n - \gamma(G) - s + 2t$. Since $G$ has a set $U_{t_0}$ of $t_0$ $(1 \leq t_0 \leq s_0)$ vertices such that $|\bigcup_{v \in U_{t_0}} N[v]| = n - \gamma(G) - s_0 + 2t_0$, by the induction hypotheses, it follows that $\gamma_k(G) = \gamma(G) + s_0$. This contradicts the fact that $\gamma_k(G) = \gamma(G) + k$.

Next we prove that condition (b) holds. $G$ is connected, so $|V_0| \geq 1$. Since $2|V_2| + |V_1| = \gamma(G) + k$, $|V_1| + |V_2| \geq \gamma(G)$, it follows that $|V_2| \leq k$. Assume that $|V_2| = 1 (1 \leq l \leq k)$. Then $|V_1| = \gamma(G) + k - 2l$. Let $W_l = V_2$. No edge joins $V_1$ and $V_2$ and $V_0 \subseteq N(V_2)$, so there exists a set $W_l$ of $(1 \leq l \leq k)$ vertices such that $|\bigcup_{v \in W_l} N[v]| = n - |V_1| = n - (\gamma(G) + k - 2l) = n - \gamma(G) - k + 2l$. Hence, condition (b) holds.

Conversely, by induction hypotheses and condition (a), $\gamma_k(G) \geq \gamma(G) + k$. We define $V_0 = \bigcup_{v \in W_l} N[v] - W_l$, $V_1 = V(G) - \bigcup_{v \in W_l} N[v]$ and $V_2 = W_l$, then $f = (V_0, V_1, V_2)$ is an RDF with $f(V(G)) = 2|W_l| + |V(G)| - |\bigcup_{v \in W_l} N[v]| = \gamma(G) + k$, so $\gamma_k(G) \leq \gamma(G) + k$. Therefore, the equality $\gamma_k(G) = \gamma(G) + k$ holds. □
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References