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COMMUNICATION

UNIMODULARITY AND CIRCLE GRAPHS*

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A property of unimodularity is introduced for antisymmetric integral matrices. It is satisfied by the adjacency matrix of a circle graph provided with a Naji orientation [8]. In a further paper we shall interprete this result in terms of symmetric matroids introduced in [2]. In this communication we give a direct proof by means of techniques used in [1] for an algorithmic solution of the Gauss problem on self-intersecting curves in the plane.

Let $A = (A_{vw}: v, w \in V)$ be an antisymmetric integral matrix. For each $W \subseteq V$ we let $A[W] = (A_{vw}: v, w \in W)$. We are interested in the following property of unimodularity:

$$\det(A[W]) \in \{-1, 0, +1\}, \quad W \subseteq V. \qquad (\alpha)$$

Our graphs will be simple. An *oriented graph* is a simple graph where an initial end and a final end have been distinguished for every edge. The *adjacency matrix* of an oriented graph G is the antisymmetric $(0, \pm 1)$-matrix $A = (A_{vw}: v, w \in V(G))$ such that $A_{vw} = +1$ if and only if vw is an edge oriented from v to w. The orientation of G is said to be *unimodular* if A satisfies Property (α).

Let m be a word such that each letter occurring in m occurs precisely twice. We say that m is a *double occurrence word.* An *alternance* of m is a non-ordered pair $v'v''$ of distinct letters such that we meet alternatively $\ldots v' \ldots v'' \ldots v' \ldots v'' \ldots$ when reading m. The *alternance graph* $G(m)$ is the simple graph whose vertices are the letters of m and whose edges are the alternances of m. From a geometric point of view, alternance graphs can be interpreted as intersection graphs of chords of a circle, and they are more widely known as *circle graphs*. These graphs are also related to stack sorting techniques in a paper by Even and Itai [4]. The reader will find in [6] a survey of some results on circle graphs. Naji [8] found recently a good characterization of circle graphs by means of particular orientations. These are precisely these orientations which we will consider.

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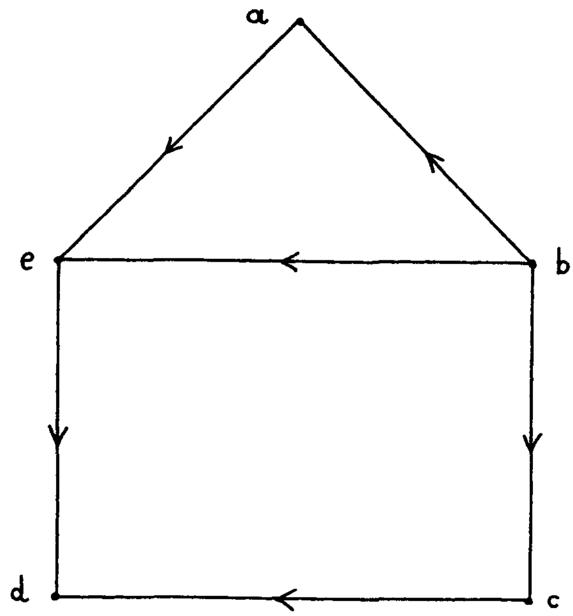

Fig. 1.

Let V be the set of letters of m, and let $V^+=\{v^+: v\in V\}$ and $V^-=\{v^-: v\in V\}$ be disjoint copies of V. A *separation* of m is any word μ over $V^+\cup V^-$ obtained by replacing in m the two occurrences of each letter v by v^+ and v^-. If $v'v''$ is an alternance of m, and $\ldots v'^{s'}\ldots v''^{s''}\ldots v'^{-s'}\ldots v''^{-s''}\ldots$ is the succession in μ of the letters belonging to $\{v'^+, v'^-, v''^+, v''^-\}$, then the edge $v'v''$ of $G(m)$ will be directed from v' to v'' if $s'=s''$, from v'' to v' otherwise. Let $D(\mu)$ be the oriented graph so defined, and let $A(\mu)=(A_{vw}: v, w\in V)$ be the adjacency matrix of $D(\mu)$.

A *rotation* of a word $x_1x_2\ldots x_r$ is its transformation into any word $x_ix_{i+1}\ldots x_rx_1\ldots x_{i-1}$. We notice that $G(m)$ and $D(\mu)$ are invariant after rotations of m and μ. Since each element of $V^+\cup V^-$ occurs precisely once in μ, we will identify μ to a cyclic permutation of $V^+\cup V^-$. Where α is the involution over $V^+\cup V^-$ which exchanges each pair $\{v^+, v^-\}$, we let $\mu^*=\mu\circ\alpha$, the composition of the permutations μ and α. If μ^* is a cyclic permutation, we define $D(\mu^*)$ and $A(\mu^*)$ like $D(\mu)$ and $A(\mu)$. To each orbit P of μ^* we attach the integral column matrix $X(P)=(X_v: v\in V)$ defined by $X_v=+1$ if $v^+\in P$ and $v^-\notin P$, $X_v=-1$ if $v^-\in P$ and $v^+\notin P$, $X_v=0$ otherwise.

Example. For $m=aebadecdbc$ and $\mu=a^+e^+b^-a^-d^+e^-c^-d^-b^+c^+$, Fig. 1 depicts $D(\mu)$. We have $\mu^*=(a^+d^+b^+a^-e^+c^-)(c^+d^-e^-b^-)$.

Property. *An orbit P of μ^* satisfies $X(P)=0$ if and only if μ^* is a cyclic permutation.*

Proof. $X(P)=0$ if and only if any pair $\{v^+, v^-\}$ which intersects P is included in P, which implies that P is a union of orbits of α. Therefore P is also a union of orbits of the composition $\mu^* \circ \alpha = \mu$. The result follows because μ is a cyclic permutation. □

Theorem. *If P is an orbit of μ^*, then $A(\mu)X(P) = 0$. If μ^* is a cyclic permutation, then $A(\mu^*) = A(\mu)^{-1}$.*

Proof. For each $v \in V$, we suppose that v^+ is the column-matrix indexed over V whose components are null at the exception of the v-component equal to $+1$, and we let $v^- = -v^+$. For each word $\mu' = p_1p_2 \ldots p_s$ which letters $p_1, p_2, \ldots, p_s$ in $V^+ \cup V^-$, let $H(\mu') = \sum (p_i: 1 \leqslant i \leqslant s)$. So we have $H(\mu) = 0$. Let us say that μ' *wraps above* μ if p_{i+1} follows p_i in μ (after eventually rotating μ) for $1 \leqslant i \leqslant s-1$. Moreover we say that μ' is *closed* if p_1 follows p_s in μ. Clearly

(i) $H(\mu') = 0$ if μ' is wrapped above μ and is closed.

For each $v \in V$ we denote by $S(v^+)$ and $S(v^-)$ the subwords of μ such that, after eventually rotating μ, $\mu = v^+S(v^+)v^-S(v^-)$.

(ii) $A(\mu)v^+ = -H(S(v^+)) = H(S(v^-)) = -A(\mu)v^-$, $v \in V$.

(iii) $S(x)S(y)$ wraps above μ if and only if $x = \mu^*(y)$.

It is easy to verify (ii). To prove (iii) we notice that the letter which follows $S(x)$ in μ is equal to $\alpha(x)$, when the letter which precedes $S(y)$ is y. Therefore $S(x)S(y)$ wraps above μ if and only if $y\alpha(x)$ is a pair of successive letters in μ, which means $x = \mu^*(y)$.

Let $(x_k \ldots x_2x_1)$ be the cyclic permutation induced over an orbit P of μ^*. Following (iii), the word $S(x_1)S(x_2)\ldots S(x_k)$ wraps above μ, and it is closed. Therefore

$$\begin{aligned} 0 &= H(S(x_1)S(x_2)\ldots S(x_k)) && \text{by (i)} \\ &= H(S(x_1)) + H(S(x_2)) + \cdots + H(S(x_k)) \\ &= -A(\mu)x_1 - A(\mu)x_2 - \cdots - A(\mu)x_k && \text{by (ii)} \\ &= -A(\mu)X(P), \end{aligned}$$

which proves the first part of the theorem.

If μ^* is a cyclic permutation, let us consider some $v \in V$ and, after an eventual rotation, let $\mu^* = v^+x_k \ldots x_2x_1v^-M$, with letters $x_1, x_2, \ldots, x_k \in V^+ \cup V^-$ and a subword M. Following (iii) the word $S(v^-)S(x_1)S(x_2)\ldots S(x_k)S(v^+)$ wraps above μ. Therefore the first letter of $S(x_1)$ is equal to v^+, and the last letter of $S(x_k)$ is also equal to v^+. Thus after removing either the first letter or the last letter of $S(x_1)S(x_2)\ldots S(x_k)$ we get a closed word wrapping above μ. Property (i) implies $v^+ = H(S(x_1)S(x_2)\ldots S(x_k))$, which implies as above $v^+ = -A(\mu)(x_1 + x_2 + \cdots + x_k)$. But $-(x_1 + x_2 + \cdots + x_k)$ is equal to the v-column of $A(\mu^*)$. Therefore $A(\mu)A(\mu^*)$ is the identity matrix. □

Corollary 1 (Jaeger [5]). *If G is an alternance graph whose adjacency matrix A considered over* GF(2) *has an inverse A^{-1}, then A^{-1} is the adjacency matrix of an alternance graph G^*.*

Proof. Use the second part of the theorem with matrices considered modulo 2, $G = G(m)$, $G^* = G(m^*)$, where m^* is the double occurrence word whose separation is μ^*. □

Corollary 2. *The orientation of $D(\mu)$ is unimodular.*

Proof. Each matrix $A[W]$ has an integral inverse when the inverse exists. □

There are two basic problems on unimodular orientations: (P1) to recognize whether a given orientation of a simple graph G is unimodular; (P2) to recognize whether a simple graph G admits an unimodular orientation. Let us consider the case where G is bipartite with chromatic classes V' and V''. If $A = (A_{vw}: v, w \in V(G))$ is the adjacency matrix of G provided with an orientation, and $B = (B_{v'v''}: v' \in V', v'' \in V'')$, then it is easy to verify that the orientation of G is unimodular if and only if B is totally unimodular. Let $M(G, V')$ be the binary matroid with a base equal to V' and the set of fundamental circuits $\{\{v'\} \cup \{v'': v'v'' \in E(G)\}: v' \in V'\}$ with respect to V'. It is easy to verify that G can be provided with an unimodular orientation if and only if $M(G, V')$ is a regular matroid. Thus Problems (P1) and (P2) are well solved (Camion [3], Tutte [9]) when G is bipartite.

Problem (P2) is of special interest because Corollary 2 says that an alternance graph admits an unimodular orientation. It can be verified directly that every nonalternance graph of lowest order, 6, does not satisfy this necessary condition. To see that the condition is not sufficient we can use the following theorem of de Fraysseix [5]: a bipartite graph G with a chromatic class V' is an alternance graph if and only if the matroid $M(G, V')$ is graphic and cographic. Figs. 2 and 3 depict two bipartite graphs G_2 and G_3 where V' is made of the circled vertices. We

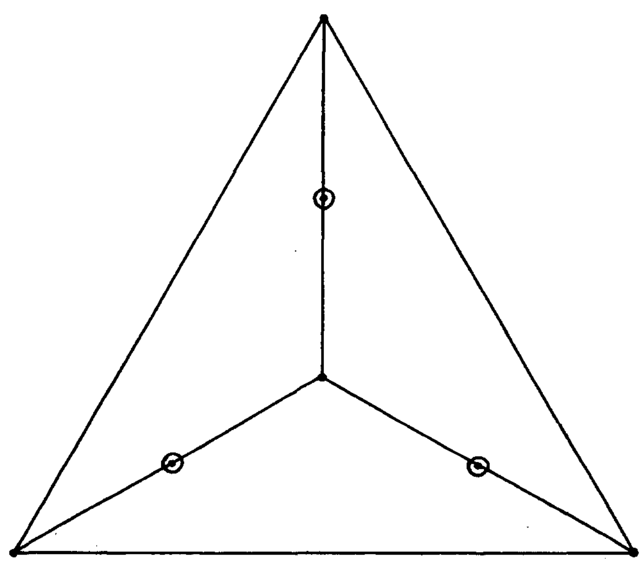

Fig. 2.

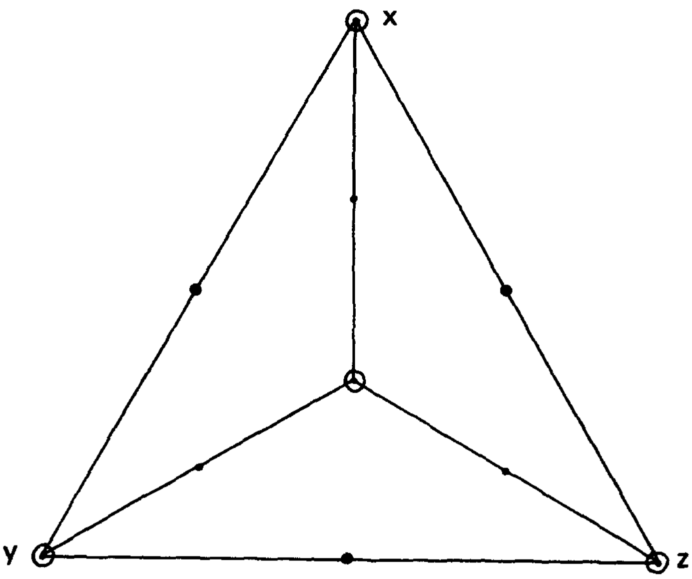

Fig. 3.

verify that $M(G_2, V')$ is the Fano matroid. This matroid is not regular, so that both de Fraysseix theorem and our necessary condition imply that G_2 is not an alternance graph. $M(G_3, V')$ is the cycle matroid of K_5 (V' corresponds in K_5 to a tree made of the four edges incident to a same vertex). This matroid is regular but it is not a cographic matroid, so that G_3 is not an alternance graph by de Fraysseix theorem when it admits an unimodular orientation.

The *local complementation* of a simple graph G at a vertex v is the operation which consists in replacing the subgraph induced on $\{w: vw \in E(G)\}$ by the complementary subgraph. A graph is *locally equivalent* to G if it is obtained through successive local complementations starting with G. If v is a letter of the double occurrence word m, and we decompose m as $AvBvC$ with suitable subwords A, B, C, and we replace the subword B by its mirror-image B', then the alternance graph of $m' = AvB'vC$ is the local complement of $G(m)$ at v. Therefore any graph locally equivalent to an alternance graph is also an alternance graph.

To see that G_3 is not an alternance graph we can make local complementations at the vertices x, y, z, obtaining so a graph G_3'. If we delete x, y, z of G_3', we get G_2 which has no unimodular orientation. If a graph has an unimodular orientation, this holds also for every induced subgraphs. Therefore G_3' has no unimodular orientation, and it cannot be a circle graph. This is also the case for G_3 which is locally equivalent to G_3'.

Conjecture. *A graph G is an alternance graph if every graph locally equivalent to G has an unimodular orientation.*

References

[1] A. Bouchet, Caractérisation des symboles croisés de genre nul, C. R. Acad. Sci. Paris 274 (1972) 724–727.
[2] A. Bouchet, Greedy algorithm and symmetric matroids, Math. Programming, to appear.
[3] P. Camion, Characterization of totally unimodular matrices, Proc. Amer. Math. Soc. 16 (1965) 1068–1073.
[4] S. Even and A. Itai, Queues, stacks and graphs, in: Theory of Machines and Computations (Academic Press, New York, 1971) 71–86.
[5] H. de Fraysseix, Local complementations and interlacement graphs, Discrete Math. 33 (1981) 29–35.
[6] M.C. Golumbic, Algorithmic Graph Theory and Perfect Graphs (Academic Press, New York, 1980).
[7] F. Jaeger, On some aglebraic properties of graphs, in: Progress in Graph Theory, J.A. Bondy and U.S.R. Murty eds. (Academic Press, New York, 1984).
[8] W. Naji, Reconnaissance des graphes de cordes, Discrete Math. 54 (1985) 329–337.
[9] W. Tutte, Lectures in matroids, J. Res. Nat. Bur. Standards Sect. B 69 (1965) 1–47.