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A new difference scheme with high accuracy and absolute stability for solving convection–diffusion equations☆

Hengfei Ding[a,∗], Yuxin Zhang[b]

[a] School of Mathematics and Statistics,Tianshui Normal University, Tianshui 741001, PR China
[b] School of Mathematics and Statistics, Lanzhou University, Lanzhou 730000, PR China

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ABSTRACT

In this paper, we use a semi-discrete and a padé approximation method to propose a new difference scheme for solving convection–diffusion problems. The truncation error of the difference scheme is $O(h^4 + \tau^5)$. It is shown through analysis that the scheme is unconditionally stable. Numerical experiments are conducted to test its high accuracy and to compare it with Crank–Nicolson method.

1. Introduction

Consider the convection–diffusion equation

$$\frac{\partial u}{\partial t} + \varepsilon \frac{\partial u}{\partial x} = \gamma \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < 1,\ t > 0 \tag{1.1}$$

subject to the initial condition

$$u(x, 0) = g(x), \quad 0 \le x \le 1$$

and boundary conditions

$$u(0, t) = 0, \quad t > 0.$$
$$u(1, t) = 0, \quad t > 0,$$

where the parameter γ is the viscosity coefficient and ε is the phase speed, and both are assumed to be positive. g is a given function of sufficient smoothness. This equation may be seen in computational hydraulics and fluid dynamics modeling convection–diffusion of quantities such as mass, heat, energy, vorticity, etc [1].

There has been much work on computing a finite difference approximation solution of equation (1.1), see [2–4]. We focus our attention on a method based on the high-order compact (HOC) finite difference discretization of equation (1.1)

☆ The project sponsored by SRF for ROCS, SEM.
∗ Corresponding author.
E-mail address: dinghf05@st.lzu.edu.cn (H. Ding).

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only with respect to the space variable. This type of discretization yields a system of ordinary differential equation. The solution of this system requires the computation of $e^{\tau A^{-1}B}\mathbf{y}$ for some vector $\mathbf{y}$, where τ is the time step-size, A and B are large Toeplitz matrixes. There are various methods to compute an approximation of $e^{\tau A^{-1}B}\mathbf{y}$. In [5–7,12], some approaches based on the Krylov subspace method were proposed. The restrictive Taylor's approximation method has been presented in [8,12]. In most of the cases, the accuracy of the difference schemes constructed by using the above methods is second order in time direction and second or fourth order in space direction. In this paper,we use padé approximation method to give an expression to compute the value of $e^{\tau A^{-1}B}$. So we get a new difference scheme for solving convection–diffusion equation (1.1) and the truncation error is $O(\tau^5 + h^4)$. Then the numerical results of our difference scheme for computing the approximate solution of Eq. (1.1) at some given time levels are compared with that of Crank–Nicolson scheme.

The present paper is organized as follows. In Section 2, we define the difference scheme and discuss the accuracy. In Section 3, we do stability analysis. Some numerical examples are presented in Section 4 and concluding remarks are given in Section 5.

2. Proposition of the difference scheme

We subdivide the interval $0 \le x \le 1$ into n equal subintervals by the grid points $x_i = ih$, $i = 0(1)n$, where $h = 1/n$,(n is a positive integer). The mesh function $u(ih, k\tau)$ is written as u_i^k at grid point $(ih, k\tau)$.

We start by examining the one-dimensional steady convection equation

$$-\gamma \frac{\mathrm{d}^2 u}{\mathrm{d}x^2} + \varepsilon \frac{\mathrm{d}u}{\mathrm{d}x} = f, \tag{2.1}$$

where f is a function of x. Using the techniques outlined in [9,10], it is easy to derive a three-point fourth-order compact scheme for Eq. (2.1) as

$$-\left(\gamma + \frac{\varepsilon^2 h^2}{12\gamma}\right)\delta_x^2 u_i + \varepsilon \delta_x u_i = \left(1 + \frac{h^2}{12}(\delta_x^2 - \frac{\varepsilon}{\gamma}\delta_x)\right) f_i + O(h^4), \tag{2.2}$$

where δ_x^2 and δ_x are the second-order and first-order center difference operators.

For convenience, we define two difference operators as follows

$$L_x = 1 + \frac{h^2}{12}(\delta_x^2 - \frac{\varepsilon}{\gamma}\,\delta_x), \qquad A_x = -\left(\gamma + \frac{\varepsilon^2 h^2}{12\gamma}\right)\delta_x^2 + \varepsilon\delta_x.$$

Eq. (2.2) can then be formulated symbolically as

$$L_x^{-1} A_x u_i = f_i + O(h^4). \tag{2.3}$$

A fourth-order semi-discrete approximation to the unsteady convection–diffusion equation in (1.1) can be obtained by replacing f with $-\frac{\partial u}{\partial t}$ in (2.3)

$$L_x^{-1} A_x u_i^k = -\frac{\partial u_i^k}{\partial t} + O(h^4). \tag{2.4}$$

Let

$$v_i^k = \frac{\partial u_i^k}{\partial t}. \tag{2.5}$$

Then we have

$$L_x^{-1} A_x u_i^k = -v_i^k + O(h^4). \tag{2.6}$$

Neglecting the high-order term $O(h^4)$ of (2.6) and then rewriting it as follows

$$\begin{aligned} &\left(\frac{1}{12} + \frac{h\varepsilon}{24\gamma}\right) v_{i-1}^k + \frac{5}{6} v_i^k + \left(\frac{1}{12} - \frac{h\varepsilon}{24\gamma}\right) v_{i+1}^k \\ &= \left(\frac{\gamma}{h^2} + \frac{\varepsilon^2}{12\gamma} + \frac{\varepsilon}{2h}\right) u_{i-1}^k + \left(-\frac{2\gamma}{h^2} - \frac{\varepsilon^2}{6\gamma}\right) u_i^k + \left(\frac{\gamma}{h^2} + \frac{\varepsilon^2}{12\gamma} - \frac{\varepsilon}{2h}\right) u_{i+1}^k. \end{aligned} \tag{2.7}$$

Along time level t, we denote $w(x_i, t)$ by $w_i(t)$, where w is u or v.

In matrix notation, (2.7) can be written as:

$$\begin{cases} \mathbf{AV}(t) = \mathbf{BU}(t), \\ \mathbf{U}(0) = \mathbf{U}_0. \end{cases} \tag{2.8}$$

From (2.5) and (2.8), we have

$$\begin{cases} \mathbf{A}\dfrac{\mathrm{d}\mathbf{U}(t)}{\mathrm{d}t} = \mathbf{B}\mathbf{U}(t), \\ \mathbf{U}(0) = \mathbf{U}_0 \end{cases} \tag{2.9}$$

where

$$\mathbf{V}(t) = [v_1(t), v_2(t), \ldots, v_{n-2}(t), v_{n-1}(t)]^{\mathrm{T}},$$
$$\mathbf{U}(t) = [u_1(t), u_2(t), \ldots, u_{n-2}(t), u_{n-1}(t)]^{\mathrm{T}},$$
$$\mathbf{U}(0) = [g(x_1), g(x_2), \ldots, g(x_{n-2}), g(x_{n-1})]^{\mathrm{T}}.$$

And A, B are the trid-diagonal matrix of order $n-1$ as below

$$A = \begin{pmatrix} \frac{5}{6} & \frac{1}{12}-\frac{\varepsilon h}{24\gamma} & 0 & \cdots & 0 & 0 \\ \frac{1}{12}+\frac{\varepsilon h}{24\gamma} & \frac{5}{6} & \frac{1}{12}-\frac{\varepsilon h}{24\gamma} & 0 & \cdots & 0 \\ 0 & \frac{1}{12}+\frac{\varepsilon h}{24\gamma} & \frac{5}{6} & \frac{1}{12}-\frac{\varepsilon h}{24\gamma} & 0 & \cdots \\ \vdots & \ddots & \ddots & \ddots & & \vdots \\ \cdots & 0 & \frac{1}{12}+\frac{\varepsilon h}{24\gamma} & \frac{5}{6} & \frac{1}{12}-\frac{\varepsilon h}{24\gamma} & 0 \\ 0 & \cdots & 0 & \frac{1}{12}+\frac{\varepsilon h}{24\gamma} & \frac{5}{6} & \frac{1}{12}-\frac{\varepsilon h}{24\gamma} \\ 0 & 0 & \cdots & 0 & \frac{1}{12}+\frac{\varepsilon h}{24\gamma} & \frac{5}{6} \end{pmatrix}$$

$$B = \begin{pmatrix} -\frac{2\gamma}{h^2}-\frac{\varepsilon^2}{6\gamma} & \frac{\gamma}{h^2}+\frac{\varepsilon^2}{12\gamma}-\frac{\varepsilon}{12} & 0 & \cdots & 0 & 0 \\ \frac{\gamma}{h^2}+\frac{\varepsilon^2}{12\gamma}+\frac{\varepsilon}{12} & -\frac{2\gamma}{h^2}-\frac{\varepsilon^2}{6\gamma} & \frac{\gamma}{h^2}+\frac{\varepsilon^2}{12\gamma}-\frac{\varepsilon}{12} & 0 & \cdots & 0 \\ 0 & \frac{\gamma}{h^2}+\frac{\varepsilon^2}{12\gamma}+\frac{\varepsilon}{12} & -\frac{2\gamma}{h^2}-\frac{\varepsilon^2}{6\gamma} & \frac{\gamma}{h^2}+\frac{\varepsilon^2}{12\gamma}-\frac{\varepsilon}{12} & 0 & \cdots \\ \vdots & \ddots & \ddots & \ddots & & \vdots \\ \cdots & 0 & \frac{\gamma}{h^2}+\frac{\varepsilon^2}{12\gamma}+\frac{\varepsilon}{12} & -\frac{2\gamma}{h^2}-\frac{\varepsilon^2}{6\gamma} & \frac{\gamma}{h^2}+\frac{\varepsilon^2}{12\gamma}-\frac{\varepsilon}{12} & 0 \\ 0 & \cdots & 0 & \frac{\gamma}{h^2}+\frac{\varepsilon^2}{12\gamma}+\frac{\varepsilon}{12} & -\frac{2\gamma}{h^2}-\frac{\varepsilon^2}{6\gamma} & \frac{\gamma}{h^2}+\frac{\varepsilon^2}{12\gamma}-\frac{\varepsilon}{12} \\ 0 & 0 & \cdots & 0 & \frac{\gamma}{h^2}+\frac{\varepsilon^2}{12\gamma}+\frac{\varepsilon}{12} & -\frac{2\gamma}{h^2}-\frac{\varepsilon^2}{6\gamma} \end{pmatrix}.$$

From [3], we know that

(i) When $\frac{1}{12}-\frac{\varepsilon h}{24\gamma} \geq 0$, the eigenvalues of the matrix A are

$$\mu_s = \frac{5}{6} + \frac{1}{12}\sqrt{4-\frac{\varepsilon^2 h^2}{\gamma^2}}\cos\frac{s\pi}{n} \neq 0, \quad (s = 1, 2, n-1).$$

(ii) When $\frac{1}{12}-\frac{\varepsilon h}{24\gamma} < 0$, the eigenvalues of the matrix A are

$$\mu_s = \frac{5}{6} - \frac{1}{12}\sqrt{\frac{\varepsilon^2 h^2}{\gamma^2}-4}\cos\frac{s\pi}{n}I \neq 0, \quad (s = 1, 2, n-1),$$

where $I = \sqrt{-1}$.

For $|A| = \prod_{s=1}^{n-1}\mu_s \neq 0$, so A is nonsingular, we easily get the exact solution of (2.9):

$$\mathbf{U}(t) = \mathrm{e}^{tA^{-1}B}\mathbf{U}_0. \tag{2.10}$$

For $t_k = k\tau$, $k = 0, 1, \ldots$, so

$$\mathbf{U}(t_{k+1}) = \mathrm{e}^{(k+1)\tau A^{-1}B}\mathbf{U}_0, \qquad \mathbf{U}(t_k) = \mathrm{e}^{k\tau A^{-1}B}\mathbf{U}_0.$$

Then we have

$$\mathbf{U}(t_{k+1}) = \mathrm{e}^{\tau A^{-1}B}\mathbf{U}(t_k). \tag{2.11}$$

Now, the problem is how to approximate $e^{\tau A^{-1}B}$ to get the numerical solution. A good approximation to e^Z is the (2, 2) padé approximation which have the form [3,11]

$$e^Z = \frac{12+6Z+Z^2}{12-6Z+Z^2} + O(Z^5). \tag{2.12}$$

Neglecting the high-order term $O(Z^5)$ of (2.12), yield to

$$e^Z \approx \frac{12+6Z+Z^2}{12-6Z+Z^2}. \tag{2.13}$$

We use $[12I - 6\tau A^{-1}B + \tau^2(A^{-1}B)^2]^{-1}[12I + 6\tau A^{-1}B + \tau^2(A^{-1}B)^2]$ to approximate the matrix series $e^{\tau A^{-1}B}$ successfully. Then we get the following difference scheme for the numerical solution of Eq. (1.1)

$$\mathbf{U}_{k+1} = [12I - 6\tau A^{-1}B + \tau^2(A^{-1}B)^2]^{-1}[12I + 6\tau A^{-1}B + \tau^2(A^{-1}B)^2]\mathbf{U}_k, \tag{2.14}$$

where $\mathbf{U}_k$ is the numerical approximation of $\mathbf{U}(t_k)$.

From (2.4) and (2.12), we know the truncation error of the difference scheme (2.14) is $O(h^4 + \tau^5)$.

3. Stability analysis

Lemma 3.1. *If the real part of Z is nonpositive, then*

$$\left|\frac{12+6Z+Z^2}{12-6Z+Z^2}\right| \le 1. \tag{3.1}$$

Proof. We suppose that $Z = a + bi$, where $a \in \mathbb{R}$, $b \in \mathbb{R}$ and $a \le 0$.

Since

$$12a(24 + 2a^2 + 2b^2) \le 0.$$

Then we have

$$\begin{aligned}
&(24 + 2a^2 - 2b^2)(12a) + (4ab)(12b) \le 0,\\
&[(12 + 6a + a^2 - b^2)^2 - (12 - 6a + a^2 - b^2)^2] \le [(-6b + 2ab)^2 - (6b + 2ab)^2],\\
&[(12 + 6a + a^2 - b^2)^2 + (6b + 2ab)^2] \le [(12 - 6a + a^2 - b^2)^2 + (-6b + 2ab)^2],\\
&\sqrt{(12 + 6a + a^2 - b^2)^2 + (6b + 2ab)^2} \le \sqrt{(12 - 6a + a^2 - b^2)^2 + (-6b + 2ab)^2},\\
&|(12 + 6a + a^2 - b^2) + (6b + 2ab)i| \le |(12 - 6a + a^2 - b^2) + (-6b + 2ab)i|,\\
&|12 + 6(a + bi) + (a + bi)^2| \le |12 - 6(a + bi) + (a + bi)^2|.
\end{aligned}$$

So

$$\left|\frac{12+6Z+Z^2}{12-6Z+Z^2}\right| \le 1.$$

This finishes the proof of Lemma 3.1. □

Lemma 3.2. *Assume that $\lambda_i (i = 1, 2, \ldots, n-1)$ are the eigenvalues of the matrix $A^{-1}B$, and $\xi_i (i = 1, 2, \ldots, n-1)$ the real vectors of dimension $(n-1)$, are corresponding eigenvectors. Then λ_i are real and satisfy*

$$\lambda_i \le 0. \tag{3.2}$$

Proof. Since λ_i and ξ_i are eigenvalues and corresponding eigenvectors of matrix $A^{-1}B$, respectively, they satisfy the following conditions

$$\lambda_i \xi_i = A^{-1}B\xi_i, \quad \text{or} \quad \lambda_i \xi_i^{\mathrm{T}} A \xi_i = \xi_i^{\mathrm{T}} B \xi_i. \tag{3.3}$$

Since

$$\begin{aligned}
\xi_i^{\mathrm{T}} B \xi_i &= \left(\xi_{i(1)}, \quad \xi_{i(2)}, \quad \ldots, \quad \xi_{i(n-1)}\right) B \left(\xi_{i(1)}, \quad \xi_{i(2)}, \quad \ldots, \quad \xi_{i(n-1)}\right)^{\mathrm{T}}\\
&= -\left(\frac{2\gamma}{h^2} + \frac{\varepsilon^2}{6\gamma}\right)(\xi_{i(1)}^2 - \xi_{i(1)}\xi_{i(2)} + \xi_{i(2)}^2 - \xi_{i(2)}\xi_{i(3)} + \xi_{i(3)}^2 - \cdots + \xi_{i(n-2)}^2 - \xi_{i(n-2)}\xi_{i(n-1)} + \xi_{i(n-1)}^2)\\
&\le -\left(\frac{2\gamma}{h^2} + \frac{\varepsilon^2}{6\gamma}\right)[\xi_{i(1)}^2 - \frac{1}{2}(\xi_{i(1)}^2 + \xi_{i(2)}^2) + \xi_{i(2)}^2 - \frac{1}{2}(\xi_{i(2)}^2 + \xi_{i(3)}^2)
\end{aligned}$$

Table 1
The absolute error for various values of the t ($h = 0.005$, $\tau = 0.001$).

t	$x = 0.1$		$x = 0.3$		$x = 0.5$		$x = 0.7$		$x = 0.9$	
	P.M	C-N.M	P.M	C-N.M	P.M	C-N.M	P.M	C-N.M	P.M	C-N.M
0.2	6.54e−010	1.88e−005	1.61e−009	3.61e−005	2.09e−009	1.39e−005	2.71e−009	2.05e−004	2.61e−008	9.67e−004
0.4	1.13e−009	3.23e−005	3.02e−009	6.76e−005	3.92e−009	2.61e−005	5.06e−009	3.84e−004	4.38e−008	1.64e−003
0.6	1.47e−009	4.15e−005	4.22e−009	9.45e−005	5.49e−009	3.66e−005	7.06e−009	5.37e−004	5.53e−008	2.10e−003
0.8	1.71e−009	4.81e−005	5.24e−009	1.17e−004	6.83e−009	4.56e−005	8.67e−009	6.64e−004	6.26e−008	2.42e−003
1	1.89e−009	5.26e−005	6.08e−009	1.35e−004	7.96e−009	5.31e−005	9.89e−009	7.68e−004	6.71e−008	2.63e−003
10	3.37e−010	7.17e−006	1.71e−009	2.87e−005	3.46e−009	2.31e−005	2.61e−009	1.11e−004	2.60e−009	2.86e−004
20	1.50e−011	2.57e−007	8.48e−011	1.13e−006	2.12e−010	1.41e−006	3.04e−010	2.10e−006	1.48e−010	7.60e−006

$$
\begin{aligned}
&+\xi_{i(3)}^2 - \cdots + \xi_{i(n-2)}^2 - \frac{1}{2}(\xi_{i(n-2)}^2 + \xi_{i(n-1)}^2) + \xi_{i(n-1)}^2] \\
&= -\left(\frac{\gamma}{h^2} + \frac{\varepsilon^2}{12\gamma}\right)(\xi_{i(1)}^2 + \xi_{i(n-1)}^2) \le 0.
\end{aligned} \tag{3.4}
$$

And

$$
\begin{aligned}
\xi_i^{\mathrm{T}} A \xi_i &= \left(\xi_{i(1)}, \quad \xi_{i(2)}, \quad \ldots, \quad \xi_{i(n-1)}\right) A \left(\xi_{i(1)}, \quad \xi_{i(2)}, \quad \ldots, \quad \xi_{i(n-1)}\right)^{\mathrm{T}} \\
&= \frac{1}{6}(5\xi_{i(1)}^2 + \xi_{i(1)}\xi_{i(2)} + 5\xi_{i(2)}^2 + \xi_{i(2)}\xi_{i(3)} + 5\xi_{i(3)}^2 + \ldots + 5\xi_{i(n-2)}^2 + \xi_{i(n-2)}\xi_{i(n-1)} + 5\xi_{i(n-1)}^2) \\
&= \frac{3}{4}\xi_{i(1)}^2 + \frac{1}{12}(\xi_{i(1)} + \xi_{i(2)})^2 + \frac{2}{3}\xi_{i(2)}^2 + \frac{1}{12}(\xi_{i(1)} + \xi_{i(2)})^2 \\
&\quad + \frac{2}{3}\xi_{i(3)}^2 + \ldots + \frac{2}{3}\xi_{i(n-2)}^2 + \frac{1}{12}(\xi_{i(n-2)} + \xi_{i(n-1)})^2 + \frac{3}{4}\xi_{i(n-1)}^2 > 0
\end{aligned} \tag{3.5}
$$

The above two results indicate that λ_i are real and $\lambda_i \le 0$. □

Theorem 3.1. *Difference scheme (2.14) is unconditionally stable.*

Proof. If $\lambda_i (i = 1, 2, \ldots, n-1)$ are the eigenvalues of $A^{-1}B$, one can see that eigenvalues of $[12I - 6\tau A^{-1}B + \tau^2(A^{-1}B)^2]^{-1}[12I + 6\tau A^{-1}B + \tau^2(A^{-1}B)^2]$ are $(12 - 6\tau\lambda_i + \tau^2\lambda_i^2)^{-1}(12 + 6\tau\lambda_i + \tau^2\lambda_i^2)(i = 1, 2, \ldots, n-1)$.

From Lemmas 3.1 and 3.2, we have

$$
|(12 - 6\tau\lambda_i + \tau^2\lambda_i^2)^{-1}(12 + 6\tau\lambda_i + \tau^2\lambda_i^2)| \le 1. \tag{3.6}
$$

Thus, the difference scheme (2.14) is unconditionally stable. □

4. Numerical examples

We present some numerical examples to compare the present method (denoted by P.M) with Crank–Nicolson method (denoted by C-N.M). In Examples 4.1–4.3, we suppose that exact solution at the first level is known.

Example 4.1. To provide some indication of accuracy of the present method, we consider the convection–diffusion equation [8]

$$
\frac{\partial u}{\partial t} + 0.1\frac{\partial u}{\partial x} = 0.01\frac{\partial^2 u}{\partial x^2}, \quad 0 \le x \le 1,\ t \ge 0,
$$

where the exact solution is given by

$$
u(x, t) = \mathrm{e}^{5x-(0.25+0.01\pi^2)t}\sin \pi x.
$$

The initial and boundary conditions are defined so as to agree with the exact solution. The accuracy of the present method and Crank–Nicolson method is compared in Table 1 for various values of the t. Table 1 gives the absolute error along $x = 0.1, 0.3, 0.5, 0.7$ and 0.9 of the domain, where $h = 0.005$, $\tau = 0.001$.

Example 4.2. We consider the convection–diffusion equation

$$
\frac{\partial u}{\partial t} + 0.22\frac{\partial u}{\partial x} = 0.5\frac{\partial^2 u}{\partial x^2}, \quad 0 \le x \le 1, t \ge 0,
$$

Table 2
The absolute error for various values of the t ($h = 0.002$, $\tau = 0.002$).

t	$x = 0.1$		$x = 0.3$		$x = 0.5$		$x = 0.7$		$x = 0.9$	
	P.M	C-N.M	P.M	C-N.M	P.M	C-N.M	P.M	C-N.M	P.M	C-N.M
0.2	1.37e−011	1.72e−007	4.01e−011	4.77e−007	5.08e−011	6.22e−007	4.25e−011	5.31e−007	1.69e−010	2.14e−007
0.4	1.69e−011	2.34e−007	5.33e−011	6.46e−007	6.82e−011	8.41e−007	5.74e−011	7.17e−007	2.29e−011	2.89e−007
0.6	1.89e−011	2.37e−007	5.34e−011	6.52e−007	6.87e−011	8.49e−007	5.80e−011	7.23e−007	2.32e−011	2.91e−007
0.8	1.70e−011	2.12e−007	4.77e−011	5.85e−007	6.15e−011	7.62e−007	5.20e−011	6.49e−007	2.08e−011	2.61e−007
1	1.43e−011	1.79e−007	3.99e−011	4.92e−007	5.16e−011	6.40e−007	4.37e−011	5.45e−007	1.75e−011	2.19e−007
10	2.48e−018	3.08e−014	6.29e−018	8.49e−014	8.87e−018	1.10e−013	7.54e−018	9.38e−014	3.02e−018	3.77e−014
20	1.17e−026	1.46e−022	3.22e−026	4.01e−022	4.19e−026	5.21e−022	3.56e−026	4.43e−022	1.43e−026	1.78e−022

Table 3
The absolute error for various values of the t ($h = 0.01$, $\tau = 0.005$).

t	$x = 0.1$		$x = 0.3$		$x = 0.5$		$x = 0.7$		$x = 0.9$	
	P.M	C-N.M	P.M	C-N.M	P.M	C-N.M	P.M	C-N.M	P.M	C-N.M
0.2	5.34e−010	3.72e−006	1.46e−009	1.02e−005	1.89e−009	1.32e−005	1.60e−009	1.12e−005	6.40e−010	4.48e−006
0.4	4.02e−010	2.80e−006	1.10e−009	7.68e−006	1.42e−009	9.93e−006	1.20e−009	8.41e−006	4.38e−010	3.36e−006
0.6	2.25e−010	1.57e−006	6.15e−010	4.29e−006	7.95e−010	5.55e−006	6.72e−010	4.69e−006	2.68e−010	1.88e−006
0.8	1.11e−010	7.77e−007	3.05e−010	2.13e−006	3.94e−010	2.75e−006	3.33e−010	2.33e−006	1.33e−010	9.29e−007
1	5.17e−011	3.61e−007	1.42e−010	9.88e−007	1.83e−010	1.28e−006	1.55e−010	1.08e−006	6.17e−011	4.31e−007
10	2.15e−029	1.50e−025	5.89e−029	4.11e−025	7.60e−029	5.31e−025	6.43e−029	4.49e−025	2.57e−029	1.79e−025
20	1.25e−050	8.75e−047	3.42e−050	2.39e−046	4.42e−050	3.09e−046	3.74e−050	2.61e−046	1.49e−050	1.04e−046

where the exact solution is given by

$$u(x,t) = e^{0.22x-(0.0242+0.5\pi^2)t}\sin \pi x.$$

The initial and boundary conditions are defined so as to agree with the exact solution. The accuracy of the present method and Crank–Nicolson method is compared in Table 2 for various values of the t. Table 2 gives the absolute error along $x = 0.1, 0.3, 0.5, 0.7$ and 0.9 of the domain, where $h = 0.01$, $\tau = 0.005$.

Example 4.3. We consider the convection–diffusion equation

$$\frac{\partial u}{\partial t} + 0.1\frac{\partial u}{\partial x} = 0.2\frac{\partial^2 u}{\partial x^2}, \quad 0 \le x \le 1,\ t \ge 0,$$

where the exact solution is given by

$$u(x,t) = e^{0.25x-(0.0125+0.2\pi^2)t}\sin \pi x.$$

The initial and boundary conditions are defined as so to agree with the exact solution. The accuracy of the present method and Crank–Nicolson method is compared in Table 3 for various values of the t. Table 3 gives the absolute error along $x = 0.1, 0.3, 0.5, 0.7$ and 0.9 of the domain, where $h = 0.002$, $\tau = 0.002$.

Solving Examples 4.1–4.3, Tables 1–3 shows comparison between the absolute error of the present method and Crank–Nicolson method. The results prove that the new method is more accurate than the Crank–Nicolson method.

5. Concluding remarks

In this work, we proposed a method to find the solution of the system of ordinary differential equations arising from discretizing the convection–diffusion equation with respect to the space variable. The method is fourth order in space and fifth order in time direction, respectively. It is shown through analysis that the difference scheme is unconditionally stable, and numerical experiments are conducted to test its high accuracy and to show its reasonableness.

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