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NEW TECHNIQUES FOR THE SOLUTION OF LINEAR SYSTEMS BY ITERATIVE METHODS

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Abstract—A new iterative method for the solution of linear systems, based upon a new splitting of the coefficient matrix A, is presented.

The method is obtained by considering splittings of the type $A = (A - M) + M$, where M^{-1} is a symmetric tridiagonal matrix, and by minimizing the Frobenius norm of the iteration matrix so derived.

Numerical examples are provided, showing that our algorithm improves the rate of convergence of Jacobi method, without increasing the order of magnitude of the computational efforts required.

1. INTRODUCTION

In this paper we show a new iterative method for the solution of linear systems, which can be efficiently implemented in a parallel computational environment. We propose the following approach.

Given an $n \times n$ nonsingular matrix A, we restrict ourselves to splittings of the type

$$A = M + (A - M),$$

where M belongs to the class of inverses of symmetric tridiagonal matrices [1–3].

We look for a matrix M such that

$$\| I - M^{-1} A \|_F$$

is minimum, where

$$\| B \|_F = \sqrt{\sum_{ij} b_{ij}^2}$$

denotes the Frobenius matrix norm.

We will present either theoretical results leading to a simple and elegant algorithm for the computation of M^{-1}, and therefore of the iteration matrix (Section 2), or an experimental discussion of the behaviour of our method (Section 3).

2. THEORETICAL RESULTS

Let $A \in R^{n \times n}$ be a nonsingular matrix, and $\mathbf{b}$ be a real n-vector. Moreover let $\mathscr{T}$ be the class of $n \times n$ nonsingular symmetric tridiagonal matrices, and let $\mathscr{M}$ be the class of $n \times n$ nonsingular symmetric matrices M such that $M^{-1} \in \mathscr{T}$ [1, 3].

Given the linear system $\cdot Ax = \mathbf{b}$, we consider the splitting

$$A = M + (A - M)$$

of A, with $M \in \mathscr{M}$, leading to the iterative method

$$x_{k+1} = (1 - M^{-1} A)\, x_k + M^{-1}\mathbf{b}.$$

We look for the matrix $\mathscr{M} \in M$ minimizing the norm $\| 1 - M^{-1} A \|_F$.

If

$$M^{-1} = \begin{bmatrix} c_1 & b_1 & & & \\ b_1 & c_2 & b_2 & & \\ & b_2 & \cdot & \cdot & \\ & & \cdot & \cdot & b_{n-1} \\ & & & b_{n-1} & c_n \end{bmatrix}.$$

we have

$$f(c_1, \ldots, c_n, b_1, \ldots, b_{n-1}) = \| 1 - M^{-1}A \|_F^2 = \sum_{ij} (\delta_{ij} - b_{i-1}a_{i-1j} - c_i a_{ij} - b_i a_{i+1j})2,$$

where

$$\delta_{ij} = \begin{cases} 1 & \text{if} \quad i = j \\ 0 & \text{otherwise'} \end{cases}$$

and $b_0 = 0$, $a_{0j} = 0$ and $a_{n+1j} = 0$, $j = 1, \ldots, n$, for convenience.

In order to find the stationary points of the function f, we compute and equate to zero the partial derivatives of f, namely

$$\frac{\partial f}{\partial c_k} = -2 \sum_j (\delta_{kj} - b_{k-1}a_{k-1j} - c_k a_{kj} - b_k a_{k+1j}) a_{kj} = 0, \quad k = 1, 2, \ldots, n,$$

and

$$\frac{\partial f}{\partial b_s} = -2 \sum_j (\delta_{s+1j} - b_s a_{sj} - c_{s+1} a_{s+1j} - b_{s+1} a_{s+2j}) a_{sj}$$

$$- 2 \sum_j (\delta_{sj} - b_{s-1} a_{s-1j} - c_s a_{sj} - b_s a_{s+1j}) a_{s+1j} = 0, \quad s = 1, 2, \ldots, n-1.$$

From the above equalities, we have:

$$\begin{bmatrix} D & L \\ L^T & T \end{bmatrix} \begin{bmatrix} \mathbf{c} \\ \mathbf{b} \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix},$$

where

$D = \mathrm{diag}\,(s_{11}, s_{22}, \ldots, s_{nn})$,

$L = (l_{ij})$, with $l_{ii} = s_{i+1i}$, $l_{i+1i} = s_{ii+1}$, $l_{ij} = 0$, if $i \neq j$ and $i \neq j-1$, $i = 1, \ldots, n$, $j = 1, \ldots, n-1$,

$T = (t_{ij})$, with $t_{ii} = s_{ii} + s_{i+1i+1}$, $t_{ii+1} = s_{i+2i}$, $t_{i+1i} = s_{ii+2}$, $t_{ij} = 0$, if $i \neq j$, $i \neq j-1$ and $i \neq j+1$, $i = 1, \ldots, n-1$, $j = 1, \ldots, n-1$,

where

$$s_{kp} = \sum_j a_{kj} a_{jp},$$

$$f_1 = [a_{11}, a_{22}, \ldots, a_{nn}],$$

$$f_2 = [a_{12} + a_{21}, \ldots, a_{n-1n} + a_{nn-1}].$$

The solution of the above linear system is a minimum point for f, since $f(c_1, \ldots, c_n, b_1, \ldots, b_{n-1})$ is convex.

The computation of the entries of matrix M^{-1} can be performed according to the following algorithm.

Algorithm

1. Compute

$$s_{ij}, \quad i=j,\ i=j-1,\ i=j-2,\ j=1,\ldots,n,$$

2. Compute

$$L^{\mathrm{T}}D^{-1}L, \quad L^{\mathrm{T}}D^{-1}f_1,$$

3. Compute

$$T-L^{\mathrm{T}}D^{-1}L,\ f_2-L^{\mathrm{T}}D^{-1}f_1,$$

4. Solve the tridiagonal linear system

$$(T-L^{\mathrm{T}}D^{-1}L)\,b=f_2-L^{\mathrm{T}}D^{-1}f_1,$$

5. Compute

$$c=D^{-1}f_1-D^{-1}L\,b.$$

Table 1 shows the sequential and the parallel cost of the algorithm, in terms of the number of operations and of the number of parallel steps (t) together with the number of processors (p), respectively.

Note that both the sequential cost $O(n \log n)$ and the parallel cost $O(\log n)$ for Step 4 can be attained by using cyclic or odd–even reduction algorithms [4].

Table 1

Step	Sequential cost	Parallel cost t	p
1	$O(n^2)$	$O(\log n)$	$O(n^2)$
2	$O(n)$	$O(1)$	$O(n)$
3	$O(n)$	$O(1)$	$O(n)$
4	$O(n \log n)$	$O(\log n)$	$O(n)$
5	$O(n)$	$O(1)$	$O(n)$

3. NUMERICAL EXPERIMENTS

In this section we present some experimental results which show the behaviour of the method described in Section 2, in comparison with the Jacobi method. We used the following approach.

The parameter chosen as a measure of the performance is the spectral radius of the iteration matrix, which determines the asymptotic rate of convergence, and therefore the number of iteration required to get a given accuracy in the result [5].

It is worth noting that the computational effort of the method based on the results of Section 2, and of the Jacobi method, are equivalent, up to constants.

Table 2. $A=uu^{\mathrm{T}}+\alpha I$, $u^{\mathrm{T}}=(1,\ldots,1)$, $\alpha=7-(2h+1)/5$, $h=0,1,\ldots,14$, $n=8$

Spectral radius of new iteration matrix $p(I-M^{-1}A)$	Spectral radius of Jacobi iteration matrix $p(J)$
0.3400	0.6250
0.3481	0.6481
0.3567	0.6731
0.3657	0.7000
0.3752	0.7292
0.3853	0.7609
0.3960	0.7955
0.4074	0.8333
0.4196	0.8750
0.4328	0.9211
0.4471	0.9722
0.4628	1.029
0.4803	1.092
0.5000	1.167
0.5230	1.250

Table 3. A is a symmetric Toeplitz matrix with random off-diagonal entries and $a_{ii} = \Sigma_{j \neq 1} a_{ij} - 5 + (5h + 1)/20$, $h = 0, 1, \ldots, 14$, $n = 32$

Spectral radius of new iteration matrix $\rho(I - M^{-1}A)$	Spectral radius of Jacobi iteration matrix $\rho(J)$
1.017	1.421
1.006	1.393
0.9953	1.366
0.9844	1.341
0.9737	1.316
0.9631	1.292
0.9526	1.269
0.9422	1.246
0.9320	1.225
0.9219	1.204
0.9120	1.184
0.9022	1.164
0.8925	1.146
0.8830	1.127
0.8736	1.110

Table 4. $A = QQ^T$ is a positive matrix, where Q has random entries, $n = 40$

Spectral radius of new iteration matrix $\rho(I - M^{-1}A)$	Spectral radius of Jacobi iteration matrix $\rho(J)$
0.9923	21.36
0.9930	114.3
0.9915	12.31
1.014	16.29
0.9976	27.16
0.9991	21.24
1.009	64.22
0.9997	102.4
0.9620	21.18
1.022	24.16
0.9925	47.14
0.9830	31.12
0.9736	13.11

Tables 2–4 show the experimental results.

It turns out that:

(1) when the Jacobi method is convergent, our method converges as well with an improved convergence rate;

(2) there exist classes of matrices for which our method converges while the Jacobi method does not.

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