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On the unimodality and combinatorics of Bessel numbers

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Abstract

The Bessel numbers are reparametrized coefficients of Bessel polynomials. The paper investigates the analogies between Stirling numbers and Bessel numbers. A generating function for the Bessel numbers is obtained, and a proof of their unimodality is given. Stirling numbers and Bessel numbers are applied to the enumeration of orbit decompositions of various powers of permutation representations.

Keywords: Bessel polynomial; Unimodal; Stirling number; Orbit decomposition

1. Introduction

The Bessel polynomials [2,4,6] are the unique polynomial solutions of the second-order differential equations

$$x^2 y_n'' + (2x+2)y_n' = n(n+1)y_n$$

(for natural numbers n) that are normalized to have unit constant term. The coefficients of these polynomials are known as Bessel coefficients. The paper is devoted to a study of the Bessel coefficients from a combinatorial standpoint. To this end, the Bessel coefficients are reparametrized. In the new form, they are called Bessel numbers. It turns out that they are closely related to the Stirling numbers of the second kind, and to the associated Stirling numbers (of the second kind) in the sense of Comtet

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[3]. These relationships, including the combinatorial interpretation of the Bessel numbers, are discussed in the first section of the paper. The second section presents an exponential generating function for the Bessel numbers in terms of falling factorials (Theorem 2.1). By analogy with the unimodality of the Stirling numbers of the second kind, the third section exhibits the unimodality of the Bessel numbers (Theorem 3.2). The remainder of the paper is concerned with the application of the Bessel numbers and Stirling numbers to the analysis of the decomposition of various powers of permutation representations of finite groups. For an action $G \to Q!$; $g \mapsto (q \mapsto qg)$ of a finite group G on a finite set Q, and for a positive integer n, the direct power G-set $(Q,G)^n$ is Q^n equipped with the *diagonal action*

$$g : (q_1, \ldots, q_n) \mapsto (q_1 g, \ldots, q_n g).$$

The nth *irredundant power* G-set $(Q,G)^{[n]}$ and the nth *bi-restricted power* G-set $(Q,G)^{[[n]]}$ are defined, respectively, by the G-subset $Q^{[n]}$ of Q^n consisting of all n-tuples of distinct elements of Q, and by the G-subset $Q^{[[n]]}$ of Q^n consisting of all n-tuples in which no element is repeated more than once. Section 4 studies the orbit decompositions of $(Q,G)^{[n]}$ and $(Q,G)^{[[n]]}$, relating them to the orbit decompositions of $(Q,G)^n$ as studied in [7]. In the final section, the exponential generating functions for the numbers of orbits on the irredundant and bi-restricted powers are related to the respective generating functions (2.2) and (2.4) for Stirling and Bessel numbers.

1. Bessel numbers and Stirling numbers

For each nonnegative integer n, the *Bessel polynomial* $y_n(x)$ is defined to be the (unique) polynomial of degree n, with unit constant term, satisfying the differential equation

$$x^2 y_n'' + (2x+2) y_n' = n(n+1) y_n.$$

[2,4,6]. The nth Bessel polynomial may be written in the form $y_n(x) = \sum_{k=0}^{\infty} B_{n,k} x^{n-k}$, in which the *Bessel coefficient* $B_{n,k}$ is defined by

$$B_{n,k} = \frac{(2n-k)!}{2^{n-k}(k)!(n-k)!} \tag{1.1}$$

for $n \geqslant k \geqslant 0$, and by $B_{n,k} = 0$ otherwise. The Bessel coefficients satisfy the recursion

$$B_{n,k} = B_{n-1,k-1} + (2n-k-1) B_{n-1,k} \tag{1.2}$$

[4]. For nonnegative integers n and k, the (n,k)th *Bessel number* $B(n,k)$ is defined to be the Bessel coefficient $B_{k,2k-n}$. By (1.1), the Bessel number $B(n,k)$ is the number of partitions of an n-set into k nonempty subsets, each of size at most 2. Rewriting (1.2), the Bessel numbers are seen to satisfy the recursion

$$B(n,k) = B(n-1,k-1) + (n-1) B(n-2,k-1). \tag{1.3}$$

The following table displays the first few Bessel numbers. The empty cells are to be filled with 0's.

9	8	7	6	5	4	3	2	$1=k$	$B(n,k)$
								1	$n=1$
							1	1	2
						1	3		3
					1	6	3		4
				1	10	15			5
			1	15	45	15			6
		1	21	105	105				7
	1	28	210	420	105				8
1	36	378	1260	945					9

The Bessel numbers

For $n \geqslant k > 0$, the Stirling number $S_2(n,k)$ of the second kind is the number of partitions of an n-set into k nonempty subsets. The Stirling numbers of second kind satisfy the recursion

$$S_2(n,k) = S_2(n-1,k-1) + kS_2(n-1,k). \tag{1.4}$$

For $n \geqslant k > 0$, the *associated Stirling number $A_2(n,k)$ of the second kind* is defined to be the number of partitions of an n-set into k nonempty subsets, each of size $\geqslant 2$ [3]. The associated Stirling numbers of second kind satisfy the recursion

$$A_2(n,k) = kA_2(n-1,k) + (n-1)A_2(n-2,k-1) \tag{1.5}$$

[3]. In terms of their combinatorial significance, the Bessel numbers are complementary to the associated Stirling numbers of the second kind. Note that $B(n,k) = S_2(n,k)$ if $2 > n-k$, and that $B(n,k) = 0$ if $k < \lceil n/2 \rceil$.

2. Generating functions for the Bessel numbers

For an indeterminate x, the *falling factorial* $[x]_n$ is given by the formula

$$[x]_n = x(x-1)(x-2)\cdots(x-n+1).$$

The Stirling numbers $S_1(n,k)$ of the first kind and $S_2(n,k)$ of the second kind then feature in the inversion formulas

$$x^n = \sum_{k=0}^{n} S_2(n,k)[x]_k \quad \text{and} \quad [x]_n = \sum_{k=0}^{n} S_1(n,k)x^k, \tag{2.1}$$

with the convention that $S_i(0,0) = 1$ and $S_i(n,0) = 0$ for each $i = 1,2$ [1]. Now consider the function

$$e^{tx} = \left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots\right)^x. \tag{2.2}$$

For each positive integer n, the nth derivative of e^{tx} with respect to t at $t=0$ is x^n. By (2.1), this is expressed as a sum of products of falling factorials with Stirling numbers of the second kind

$$\left.\frac{d^n}{dt^n} e^{tx}\right|_{t=0} = \sum_{k=0}^{n} S_2(n,k)[x]_k. \tag{2.3}$$

Now consider the following drastic truncation of (2.2):

$$f(t) = \left(1 + t + \frac{t^2}{2!}\right)^x. \tag{2.4}$$

Then for each positive integer n, the nth derivative of $f(t)$ with respect to t at $t=0$ is expressed as a sum of products of falling factorials with Bessel numbers:

Theorem 2.1. *For an indeterminate x, let $f(t)$ be given by (2.4). Then*

$$f^{(n)}(0) = \sum_{k=1}^{n} B(n,k)[x]_k \tag{2.5}$$

for each positive integer n.

Proof. Since $f^{(n)}(0)$ is a polynomial with respect to x, and $\{[x]_k\}$ is a basis for the ring of polynomials, the coefficient of $[x]_k$ is uniquely defined. By the binomial formula, we have

$$\begin{aligned} f(t) = \left[1 + \left(t + \frac{t^2}{2!}\right)\right]^x &= \sum_{k=0}^{\infty} \binom{x}{k} \left(t + \frac{t^2}{2!}\right)^k \\ &= \sum_{k=0}^{\infty} \binom{x}{k} \sum_{l=0}^{k} \binom{k}{l} t^{k-l} \left(\frac{t^2}{2!}\right)^l. \end{aligned}$$

Writing the binomial coefficient $\binom{x}{k}$ in the form $[x]_k/k!$ gives

$$f(t) = \sum_{k=0}^{\infty} \sum_{l=0}^{k} \frac{[x]_k t^{k+l}}{2^l l!(k-l)!} = \sum_{n=0}^{\infty} \sum_{k=\lceil n/2 \rceil}^{n} \frac{[x]_k}{2^{n-k}(n-k)!(2k-n)!} t^n. \tag{2.6}$$

Since $f(t) = \sum_{k=0}^{\infty} f^{(n)}(0)t^n/n!$, (2.6) yields

$$f^{(n)}(0) = \sum_{k=\lceil n/2 \rceil}^{n} \frac{n!}{2^{n-k}(n-k)!(2k-n)!} [x]_k.$$

The result then follows by the definition of the Bessel numbers. □

3. Unimodality of the Bessel numbers

The Stirling numbers of the second kind are *unimodal*: for each positive integer n, there is an integer k such that

$$S_2(n,0) \leqslant S_2(n,1) \leqslant \cdots \leqslant S_2(n,k) \geqslant S_2(n,k+1) \geqslant \cdots \geqslant S_2(n,n) \tag{3.1}$$

[3]. It will be shown that the Bessel numbers display similar unimodality. The *sequence of* nth *Bessel numbers* is defined as $\{B(n,k) \mid k>0\}$.

Theorem 3.1. *The sequence of nth Bessel numbers is unimodal*: *for each positive integer n*, *there is an integer k such that*

$$B(n,1) \leqslant B(n,2) \leqslant \cdots \leqslant B(n,k-1) \leqslant B(n,k) \geqslant B(n,k+1) \geqslant \cdots \geqslant B(n,n). \tag{3.2}$$

Proof. Since the table of Bessel numbers shows that the sequence of nth Bessel numbers is unimodal for each $n \leqslant 7$, we assume $n>7$. By (1.1) and the definition of the Bessel numbers, we have

$$B(n,k) = \frac{n!}{2^{n-k}(n-k)!(2k-n)!} \quad \text{for all } k \in [\lceil n/2 \rceil, n]. \tag{3.3}$$

For the fixed value of n under consideration, set $a_m := B(n,m)/B(n,m-1)$ for each m in the interval $[\lceil n/2 \rceil + 1, n]$. Then the sequence $\{a_m \mid m \in [\lceil n/2 \rceil + 1, n]\}$ is strictly decreasing, since

$$\begin{aligned} \frac{a_m}{a_{m+1}} &= \frac{B(n,m)^2}{B(n,m-1)B(n,m+1)} \\ &= \frac{(n-m+1)(2m-n+2)(2m-n+1)}{(n-m)(2m-n)(2m-n-1)} > 1 \end{aligned} \tag{3.4}$$

by (3.3). Since

$$a_n = \frac{2}{n(n-1)} \quad \text{and} \quad a_{\lceil n/2 \rceil + 1} = \begin{cases} n/2 & \text{if } n \text{ is even,} \\ \lfloor n/2 \rfloor /3 & \text{otherwise,} \end{cases} \tag{3.5}$$

one has $a_n < 1$ and $a_{\lceil n/2 \rceil+1} > 1$ for all $n>7$. Since a_m is strictly decreasing, there is an integer k such that

$$a_{\lceil n/2 \rceil + 1} > \cdots > a_k \geqslant 1 > a_{k+1} > \cdots > a_n.$$

Thus

$$B(n, \lceil n/2 \rceil) < \cdots < B(n,k-1) \leqslant B(n,k) > B(n,k+1) > \cdots > B(n,n).$$

Recalling $B(n,m)=0$ for all $m < \lceil n/2 \rceil$, one obtains (3.2). □

Corollary 3.2. *The sequence of non-zero Bessel numbers is nonincreasing,*

$$B\left(n,\left\lceil\frac{n}{2}\right\rceil\right)\geqslant\cdots\geqslant B(n,n),$$

precisely for $n=1,2,3,5,7$.

Proof. From (3.5), $a_{\lceil n/2\rceil+1}\leqslant 1$ iff $n=2,3,5,7$. □

4. Powers of permutation representations

Consider a permutation representation of a finite group G on a set Q. For each positive integer n, the *irredundant power* G-set $Q^{[n]}$ is defined as follows (cf. [5, II.1.10]):

Definition 4.1. The nth *irredundant power* of a set Q is

$$Q^{[n]}=\{(q_1,q_2,\ldots,q_n)\in Q^n \mid \forall i\neq j, q_i\neq q_j\}. \tag{4.1}$$

For a G-set (Q,G), the restriction of the direct power action of G on Q^n to $Q^{[n]}$ is called the nth *irredundant power* of (Q,G), and denoted by $(Q,G)^{[n]}$.

The G-isomorphism type of a G-set (A,G) is denoted by $[A]$. Defining $[A]+[B]:=[A+B]$, $[A].[B]:=[A\times B]$, $0:=[\emptyset]$ and $1:=[\{1\}]$, the set of G-isomorphism types of finite G-sets becomes a unital, commutative semiring. This semiring embeds into its Grothendieck group, the *Burnside ring* of G [7,9]. Application of the Stirling inversion (2.1) inside the Burnside ring of G then yields the following relationship between the isomorphism types of the direct and irredundant powers of (Q,G).

Proposition 4.2. *For any G-set Q,*

$$[Q^n]=\sum_{k=1}^{n}S_2(n,k)[Q^{[k]}] \quad \textit{and} \quad [Q^{[n]}]=\sum_{k=1}^{n}S_1(n,k)[Q^k]. \tag{4.2}$$

Consider the nth direct power Q^n as the set of all functions from an n-set to Q. Then the nth irredundant power $Q^{[n]}$ is the subset of injective functions from the n-set into Q. We now consider a certain intermediate set, included in Q^n and including $Q^{[n]}$, as follows.

Definition 4.3. The nth *bi-restricted power* of a set Q is

$$Q^{[[n]]}=\{f:\{1,2,3,\ldots,n\}\to Q \mid \forall q\in Q, |f^{-1}\{q\}|\leqslant 2\}. \tag{4.3}$$

For a G-set (Q,G), the restriction of the direct power action of G on Q^n to $Q^{[[n]]}$ is called the nth *bi-restricted power* of (Q,G), and denoted by $(Q,G)^{[[n]]}$.

The formula $[Q^{[n]}] = \sum_{k=1}^{n} S_1(n,k)[Q^k]$ of Proposition 4.2 gives a relation between the irredundant powers and the direct powers. In similar fashion, the bi-restricted powers are related to the irredundant powers and the direct powers.

Proposition 4.4.

$$[Q^{[[n]]}] = \sum_{k=1}^{n} B(n,k)[Q^{[k]}]. \tag{4.4}$$

Proof. For each $1 \leqslant k \leqslant n$, let $A_k^n = \{f \in Q^n \mid \forall q \in Q,\ |f^{-1}(q)| \leqslant 2 \text{ and } k = |\mathrm{Im}(f)|\}$. For each partition π of an n-set of type $1^{2k-n}2^{n-k}3^0 \ldots n^0$, let $Q_\pi = \{f \in Q^n \mid \pi = \ker(f)\}$. Then Q_π is in A_k^n and is G-isomorphic to $Q^{[k]}$. Since there are $B(n,k)$ many partitions of an n-set of the given type, $[A_k^n] = B(n,k)[Q^{[k]}]$. Since $Q^{[[n]]} = \bigcup_{k=1}^{n} A_k^n$, one obtains

$$[Q^{[[n]]}] = \sum_{k=1}^{n} [A_k^n] = \sum_{k=1}^{n} B(n,k)[Q^{[k]}],$$

as required. □

Since the matrices $[B(n,k)]$ and $[(-1)^{n-k}B(2n-k-1,n-1)]$ are mutually inverse [4], each irredundant power may be expressed as a sum of bi-restricted powers as follows.

Proposition 4.5.

$$[Q^{[n]}] = \sum_{k=1}^{n} (-1)^{n-k} B(2n-k-1,n-1)[Q^{[[k]]}]. \tag{4.5}$$

In Propositions 4.4 and 4.5, we have the relations between the bi-restricted and the irredundant powers. Using Proposition 4.2, we can get the relations between the bi-restricted and direct powers.

Corollary 4.6.

(1) $[Q^{[[n]]}] = \sum_{k=1}^{n} (\sum_{m=k}^{n} B(n,m)S_1(m,k))[Q^k]$;
(2) $[Q^n] = \sum_{k=1}^{n} (\sum_{m=k}^{n} (-1)^{m-k} S_2(n,m)B(2m-k-1,m-1))[Q^{[[k]]}]$.

5. Generating functions for numbers of orbits

For a G-set (Q,G), let $\pi(g)$ be the number of points of Q fixed by an element g of G. By Burnside's Lemma, the average number of fixed points

$$\frac{1}{|G|} \sum_{g \in G} \pi(g)^n \tag{5.1}$$

is the number of orbits of G on Q^n [8]. Similarly, the number of orbits of G on the nth irredundant power set may be calculated as follows.

Lemma 5.1. *For each positive integer n, the number of orbits of G on $Q^{[n]}$ is*

$$\frac{1}{|G|}\sum_{g\in G}[\pi(g)]_n. \tag{5.2}$$

Proof. Applying (5.1) to Proposition 4.2, the number of orbits of G on the irredundant power $Q^{[n]}$ is

$$\sum_{k=1}^{n} S_1(n,k)\left(\frac{1}{|G|}\sum_{g\in G}\pi(g)^k\right)=\frac{1}{|G|}\sum_{g\in G}\left(\sum_{k=1}^{n} S_1(n,k)\pi(g)^k\right), \tag{5.3}$$

and by (2.1) this is the same as (5.3). □

The exponential generating function for the number of orbits on the direct power G-sets Q^n is

$$\frac{1}{|G|}\sum_{g\in G}\mathrm{e}^{t\pi(g)} \tag{5.4}$$

(where the exponential generating function for a sequence $(a_n)_{n=0}^{\infty}$ is $\sum_{n=0}^{\infty} a_n t^n/n!$) [8, (5.1)]. Similarly, the exponential generating functions for the numbers of orbits on $Q^{[n]}$ and on $Q^{[[n]]}$ are obtained as follows (cf. [5, V.20.4]).

Theorem 5.2. *The exponential generating function for the number of orbits on the nth irredundant power G-set $(Q,G)^{[n]}$ is*

$$f(t)=\frac{1}{|G|}\sum_{g\in G}(1+t)^{\pi(g)}, \tag{5.5}$$

where $\pi(g)$ is the number of points of Q fixed by an element g of G.

Proof. Expansion of $f(t)$ in a Taylor series at $t=0$ yields

$$f(t)=\sum_{n=0}^{\infty} f^{(n)}(0)\frac{t^n}{n!}=\sum_{n=0}^{\infty}\left(\frac{1}{|G|}\sum_{g\in G}[\pi(g)]_n\right)\frac{t^n}{n!}. \tag{5.6}$$

By Lemma 5.1, $f(t)$ is the exponential generating function for the sequence of numbers of orbits on $(Q,G)^{[n]}$. □

Theorem 5.3. *The exponential generating function for the number of orbits on the nth bi-restricted power G-set* $(Q, G)^{[[n]]}$ *is*

$$f(t) = \frac{1}{|G|} \sum_{g \in G} \left(1 + t + \frac{t^2}{2!}\right)^{\pi(g)}, \tag{5.7}$$

where $\pi(g)$ *is the number of points of* Q *fixed by an element* g *of* G.

Proof. By Theorem 2.1, the nth derivative of f with respect to t at $t = 0$ is

$$f^{(n)}(0) = \frac{1}{|G|} \sum_{g \in G} \left(\sum_{k=1}^{n} B(n,k)[\pi(g)]_k \right)$$

$$= \sum_{k=1}^{n} B(n,k) \left(\frac{1}{|G|} \sum_{g \in G} \pi(g)]_k \right). \tag{5.8}$$

By Lemma 5.1 and Proposition 4.4, it is readily seen that $f^{(n)}(0)$ is the number of orbits of G on the nth bi-restricted power $Q^{[[n]]}$. □

References

[1] M. Aigner, Combinatorial Theory, Springer, Berlin, 1979.
[2] J.L. Burchnall, The Bessel polynomials, Canad. J. Math. 3 (1951) 62–68.
[3] L. Comtet, Analyse Combinatoire, Les Presses Universitaires de France, Paris, 1970 (English version: Advanced Combinatorics, Reidel, Dordrecht, 1974).
[4] E. Grosswald, Bessel Polynomials, Springer, New York, 1978.
[5] B. Huppert, Endliche Gruppen I, Springer, Berlin, 1967.
[6] H.L. Krall, O. Frink, A new class of orthogonal polynomials: the Bessel polynomials, Trans. Amer. Math. Soc. 65 (1949) 100–115.
[7] J.D.H. Smith, Speed's orbit problem: variations on themes of Burnside, European J. Combin. 20 (1999) 867–863.
[8] J.D.H. Smith, Classical and quantum statistical mechanics of permutation representations, in: Y.G. Baik, D.L. Johnson, A.C. Kim (Eds.), Groups–Korea 98, De Gruyter, Berlin, 2000.
[9] T. tom Dieck, Transformation Groups and Representation Theory, Springer, Berlin, 1979.