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Dirac's minimum degree condition restricted to claws

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Abstract

Let G be a graph on $n \geqslant 3$ vertices. Dirac's minimum degree condition is the condition that all vertices of G have degree at least $\frac{1}{2}n$. This is a well-known sufficient condition for the existence of a Hamilton cycle in G. We give related sufficiency conditions for the existence of a Hamilton cycle or a perfect matching involving a restriction of Dirac's minimum degree condition to certain subsets of the vertices. For this purpose we define G to be 1-heavy (2-heavy) if at least one (two) of the end vertices of each induced subgraph of G isomorphic to $K_{1,3}$ (a claw) has (have) degree at least $\frac{1}{2}n$. Thus, every claw-free graph is 2-heavy, and every 2-heavy graph is 1-heavy. We show that a 1-heavy or a 2-heavy graph G has a Hamilton cycle or a perfect matching if we impose certain additional conditions on G involving numbers of common neighbours, (local) connectivity, and forbidden induced subgraphs. These results generalize or extend previous work of Broersma & Veldman, Dirac, Fan, Faudree et al., Goodman & Hedetniemi, Las Vergnas, Oberly & Sumner, Ore, Shi, and Sumner.

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1. Terminology and notation

We use [5] for terminology and notation not defined here and consider finite simple graphs only.

Let G be a graph of n vertices. We say that G is *hamiltonian* if G has a *Hamilton cycle*, i.e. a cycle containing all vertices of G. If $S \subseteq V(G)$, then $\langle S \rangle$ denotes the subgraph of G induced by S. A graph H is an *induced subgraph* of G if $H = \langle S \rangle$ for

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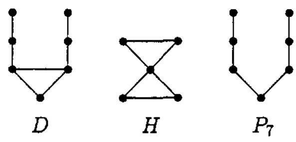

Fig. 1.

some $S \subseteq V(G)$. An induced subgraph of G with vertex set $\{u, v, w, x\}$ and edge set $\{uv, uw, ux\}$ is called a *claw* of G, with *center* u and *end vertices* v, w, x. Throughout the paper, whenever the vertices of a claw of G are listed, its center will always be listed first. A vertex v of G is called *heavy* if $d(v) \geqslant \frac{1}{2}n$. A claw of G is called *1-heavy* if at least one of its end vertices is heavy, and it is called *2-heavy* if at least two of its end vertices are heavy. A graph is *1-heavy* (*2-heavy*) if all its claws are 1-heavy (2-heavy). If X is a graph, we say that G is *X-free* if G does not contain an induced subgraph isomorphic to X. Instead of $K_{1,3}$-free, we use the more common term *claw-free*. Note that every claw-free graph is 2-heavy, and that every 2-heavy graph is 1-heavy. An induced subgraph of G isomorphic to $K_{1,3}$ with one additional edge is called a *modified claw*. We use $\omega(G)$ to denote the number of components of G. G is *1-tough* if $\omega(G-S) \leqslant |S|$ for every subset S of $V(G)$ with $\omega(G-S) > 1$. We use D (of *deer*) and H (of *hourglass*) to denote the graphs of Fig. 1, and P_7 for a path on 7 vertices.

If $v \in V(G)$, then $N(v)$ denotes the set of vertices adjacent to v (the *neighbourhood* of v). A vertex $v \in V(G)$ is *locally-connected* if $\langle N(v) \rangle$ is connected, and the graph G is *locally-connected* if all vertices of G are locally-connected. G is called *even* (*odd*) if n is even (odd). A *perfect matching* or *1-factor* of G is a set of $\frac{1}{2}n$ edges of G no two of which have a vertex in common.

2. Introduction

Generally speaking, one can distinguish two types of sufficiency conditions with respect to cyclic properties of graphs. On one hand, there are the so-called numerical conditions, of which probably degree conditions are the most well known; on the other hand, there are what we call structural conditions, of which forbidden subgraph conditions form a good example. We give examples of both types of conditions in the sequel.

Our main objective here is to generalize existing results by combining the two types of conditions, or, to be more precise, by restricting the numerical conditions to certain substructures. The following example should give the reader the general flavour of the results. Consider the following two results in hamiltonian graph theory.

Theorem 1 (Dirac [8]). *Let G be a graph on $n \geqslant 3$ vertices with $\delta \geqslant \frac{1}{2}n$. Then G is hamiltonian.*

Theorem 2 (Shi [16]). *Let G be a 2-connected graph on $n \geqslant 3$ vertices. If G is claw-free and $|N(u) \cap N(v)| \geqslant 2$ for every pair of vertices u, v with $d(u,v) = 2$, then G is hamiltonian.*

Since the hypothesis of Theorem 1 implies that G is 2-connected and that $|N(u) \cap N(v)| \geqslant 2$ for every pair of vertices u, v with $d(u,v) = 2$, the following result, which we prove in Section 5, obviously is a common generalization of Theorem 1 and Theorem 2.

Theorem 3. *Let G be a 2-connected graph on $n \geqslant 3$ vertices. If G is 2-heavy and $|N(u) \cap N(v)| \geqslant 2$ for every pair of vertices u, v with $d(u,v) = 2$ and $\max(d(u), d(v)) < \frac{1}{2}n$, then G is hamiltonian.*

In fact, we can prove a slightly stronger version of the above theorem, in which we require $|N(u) \cap N(v)| \geqslant 2$ only for every pair of vertices u, v in a modified claw of G with $d(u,v) = 2$ and $\max(d(u), d(v)) < \frac{1}{2}n$. This stronger version also generalizes the result of Goodman and Hedetniemi [11], that every 2-connected graph on at least 3 vertices is hamiltonian if it does not contain an induced claw or modified claw.

Using similar ideas we extend several known results on the existence of Hamilton cycles and perfect matchings in claw-free graphs to the larger classes of 2-heavy or 1-heavy graphs. We also discuss the sharpness of the results and pose some open problems. The results on hamiltonicity are presented in Section 3, those on perfect matchings and toughness in Section 4. We postpone most of the proofs to Section 5.

Related recent work is due to Bedrossian et al. [1]. They impose degree conditions on all nonadjacent vertices of induced claws and modified claws to guarantee (strongly) hamiltonian properties of graphs.

3. Hamilton cycles

In the previous section we stated our first result on hamiltonicity (Theorem 3), and we remarked that it is a common generalization of known results by Dirac [8] and Shi [16]. Theorem 3 also generalizes the following result.

Corollary 4 (Fan [9]). *If G is a 2-connected graph of order $n \geqslant 3$ such that $\max(d(u), d(v)) \geqslant \frac{1}{2}n$ for each pair of vertices u, v with $d(u,v) = 2$, then G is hamiltonian.*

Proof. The hypothesis of Corollary 4 implies that there are no pairs of vertices u, v with $d(u,v) = 2$ and $\max(d(u), d(v)) < \frac{1}{2}n$. Next, considering the three different pairs of end vertices of a claw, the hypothesis of Corollary 4 implies that at least two of the end vertices are heavy. □

Corollary 4 (and Theorem 3) also generalizes the following well-known result (cf. [9]).

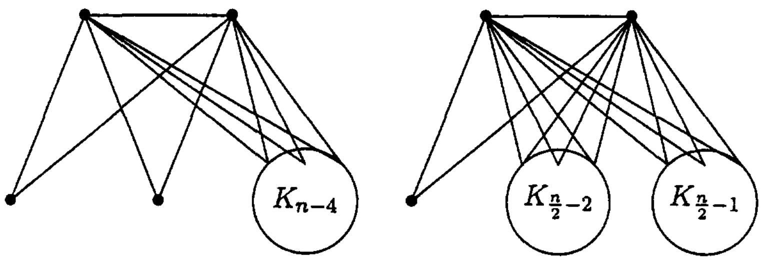

Fig. 2.

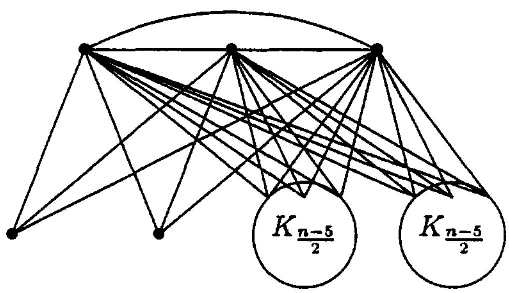

Fig. 3.

Corollary 5 (Ore [15]). *If G is a graph of order $n \geqslant 3$ such that $d(u)+d(v) \geqslant n$ for each pair of nonadjacent vertices u, v, then G is hamiltonian.*

The condition on the vertices at distance 2 in Theorem 3 cannot be omitted, since there exist 2-connected nonhamiltonian claw-free graphs. The graphs $K_2 \vee (2K_1 + K_{n-4})$ and $K_2 \vee (K_1 + K_{n/2-2} + K_{n/2-1})$ (where + denotes the disjoint union and $\vee$ denotes the join of graphs) sketched in Fig. 2, respectively, show one cannot relax 2-heavy to 1-heavy in Theorem 3, and one cannot relax the bound $\frac{1}{2}n$ on the end vertices of claws in Theorem 3.

However, imposing a stronger connectivity condition, one can replace 2-heavy in Theorem 3 by the weaker condition 1-heavy.

Theorem 6. *Let G be a 3-connected graph. If G is 1-heavy and $|N(u) \cap N(v)| \geqslant 2$ for every pair of vertices u, v with $d(u,v) = 2$ and $\max(d(u), d(v)) < \frac{1}{2}n$, then G is hamiltonian.*

The condition on the vertices at distance 2 in Theorem 6 cannot be omitted, since there exist 3-connected nonhamiltonian claw-free graphs (See e.g. [13]). The graph $K_3 \vee (2K_1 + 2K_{(n-5)/2})$ sketched in Fig. 3 shows one cannot relax the bound $\frac{1}{2}n$ on the end vertices of claws in Theorem 6.

It is an open question whether the conclusion of Theorem 6 remains valid if one replaces 3-connected by 1-tough.

Next we examined whether we could replace the condition on the vertices at distance 2 in the previous results by another condition.

The first alternative was motivated by the following result on claw-free graphs.

Theorem 7 (Oberly and Sumner [14]). *Let G be a graph on $n \geqslant 3$ vertices. If G is claw-free, connected and locally-connected, then G is hamiltonian.*

We extended Theorem 7 to the class of 2-heavy graphs.

Theorem 8. *Let G be a graph on $n \geqslant 3$ vertices. If G is 2-heavy, connected and locally-connected, then G is hamiltonian.*

The local connectivity condition in Theorem 8 cannot be omitted, since there exist connected nonhamiltonian claw-free graphs. The graphs sketched in Fig. 2 show one cannot relax 2-heavy to 1-heavy, and one cannot relax the bound $\frac{1}{2}n$ on the end vertices of claws in Theorem 8.

It is an open question whether the conclusion of Theorem 8 remains valid if one replaces connected by 1-tough, and 2-heavy by 1-heavy.

The following result on claw-free graphs is implicit in [6], and motivated us to consider forbidden subgraph conditions.

Theorem 9 (Broersma and Veldman [6]). *Let G be a 2-connected graph. If G is claw-free, P_7-free and D-free, then G is hamiltonian.*

A similar result can be found in [10].

Theorem 10 (Faudree, Ryjáček and Schiermeyer [10]). *Let G be a 2-connected graph. If G is claw-free, P_7-free and H-free, then G is hamiltonian.*

We extended Theorem 9 and Theorem 10 to the class of 2-heavy graphs.

Theorem 11. *Let G be a 2-connected graph. If G is 2-heavy, and moreover P_7-free and D-free, or P_7-free and H-free, then G is hamiltonian.*

The graph $K_2 \vee (2K_1 + K_{n-4})$ of Fig. 2 shows one cannot relax 2-heavy to 1-heavy in Theorem 11.

4. Perfect matchings and toughness

We start this section with a result that was proved independently in [12] and [17].

Theorem 12 (Las Vergnas [12], Sumner [17]). *Let G be an even connected graph. If G is claw-free, then G has a perfect matching.*

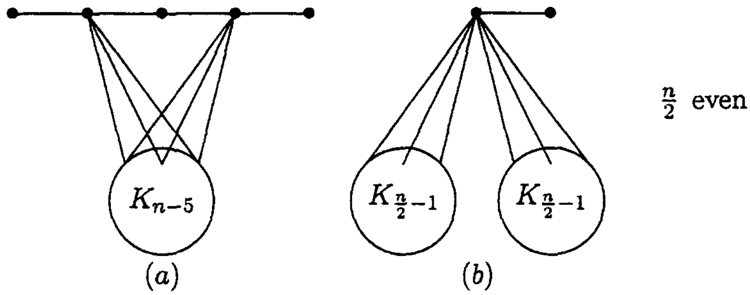

Fig. 4.

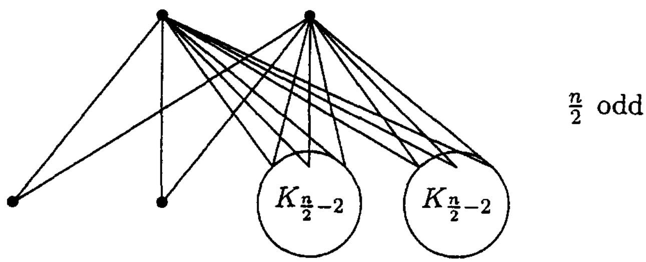

Fig. 5.

We extended Theorem 12 to the class of 2-heavy graphs.

Theorem 13. *Let G be an even connected graph. If G is 2-heavy, then G has a perfect matching.*

The graph sketched in Fig. 4(a) shows that an even connected 1-heavy graph need not have a perfect matching. The graph in Fig. 4(b) shows one cannot relax the degree bound $\frac{1}{2}n$ on the end vertices of claws in Theorem 13. However, imposing a stronger connectivity condition, one can replace 2-heavy in Theorem 13 by the weaker condition 1-heavy.

Theorem 14. *Let G be an even 2-connected graph. If G is 1-heavy, then G has a perfect matching.*

The graph of Fig. 4(a) shows one cannot replace 2-connected by connected in Theorem 14; the graph $2K_1 \vee (2K_1 + 2K_{n/2-2})$ sketched in Fig. 5 shows one cannot relax the bound $\frac{n}{2}$ on the end vertices of claws in Theorem 14.

Using similar techniques as in the proof of Theorem 14, we prove the following two results on toughness.

Theorem 15. *Every 2-connected 2-heavy graph is 1-tough.*

Theorem 16. *Every 3-connected 1-heavy graph is 1-tough.*

The above results show that the condition on the vertices at distance 2 in Theorems 3 and 6 is not necessary if one replaces the conclusion in these theorems by the weaker conclusion that G is 1-tough.

The graph $K_2 \vee (2K_1 + K_{n-4})$ of Fig. 2 shows that a 2-connected 1-heavy graph need not be 1-tough.

5. Proofs

We start this section with some preliminary results. But first we introduce some additional terminology and notation.

Let G be a graph on n vertices and let C be a cycle of G. We denote by $\overrightarrow{C}$ the cycle C with a given orientation, and by $\overleftarrow{C}$ the cycle C with the reverse orientation. If $u, v \in V(C)$, then $u \overrightarrow{C} v$ denotes the consecutive vertices of C from u to v in the direction specified by $\overrightarrow{C}$. The same vertices, in reverse order, are given by $v \overleftarrow{C} u$. We will consider $u \overrightarrow{C} v$ and $v \overleftarrow{C} u$ both as paths and as vertex sets. We use u^+ to denote the successor of u on $\overrightarrow{C}$ and u^- to denote its predecessor. If $A \subseteq V(C)$, then $A^+ = \{v^+ \mid v \in A\}$ and $A^- = \{v^- \mid v \in A\}$. Recall that a vertex v of G is heavy if $d(v) \geqslant \frac{1}{2}n$; if v is not heavy we call it *light*. The cycle C is called *heavy* if it contains all the heavy vertices of G; it is called *extendable* if there exists a longer cycle in G containing all vertices of C. A set $S \subseteq V(G)$ is called an *antifactor set* if the number of odd components in $G - S$ exceeds $|S|$.

Lemma 17 (Bollobás and Brightwell [2], Shi [16]). *Every 2-connected graph contains a heavy cycle.*

The two observations in the following lemma are implicit in the works of Chvátal and Erdös [7] and Bondy [3], respectively.

Lemma 18. *Let $\overrightarrow{C}$ be a nonextendable cycle in a graph G of order n, H a component of $G - V(C)$, and A the set of neighbours of H on C. Then*

(a) *$A \cap A^- = \emptyset$, $A \cap A^+ = \emptyset$, and A^- and A^+ are independent sets.*

(b) *Each pair of vertices from A^- or A^+ has degree sum smaller than n.*

The following lemma is a variation of the closure lemma by Bondy and Chvátal [4].

Lemma 19. *Let G be a graph and $u, v \in V(G)$ be two nonadjacent heavy vertices. If $G + uv$ has a cycle C containing all heavy vertices of G, then G has a cycle containing all vertices of C.*

Proof. Assume G does not have a cycle containing all vertices of C. Consider a path P from u to v in G containing all vertices of C. Clearly, u and v have no common neighbour in $V(G) \setminus V(P)$, and by a standard argument (See e.g. [4]) the degree sum of u and v on P is smaller than $|V(P)|$. Hence at most one of u and v is heavy. □

Proof of Theorem 3. By Lemma 17, G contains a heavy cycle. Consider a longest heavy cycle C of G, fix an orientation on C, and assume G is not hamiltonian. Since G is 2-connected, there exists a path P between two vertices $w_1 \in V(C)$ and $w_2 \in V(C)$ internally disjoint with C and such that $|V(P)| \geqslant 3$. By the choice of C, all internal vertices on P are light, and by Lemma 18(b) we may assume w_1^+ is light. Since G is 2-heavy, w_1 is not a center of a claw, implying that $w_1^- w_1^+ \in E(G)$. Let v denote the successor of w_1 on P, and let x denote a vertex in $(N(w_1^+) \cap N(v)) \setminus \{w_1\}$. It is clear that $x \in V(C)$. If $x^- x^+ \in E(G)$, then $w_1^+ \overrightarrow{C} x^- x^+ \overrightarrow{C} w_1 v x w_1^+$ contradicts the choice of C. So $x^- x^+ \notin E(G)$. By Lemma 18(a) $w_1^+ x^+ \notin E(G)$. Hence $\{x, v, w_1^+, x^+\}$ induces a claw such that both v and w_1^+ are light, contradicting that G is 2-heavy. □

For a proof of the stronger version mentioned in Section 2 we only need to add the observation that $\{w_1, w_1^-, w_1^+, v\}$ induces a modified claw in G.

Proof of Theorem 6. By Lemma 17, G contains a heavy cycle. Consider a longest heavy cycle C of G, fix an orientation on C, and assume G is not hamiltonian. Let H be a component of $G - V(C)$. Since G is 3-connected, there are at least 3 distinct neighbours w_1, w_2, w_3 of H on C. By Lemma 18(b), we know that for at least one $i \in \{1, 2, 3\}$ both w_i^- and w_i^+ are light. Assume without loss of generality that w_1^- and w_1^+ are light. Denote a neighbour of w_1 in H by v. Since G is 1-heavy and v is light, $w_1^- w_1^+ \in E(G)$. As in the proof of Theorem 3, the hypothesis of Theorem 6 implies there exists a vertex $x \in (N(w_1^+) \cap N(v)) \setminus \{w_1\}$ on C such that $x^- x^+ \notin E(G)$. Now since G is 1-heavy, using Lemma 18(b) and considering $\{x, v, w_1^+, x^+\}$, we obtain that x^+ is heavy. Since G is 3-connected, there is a neighbour $y \neq w_1, x$ of H on C. Since x^+ is heavy, Lemma 18(b) yields that y^+ is light. Denote by z a neighbour of y in H. As before, the hypothesis of the theorem implies there exists a vertex $p \in N(z) \cap N(y^+)$ on $V(C)$ such that $p^- p^+ \notin E(G)$. Now since G is 1-heavy, using Lemma 18(b) and considering $\{p, z, p^+, y^+\}$, we obtain that p^+ is heavy. This leads to a contradiction with Lemma 18(b) unless $p = x$. In the latter case, $\{x, w_1^+, y^+, v\}$ induces a claw with light end vertices only, contradicting that G is 1-heavy. □

Proof of Theorem 8. Since G is connected and locally-connected, G is 2-connected. By Lemma 17, G contains a heavy cycle. Consider a longest heavy cycle C of G, fix an orientation on C, and assume G is not hamiltonian. As in the former proofs, we can find a vertex $x \in V(G) \setminus V(C)$ in such a way that for some $w \in V(C)$, $xw \in E(G)$, $w^- w^+ \in E(G)$, and w^- or w^+ is light. Assume without loss of generality that w^+ is light. Since $N(w)$ induces a connected graph, denoted by W, there is a path in W connecting x and w^+. Choose a shortest path P in W between w^+ and a vertex y in the component of $G - V(C)$ containing x. Observe that all vertices of P except for y are on C. Denote $P : y = y_0 y_1 \ldots y_l = w^+$. By Lemma 18(a), $l \geqslant 2$. We claim that $l = 3$. Otherwise, if $l \geqslant 4$, then $\{w, y, y_2, w^+\}$ induces a claw with y and w^+ light, a contradiction; if $l = 2$, then $\{y_1, y_1^+, y, w^+\}$ induces a claw contradicting the hypothesis that G is 2-heavy. Suppose $w^- \in V(P)$. Considering the claw induced by $\{y_1, y_1^-, y, w^-\}$, since G is 2-heavy and y is light, we obtain that y_1^- and w^- are heavy, contradicting Lemma 18(b). Hence $w^- \notin V(P)$. We next observe that $y_1 y_2 \notin E(C)$. Otherwise, if $y_2 = y_1^+$, we contradict Lemma 18(a); if $y_1 = y_2^+$, the cycle $wyy_1 \overrightarrow{C} w^- w^+ \overrightarrow{C} y_2 w$ contradicts the choice of C. Now we distinguish two cases.

1. $y_1^- y_1^+ \in E(G)$.

We claim that $y_2^+ w^+, y_1 y_2^+ \notin E(G)$. Otherwise, if $y_2^+ w^+ \in E(G)$, the cycle $y_2 \overleftarrow{C} w^+ y_2^+ \overrightarrow{C} y_1^- y_1^+ \overrightarrow{C} wyy_1y_2$ (if $y_2 \in w^+ \overrightarrow{C} y_1^{--}$) or $y_2 y_1 yw \overleftarrow{C} y_2^+ w^+ \overrightarrow{C} y_1^- y_1^+ \overrightarrow{C} y_2$ (if $y_2 \in y_1^{++} \overrightarrow{C} w^-$) contradicts the choice of C; if $y_1 y_2^+ \in E(G)$, the cycle $y_2 wyy_1 y_2^+ \overrightarrow{C} y_1^- y_1^+ \overrightarrow{C} w^- w^+ \overrightarrow{C} y_2$ (if $y_2 \in w^+ \overrightarrow{C} y_1^{--}$) or $y_2 wyy_1 y_2^+ \overrightarrow{C} w^- w^+ \overrightarrow{C} y_1^- y_1^+ \overrightarrow{C} y_2$ (if $y_2 \in y_1^{++} \overrightarrow{C} w^-$) contradicts the choice of C (recall that we already know that $y_1 \neq y_2$, $y_1 y_2 \notin E(C)$ and $w^- \notin V(P)$). Hence $\{y_2, y_1, y_2^+, w^+\}$ induces a claw. Since G is 2-heavy and w^+ is light, we obtain that y_1 and y_2^+ are heavy. Clearly, $G + y_1 y_2^+$ has a cycle C' containing all heavy vertices of G, and such that C' is longer than C. By Lemma 19, G has a cycle containing all vertices of C', a contradiction with the choice of C.

2. $y_1^- y_1^+ \notin E(G)$.

Consider the claw induced by $\{y_1, y_1^-, y_1^+, y\}$. Since G is 2-heavy and y is light, we conclude that y_1^- and y_1^+ are heavy. The arguments we used in Case 1 can now be applied to the graph $G' = G + y_1^- y_1^+$ to conclude that G' has a cycle C' containing all heavy vertices of G and such that C' is longer than C. Now Lemma 19 again yields a contradiction with the choice of C. (Note that the degrees of y_1 and y_2^+ do not change if we add the edge $y_1^- y_1^+$.) □

Proof of Theorem 11. By Lemma 17, G contains a heavy cycle. Consider a longest heavy cycle C of G, fix an orientation on C, and assume G is not hamiltonian. Since G is 2-connected, there exists a path of length at least 2, internally-disjoint with C, that connects two vertices of C. Let $P = w_1 x_1 x_2 \ldots x_r w_2$ be such a path of minimum length, implying that P is an induced path unless $w_1 w_2 \in E(G)$. For $i = 1, 2$, let y_i be the first vertex in $w_i^+ \overrightarrow{C} w_{3-i}^-$ satisfying $y_i w_i \notin E(G)$. Such a vertex exists; otherwise without loss of generality assume $w_2^- w_1 \in E(G)$. If $w_1^- w_1^+ \in E(G)$, then $C' = w_1^+ \overrightarrow{C} w_2^- w_1 x_1 \ldots x_r w_2 \overrightarrow{C} w_1^- w_1^+$ contradicts the choice of C; if $w_1^- w_1^+ \notin E(G)$, then, since x_1 is light and G is 2-heavy, both w_1^- and w_1^+ are heavy. Then $G + w_1^- w_1^+$ contains the cycle C', and, by Lemma 19, G has a cycle containing all vertices of C', contradicting the choice of C. By Lemma 18(b), at least one of the pairs $\{w_1^-, w_1^+\}$ and $\{w_2^-, w_2^+\}$ contains a light vertex. Without loss of generality assume $\{w_1^-, w_1^+\}$ contains a light vertex. Then, since G is 2-heavy and x_1 is light, $w_1^- w_1^+ \in E(G)$. We distinguish two cases.

1. $w_2^- w_2^+ \in E(G)$.

Let z_i be an arbitrary vertex in $w_i^+ \overrightarrow{C} y_i$ $(i = 1,2)$ and let x be a vertex in $V(P) \setminus \{w_1, w_2\}$. Then we first show

$$xz_1, xz_2, z_1w_2, z_2w_1, z_1z_2 \notin E(G). \tag{1}$$

If $xz_1 \in E(G)$, then $w_1x_1 \dots xz_1 \overrightarrow{C} w_1^- w_1^+ \overrightarrow{C} z_1^- w_1$ (if $z_1 \neq w_1^+$) or $w_1x_1 \dots xz_1 \overrightarrow{C} w_1$ (if $z_1 = w_1^+$) is a cycle contradicting the choice of C. Hence $xz_1 \notin E(G)$. Similarly, $xz_2 \notin E(G)$. If $z_1w_2 \in E(G)$, then the cycle $w_1x_1 \dots x_rw_2z_1 \overrightarrow{C} w_2^- w_2^+ \overrightarrow{C} w_1^- w_1^+ \overrightarrow{C} z_1^- w_1$ (if $z_1 \neq w_1^+$) or $w_1x_1 \dots x_rw_2z_1 \overrightarrow{C} w_2^- w_2^+ \overrightarrow{C} w_1$ (if $z_1 = w_1^+$) contradicts the choice of C. Hence $z_1w_2 \notin E(G)$. Similarly, $z_2w_1 \notin E(G)$. Suppose $z_1z_2 \in E(G)$. If $z_1 \neq w_1^+$ and $z_2 \neq w_2^+$, then the cycle $w_1x_1 \dots x_rw_2z_2^- \overleftarrow{C} w_2^+ w_2^- \overleftarrow{C} z_1z_2 \overrightarrow{C} w_1^- w_1^+ \overrightarrow{C} z_1^- w_1$ contradicts the choice of C. Similarly, a heavy cycle longer than C can be indicated if $z_1 = w_1^+$ or $z_2 = w_2^+$. Hence $z_1z_2 \notin E(G)$.

Now if $r = 1$, then by (1) and the choice of y_1 and y_2, $\{x_1, w_1, y_1^-, y_1, w_2, y_2^-, y_2\}$ induces P_7 (if $w_1w_2 \notin E(G)$) or D (if $w_1w_2 \in E(G)$). In the latter case, it is easy to check that $\{x_1, w_1, w_2, w_1^-, w_1^+\}$ induces H. Next assume $r \geqslant 2$. Then $x_1w_2 \notin E(G)$. Suppose $w_1w_2 \in E(G)$. Then, by (1), using that G is 2-heavy and x_1 and x_r are light, the claws induced by $\{w_1, w_1^+, x_1, w_2\}$ and $\{w_2, w_2^+, x_r, w_1\}$ yield that both w_1^+ and w_2^+ are heavy, contradicting Lemma 18(b). Hence $w_1w_2 \notin E(G)$. Now $\{x_1, \dots, x_r, w_1, y_1^-, y_1, w_2, y_2^-, y_2\}$ induces P_{r+6}. So in all cases we find an induced subgraph isomorphic to P_7 or one isomorphic to D and one isomorphic to H, contradicting the hypothesis of Theorem 11.

2. $w_2^- w_2^+ \notin E(G)$.

Then $\{w_2, w_2^-, w_2^+, x_r\}$ induces a claw. Since G is 2-heavy and x_r is light, w_2^- and w_2^+ are both heavy. If we apply the arguments of Case 1 to the graph $G' = G + w_2^- w_2^+$, we find a cycle C'' in G' containing all vertices of C and longer than C. (Note that the edge $w_2^- w_2^+$ is not an edge of one of the induced subgraphs considered in Case 1.) By Lemma 19, G has a cycle containing all vertices of C'', contradicting the choice of C. □

The following lemma is implicit in [18].

Lemma 20 (Sumner [18]). *Let G be an even connected graph without a perfect matching, and let S be a minimum antifactor set of G. Then every vertex of S is adjacent to vertices of at least three odd components of $G - S$ (and therefore centers a claw).*

Lemma 21. *Let G be a graph, and let S be a nonempty set of vertices of G such that $\omega(G - S) > |S|$. Then at most one component of $G - S$ contains a heavy vertex of G.*

Proof. Let $G_1, G_2, \dots, G_k$ be the components of $G - S$ for some $k \geqslant |S| + 1$, and suppose that at least two components of $G - S$ contain a heavy vertex of G. Without loss of generality assume $x_1 \in V(G_1)$ and $x_2 \in V(G_2)$ are heavy. It is clear that each neighbour of x_i is in G_i or in S $(i = 1,2)$, hence $|V(G_i)| - 1 + |S| \geqslant \frac{1}{2}|V(G)|$ $(i = 1,2)$, so that $|V(G)| = |V(G_1)| + |V(G_2)| + |S| + |V(G_3)| + \cdots + |V(G_k)| \geqslant |V(G_1)| + |V(G_2)| + |S| + k - 2 \geqslant |V(G_1)| + |V(G_2)| + 2|S| - 1 \geqslant |V(G)| + 1$, a contradiction. □

Proof of Theorem 13. Suppose that G has no perfect matching. Let S denote a minimum antifactor set. By Lemma 20 every vertex of S centers a claw with end vertices in different components of $G - S$. By Lemma 21 such a claw has at most one heavy end vertex, contradicting the hypothesis that G is 2-heavy. □

Proof of Theorem 14. Suppose that G has no perfect matching. Let S denote a minimum antifactor set. Then S is not empty. Let $s = |S|$ and let $G_1, G_2, \ldots, G_k$ denote (all) the components of $G - S$. Since G is 2-connected, $s \geqslant 2$, and by Tutte's Theorem and parity arguments, $k \geqslant s + 2$. By Lemma 20, every vertex of S centers a claw with end vertices in different components of $G - S$. By Lemma 21 and the hypothesis that G is 1-heavy, exactly one of the components of $G - S$ contains a heavy vertex of G, G_1 say, and every vertex of S has a (heavy) neighbour in G_1. Moreover, by the same arguments, every vertex of S has neighbours in exactly two other components of $G - S$. So, if we denote by $r(x)$ the number of components of $G - S$ containing at least one neighbour of a vertex $x \in S$, then we have $\sum_{x \in S} r(x) = 3s$. On the other hand, since G is 2-connected, every component of $G - S$ except G_1 contributes one to $r(x)$ and $r(y)$ for at least two distinct vertices $x, y \in S$, while G_1 contributes one to $r(x)$ for all $x \in S$. This implies $\sum_{x \in S} r(x) \geqslant s + 2(k - 1)$. Combining the inequality and equality we obtain that $k \leqslant s + 1$, a contradiction. □

We now prove the following variation of Lemma 20.

Lemma 22. *Let G be a graph, and let S be a minimum set of vertices of G such that $\omega(G - S) > |S|$. Then either $|S| \leqslant 1$ or every vertex of S is adjacent to vertices of at least three components of $G - S$ (and therefore centers a claw).*

Proof. Let $G_1, G_2, \ldots, G_k$ be the components of $G - S$ and suppose that $s = |S| \geqslant 2$. First suppose there exists a vertex $x \in S$ having neighbours in S and exactly one component of $G - S$. Then $\omega(G - (S \setminus \{x\})) = \omega(G - S)$, contradicting the minimality of S. Next suppose there exists a vertex $x \in S$ having neighbours in S and exactly two components of $G - S$. Then $\omega(G - (S \setminus \{x\})) = \omega(G - S) - 1$, again contradicting the minimality of S. □

Proof of Theorem 15. Suppose G is a 2-connected graph and G is not 1-tough. Choose a minimum set S for which $\omega(G - S) > |S| \geqslant 2$. By Lemma 22, every $x \in S$ centers a claw with end vertices in different components of $G - S$. But then, by Lemma 21, G is not 2-heavy. □

Proof of Theorem 16. Suppose G is a 3-connected graph and G is not 1-tough. Choose a minimum set S for which $\omega(G-S) > |S| \geqslant 3$. Let $s = |S|$ and let $G_1, G_2, \ldots, G_k$ be the components of $G-S$, implying that $k \geqslant s+1$. By Lemma 22, every $x \in S$ centers a claw with end vertices in different components of $G-S$. By Lemma 21 and the hypothesis that G is 1-heavy, exactly one of the components of $G-S$, G_1 say, contains a heavy vertex, and every $x \in S$ has a (heavy) neighbour in G_1. Moreover, by the same arguments, every $x \in S$ has neighbours in exactly two other components of $G-S$. As in the proof of Theorem 14, if we let $r(x)$ denote the number of components of $G-S$ containing at least one neighbour of $x \in S$, we obtain $\sum_{x \in S} r(x) = 3s$. On the other hand, since G is 3-connected, every component of $G-S$ has at least three neighbours in S. Hence $\sum_{x \in S} r(x) \geqslant 3k \geqslant 3s+3$, a contradiction. □

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