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Conformally invariant fully nonlinear elliptic equations and isolated singularities

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Abstract

We study properties of solutions with isolated singularities to general conformally invariant fully nonlinear elliptic equations of second order. The properties being studied include radial symmetry and monotonicity of solutions in the punctured Euclidean space and the asymptotic behavior of solutions in a punctured ball. Some results apply to more general situations including more general fully nonlinear elliptic equations of second order, and some have been used in a companion paper to establish comparison principles and Liouville type theorems for degenerate elliptic equations.

MSC: 35J60; 53A30

Keywords: Fully nonlinear elliptic equations; Isolated singularity

1. Introduction

There has been much work on conformally invariant fully nonlinear elliptic equations and applications to geometry and topology. See for instance [20,5,3,13,16,11], and the references therein. In this and a companion paper [17] we address some analytical issues concerning these equations.

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For $n \geqslant 3$, consider

$$-\Delta u = n(n-2)u^{\frac{n+2}{n-2}} \quad \text{on } \mathbb{R}^n. \tag{1}$$

The method of moving planes was used by Gidas et al. [8] in proving that any positive C^2 solution of (1) satisfying $\int_{\mathbb{R}^n} u^{\frac{2n}{n-2}} < \infty$ must be of the form

$$u(x) = \left(\frac{a}{1 + a^2|x - \bar{x}|^2} \right)^{\frac{n-2}{2}},$$

where $a > 0$ and $\bar{x} \in \mathbb{R}^n$. The hypothesis $\int_{\mathbb{R}^n} u^{\frac{2n}{n-2}} < \infty$ was removed by Caffarelli et al. [1]; this is important for applications. This latter result was extended to general conformally invariant fully nonlinear second-order elliptic equations in joint work with Li [13,16], see also [14,15]. For earlier results on the Liouville-type theorems, see [16] for a description. Behavior near the origin of positive solutions of $-\Delta u = u^{\frac{n+2}{n-2}}$ in a punctured ball is also analyzed in [1]. Among other things, we extend in this paper a number of results in [1] to general conformally invariant second-order fully nonlinear elliptic equations. New techniques are developed in the present paper. Some of these, in particular Theorem 1.11, have been used in the companion paper [17] to study general degenerate conformally invariant fully nonlinear elliptic equations.

Let $\mathcal{S}^{n\times n}$ denote the set of $n \times n$ real symmetric matrices, $\mathcal{S}_+^{n\times n}$ denote the subset of $\mathcal{S}^{n\times n}$ consisting of positive definite matrices, $O(n)$ denote the set of $n \times n$ real orthogonal matrices, $U \subset \mathcal{S}^{n\times n}$ be an open set, and $F \in C^1(U) \cap C^0(\overline{U})$.

We list below a number of properties of (F, U). Subsets of these properties are used in various lemmas, propositions and theorems:

$$O^{-1}UO = U, \quad \forall O \in O(n), \tag{2}$$

$$U \cap \{M + tN \mid 0 < t < \infty\} \quad \text{is convex } \forall M \in \mathcal{S}^{n\times n}, N \in \mathcal{S}_+^{n\times n}, \tag{3}$$

$$M \in U \text{ and } N \in \mathcal{S}_+^{n\times n} \text{ implies } M + N \in U, \tag{4}$$

$$M \in U \text{ and } a > 0 \text{ implies } aM \in U, \tag{5}$$

$$F(O^{-1}MO) = F(M), \quad \forall M \in U, \forall O \in O(n), \tag{6}$$

$$(F_{ij}(M)) > 0, \quad \forall M \in U, \tag{7}$$

where $F_{ij}(M) := \frac{\partial F}{\partial M_{ij}}(M)$, and, for some $\delta > 0$,

$$F(M) \neq 1, \quad \forall M \in U \cap \left\{ M \in \mathcal{S}^{n\times n} \mid \|M\| := \left(\sum_{i,j} M_{ij}^2 \right)^{\frac{1}{2}} < \delta \right\}. \tag{8}$$

Examples of such (F, U) include those given by the elementary symmetric functions. For $1 \leqslant k \leqslant n$, let

$$\sigma_k(\lambda) = \sum_{1 \leqslant i_1 < \cdots < i_k \leqslant n} \lambda_{i_1} \cdots \lambda_{i_k}$$

be the kth elementary symmetric function and let Γ_k be the connected component of $\{\lambda \in \mathbb{R}^n \mid \sigma_k(\lambda) > 0\}$ containing the positive cone $\Gamma_n := \{\lambda = (\lambda_1, \ldots, \lambda_n) \mid \lambda_i > 0\}$. Let

$$U_k := \{M \in \mathcal{S}^{n \times n} \mid \lambda(M) \in \Gamma_k\},$$

and

$$F_k(M) := \sigma_k(\lambda(M))^{\frac{1}{k}},$$

where $\lambda(M)$ denotes the eigenvalues of M. Then $(F, U) = (F_k, U_k)$ satisfy all the above listed properties, see for instance [2]. Taking $k = 1$, equation

$$F_1(A^u) = 1$$

amounts to, modulo a harmless positive constant,

$$-\Delta u = u^{\frac{n+2}{n-2}}.$$

Here and throughout the paper we use notation

$$A^u = -\frac{2}{n-2} u^{-\frac{n+2}{n-2}} \nabla^2 u + \frac{2n}{(n-2)^2} u^{-\frac{2n}{n-2}} \nabla u \otimes \nabla u - \frac{2}{(n-2)^2} u^{-\frac{2n}{n-2}} |\nabla u|^2 I,$$

where ∇u denotes the gradient of u and $\nabla^2 u$ denotes the Hessian of u.

Other, much more general, examples are as follows. Let

$$\Gamma \subset \mathbb{R}^n \text{ be an open convex symmetric cone with vertex at the origin}$$

satisfying

$$\Gamma_n \subset \Gamma \subset \Gamma_1 := \left\{ \lambda \in \mathbb{R}^n \middle| \sum_i \lambda_i > 0 \right\}.$$

Naturally, Γ being symmetric means $(\lambda_1, \lambda_2, \ldots, \lambda_n) \in \Gamma$ implies $(\lambda_{i_1}, \lambda_{i_2}, \ldots, \lambda_{i_n}) \in \Gamma$ for any permutation $(i_1, i_2, \ldots, i_n)$ of $(1, 2, \ldots, n)$.

Let

$$f \in C^1(\Gamma) \cap C^0(\overline{\Gamma})$$

satisfy

$$f|_{\partial\Gamma} = 0, \quad \nabla f \in \Gamma_n \text{ on } \Gamma,$$

and

$$f(s\lambda) = sf(\lambda) \quad \forall s > 0 \text{ and } \lambda \in \Gamma.$$

With such (f, Γ), let

$$U := \{M \in \mathcal{S}^{n\times n} \mid \lambda(M) \in \Gamma\},$$

and

$$F(M) := f(\lambda(M)).$$

Then (F, U) satisfies all the above listed properties. In fact, for all these (F, U), $A^u \in U$ implies $\Delta u \leqslant 0$. So for these (F, U), the assumption $\Delta u \leqslant 0$ in various theorems in this paper is automatically satisfied. We note that in all these examples, F is actually concave in U, but this property is not needed in results in this paper.

Throughout the paper we use $B_a(x) \subset \mathbb{R}^n$ to denote the ball of radius a and centered at x, and $B_a = B_a(0)$. Also, unless otherwise stated, the dimension n is bigger than 2.

Theorem 1.1. *Let $U \subset \mathcal{S}^{n\times n}$ be an open set satisfying* (2) *and* (3), *let $F \in C^1(U)$ satisfy* (6)–(8). *Assume that $u \in C^2(\mathbb{R}^n \setminus B_{\frac{1}{2}})$ satisfy*

$$u > 0, \quad \Delta u \leqslant 0 \quad \text{in } \mathbb{R}^n \setminus B_{\frac{1}{2}}, \tag{9}$$

and

$$F(A^u) = 1, \quad A^u \in U \quad \text{in } \mathbb{R}^n \setminus B_{\frac{1}{2}}. \tag{10}$$

Then

$$\limsup_{|x|\to\infty} |x|^{\frac{n-2}{2}} u(x) < \infty. \tag{11}$$

Remark 1.1. For $(F, U) = (F_1, U_1)$, (11) was proved in [1].

Remark 1.2. Gonzalez in [10] and Han in [12] studied for certain (F_k, U_k) solutions with isolated singularities which have finite volume, and Gonzalez in [9] studied sub-critical (F_k, U_k) solutions with isolated singularities. Chang, Han and Yang studied in [6] radial solutions on annular domains including punctured balls and $\mathbb{R}^n$. See these papers for precise statements and details.

Remark 1.3. If $diag(-\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2}) \in U$, then the upper bound (11) is sharp in the sense that the exponent $\frac{n-2}{2}$ cannot be larger. This is because

$$\lambda(A^u) \equiv \left\{-\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2}\right\} \quad \text{on } \mathbb{R}^n \setminus \{0\}$$

for $u(x) = |x|^{\frac{2-n}{2}}$. In particular, (11) is sharp for $(F, U) = (F_k, U_k)$ for $1 \leqslant k < \frac{n}{2}$. See Section 8 for details.

Remark 1.4. Condition (8) cannot be dropped since $u \equiv$ constant could be a solution.

Remark 1.5. Instead of (11), what we have actually proved is

$$\sup_{|x| \geqslant 1} |x|^{\frac{n-2}{2}} u(x) \leqslant C,$$

for some C explicitly given in terms of $\min_{\partial B_1} u$ and n. This can be seen from the proof of Theorem 1.1.

Replacing u by $|x|^{2-n} u(\frac{x}{|x|^2})$ and using the conformal invariance property of $F(A^u)$—see for example line 9 on page 1431 of [13], it is easy to see that Theorem 1.1 is equivalent to

Theorem 1.1′. *Let $U \subset \mathcal{S}^{n\times n}$ be an open set satisfying* (2) *and* (3), *let $F \in C^1(U)$ satisfy* (6)–(8). *Assume that $u \in C^2(B_2 \setminus \{0\})$ satisfy*

$$u > 0, \quad \Delta u \leqslant 0 \quad in\ B_2 \setminus \{0\}, \tag{12}$$

and

$$F(A^u) = 1, \quad A^u \in U \quad in\ B_2 \setminus \{0\}. \tag{13}$$

Then

$$\limsup_{|y| \to 0} |y|^{\frac{n-2}{2}} u(y) < \infty.$$

Theorem 1.2. *Let* $U \subset \mathcal{S}^{n\times n}$ *be an open set satisfying* (2) *and* (3), *and let* $F \in C^1(U)$ *satisfy* (6) *and* (7). *Assume that* $u \in C^2(\mathbb{R}^n \setminus \{0\})$ *satisfy*

$$u > 0, \quad \Delta u \leqslant 0 \quad \text{in } \mathbb{R}^n \setminus \{0\},$$

$$F(A^u) = 1, \quad A^u \in U \quad \text{in } \mathbb{R}^n \setminus \{0\},$$

and

$$u \text{ cannot be extended as a } C^2 \text{ positive function satisfying } A^u \in U \text{ near the origin.} \tag{14}$$

Then u *is radially symmetric about the origin and* $u'(r) < 0$ *for all* $0 < r < \infty$.

Remark 1.6. For $(F, U) = (F_1, U_1)$, the result was proved in [1].

Theorem 1.3. *Let* $U \subset \mathcal{S}^{n\times n}$ *be an open set satisfying* (2) *and* (3), *and let* $F \in C^1(U)$ *satisfy* (6)–(8). *Assume that* $u \in C^2(B_2 \setminus \{0\})$ *satisfies* (12) *and* (13). *Then*, *for some constant* $\varepsilon > 0$,

$$u_{x,\lambda}(y) \leqslant u(y) \quad \forall 0 < \lambda < |x| \leqslant \varepsilon,\ |y-x| \geqslant \lambda, 0 < |y| \leqslant 1. \tag{15}$$

Consequently, *for some positive constant* C,

$$\left|\frac{u(x)}{u(y)} - 1\right| \leqslant Cr \quad \forall 0 < r = |x| = |y| < 1. \tag{16}$$

Remark 1.7. For $(F, U) = (F_1, U_1)$, the result was proved in [1].

Remark 1.8. In view of Remark 1.5, we can obtain explicit dependence of ε and C in terms of $\min_{\partial B_1} u$ and n. With such explicit dependence, Theorem 1.2 follows from Theorem 1.3 by rescaling a large ball to B_2 and then sending the radius of the large ball to infinity. In doing this, the minimum of ∂B_1 of the rescaled function is under control due to the fact $\liminf_{|y|\to\infty} |y|^{n-2}u(y) > 0$. We leave the details to interested readers.

Theorem 1.4. *Let*

$$U \subset \mathcal{S}_+^{n\times n} \tag{17}$$

be an open set satisfying (2) *and* (3), *let* $F \in C^1(U)$ *satisfy* (6) *and* (7), *and let* $u \in C^2(B_2 \setminus \{0\})$ *satisfy*

$$u > 0 \quad \text{in } B_2 \setminus \{0\},$$

and

$$F(A^u)=1,\, A^u\in U \quad in\ B_2\setminus\{0\}.$$

Then u can be extended as a positive Lipschitz function in B_1.

Corollary 1.1. *The conclusion of Theorem* 1.4 *holds for $(F,U)=(F_n,U_n)$.*

Theorem 1.5. *Let $U\subset\mathcal{S}^{n\times n}$ be an open set satisfying* (4) *and* (5). *We assume that there exists some $\eta\in C^2(B_2\setminus\{0\})\cap C^0(B_2)$ satisfying*

$$\eta(0)=0,\quad \eta(x)>0,\quad \forall x\in B_2\setminus\{0\}, \tag{18}$$

$$D^2\eta(x)\ \text{does not belong to}\ U,\quad \forall x\in B_2\setminus\{0\}. \tag{19}$$

Suppose that $\xi\in C^0_{\text{loc}}(B_2\setminus\{0\})\cap L^\infty(B_2\setminus\{0\})$ satisfies

$$\xi>0\ \text{in}\ B_2\setminus\{0\},\quad \Delta\xi\geqslant 0\ \text{in}\ B_2\setminus\{0\}\ \text{in the distribution sense}, \tag{20}$$

and there exist $\{\xi_i\}$ in $C^2(B_2\setminus\{0\})$ such that

$$\Delta\xi_i\geqslant 0\quad \text{in}\ B_2\setminus\{0\}, \tag{21}$$

$$D^2\xi_i\in\overline{U}\quad \text{in}\ B_2\setminus\{0\}, \tag{22}$$

$$\xi_i\to\xi\quad \text{in}\ C^0_{\text{loc}}(B_2\setminus\{0\}). \tag{23}$$

Then ξ can be extended as a function in $C^0(B_1)$ which satisfies

$$\sup_{B_1\setminus\{0\}}\xi\leqslant\max_{\partial B_1}\xi, \tag{24}$$

$$|\xi(x)-\xi(y)|\leqslant C(\eta)\left[\sup_{B_1\setminus\{0\}}\xi-\inf_{B_1\setminus\{0\}}\xi\right]\big[\eta(x-y)+\eta(y-x)\big]\quad \forall x,y\in B_{\frac14}, \tag{25}$$

where $C(\eta)$ denotes some positive constant depending on η.

Corollary 1.2. *For $B_2\subset\mathbb{R}^n$, $n\geqslant 1$, let k be an integer satisfying $\frac{n}{2}<k\leqslant n$. We assume that $\xi\in C^2(B_2\setminus\{0\})\cap L^\infty(B_2\setminus\{0\})$ and*

$$\lambda(D^2\xi)\in\overline{\Gamma}_k\quad \text{in}\ B_2\setminus\{0\}.$$

Then, for $\alpha = \frac{2k-n}{k}$, ξ *can be extended as a function in* $C^{0,\alpha}(B_1)$ *and, for any* $0 < a < 2$,

$$\|\xi\|_{C^{0,\alpha}(B_a)} \leqslant C(n,a)\left(\sup_{B_2\setminus\{0\}}\xi - \inf_{B_2\setminus\{0\}}\xi\right), \tag{26}$$

where $C(n,a)$ *is some positive constant depending only on* n *and* a.

Remark 1.9. Without the possible singularity of ξ at the origin, (26) was known, see theorem 2.7 in [19] by Trudinger and Wang.

Corollary 1.3. *Let* U *and* η *be as in Theorem* 1.5. *Suppose that* $u \in C^0_{\text{loc}}(B_2 \setminus \{0\})$ *satisfies* $u > 0$ *in* $B_2 \setminus \{0\}$ *and there exist* $\{u_i\}$ *in* $C^2(B_2 \setminus \{0\})$,

$$\Delta u_i \leqslant 0, \quad A^{u_i} \in \overline{U} \quad \text{in } B_2 \setminus \{0\},$$

$$u_i \to u \quad \text{in } C^0_{\text{loc}}(B_2 \setminus \{0\}).$$

Then $\xi := u^{-\frac{2}{n-2}}$ *can be extended as a function in* $C^0(B_1)$ *and*

$$\sup_{B_1\setminus\{0\}}\xi \leqslant \max_{\partial B_1}\xi = \left[\max_{\partial B_1} u\right]^{-\frac{2}{n-2}}, \tag{27}$$

$$|\xi(x)-\xi(y)| \leqslant C(\eta)\left[\min_{\partial B_1} u\right]^{-\frac{2}{n-2}}[\eta(x-y)+\eta(y-x)] \quad \forall x,y \in B_{\frac{1}{4}}. \tag{28}$$

Consequently, either

$$0 < \inf_{B_1\setminus\{0\}} u \leqslant \sup_{B_1\setminus\{0\}} u < \infty \text{ and } u \in C^0(B_1), \tag{29}$$

or

$$\inf_{x\in B_1\setminus\{0\}} [\eta(x)+\eta(-x)]^{\frac{n-2}{2}} u(x) > 0. \tag{30}$$

Corollary 1.4. *Let* $B_2 \subset \mathbb{R}^n$ *and let* k *be an integer satisfying* $\frac{n}{2} < k \leqslant n$. *We assume that* $u \in C^2(B_2 \setminus \{0\})$, $u > 0$ *and* $\lambda(A^u) \in \overline{\Gamma}_k$ *on* $B_2 \setminus \{0\}$. *Then* $\xi := u^{-\frac{2}{n-2}}$ *can be extended as a function in* $C^{0,\alpha}(B_1)$, *with* $\alpha = \frac{2k-n}{k} \in (0,1]$, *and*

$$\|\xi\|_{C^{0,\alpha}(B_{\frac{1}{2}})} \leqslant C(n)\left[\min_{\partial B_1} u\right]^{-\frac{2}{n-2}}.$$

Consequently, either

$$0 < \inf_{B_1 \setminus \{0\}} u \leqslant \sup_{B_1 \setminus \{0\}} u < \infty \quad \textit{and } u \in C^{0,\alpha}(B_1),$$

or

$$|x|^{\frac{n-2}{2}\alpha} u(x) \geqslant \frac{1}{C(n)} \left[\min_{\partial B_1} u \right] \quad \forall\ |x| < \frac{1}{2}.$$

Remark 1.10. The Hölder regularity of ξ was independently proved by Gursky and Viaclovsky in [11], which contains some more general and other very nice results. Our proof is different.

Remark 1.11. The Hölder exponent in Theorem 1.4 is sharp, compare for instance results in [6].

Our proofs of Theorems 1.1, 1.2, 1.4 and 1.5 make use of the following theorem and its generalizations.

Theorem 1.6. *Let $U \subset \mathcal{S}^{n\times n}$ be an open set satisfying* (2) *and* (3), *and let $F \in C^1(U)$ satisfy* (6) *and* (7). *We assume that $u \in C^2(B_2 \setminus \{0\})$ and $v \in C^2(B_2)$ satisfy*

$$u > v \quad \textit{in } B_2 \setminus \{0\},$$

$$F(A^u) \geqslant 1, \quad A^u \in U, \Delta u \leqslant 0 \quad \textit{in } B_2 \setminus \{0\},$$

$$F(A^v) \leqslant 1, \quad A^v \in U, v > 0 \quad \textit{in } B_2.$$

Then

$$\liminf_{|x|\to 0} [u(x) - v(x)] > 0. \tag{31}$$

Remark 1.12. As pointed out in [16], the arguments in [13] together with Theorem 1.6 yield the Liouville-type theorem in [16]. The proof of the Liouville-type theorem in [16] avoids such local result by using global information of the entire solution u. Our proof of Theorem 1.6 makes use of the crucial idea in the proof of the Liouville-type theorem in [16]—a delicate use of Lemma 1.2.

The conclusion of Theorem 1.6 holds for elliptic operators with less invariance than the Möbius group. Let $T \in C^1(\mathbb{R}^+ \times \mathbb{R}^n \times \mathcal{S}^{n\times n})$ satisfy

$$\left(-\frac{\partial T}{\partial u_{ij}} \right) > 0 \quad \text{on } \mathbb{R}_+ \times \mathbb{R}^n \times \mathcal{S}^{n\times n}, \tag{32}$$

where $\mathbb{R}_+ = (0, \infty)$. With (32), the operator $T(u, \nabla u, \nabla^2 u)$ is elliptic.

For a positive function v, and for $x \in \mathbb{R}^n$ and $\lambda > 0$, let

$$v^{x,\lambda}(y) := \lambda^{\frac{n-2}{2}} v(x + \lambda y), \quad v^{\lambda}(y) := v^{0,\lambda}(y).$$

We assume that the operator T has the following invariance: For any positive function $v \in C^2(\mathbb{R}^n)$ and for any $\lambda > 0$,

$$T(v^{\lambda}, \nabla v^{\lambda}, \nabla^2 v^{\lambda})(\cdot) \equiv T(v, \nabla v, \nabla^2 v)(\lambda \cdot) \quad \text{in } \mathbb{R}^n. \tag{33}$$

Remark 1.13. Let $T(t, p, M) := S(t^{-\frac{n}{n-2}} p, t^{-\frac{n+2}{n-2}} M)$ for some $S \in C^0(\mathbb{R}^n \times \mathcal{S}^{n\times n})$. Then T satisfies (33). See Lemma 9.1.

Theorem 1.7. *Let $B_2 \subset \mathbb{R}^n$ and let $T \in C^1(\mathbb{R}^+ \times \mathbb{R}^n \times \mathcal{S}^{n\times n})$ satisfy* (33). *We assume that $u \in C^2(B_2 \setminus \{0\})$ and $v \in C^2(B_2)$ satisfy*

$$v > 0 \quad \text{in } B_2, \tag{34}$$

$$u > v \quad \text{in } B_2 \setminus \{0\}, \tag{35}$$

$$\Delta u \leqslant 0 \quad \text{in } B_2 \setminus \{0\}, \tag{36}$$

$$T(u, \nabla u, \nabla^2 u) \geqslant 0 \geqslant T(v, \nabla v, \nabla^2 v) \quad \text{in } B_2 \setminus \{0\}. \tag{37}$$

Then

$$\liminf_{|x|\to 0} [u(x) - v(x)] > 0. \tag{38}$$

Remark 1.14. It is not difficult to see from the proof of Theorem 1.7 that we have only used the following properties of u, v and T: $T \in C^1(\mathbb{R}^+ \times \mathbb{R}^n \times \mathcal{S}^{n\times n})$, $u \in C^2(B_2 \setminus \{0\})$ and $v \in C^2(B_2)$ satisfy (34)–(36), and there exists some $\varepsilon_5 > 0$ such that

$$T(u, \nabla u, \nabla^2 u) \geqslant T(v^{x,\lambda}, \nabla v^{x,\lambda}, \nabla^2 v^{x,\lambda}) \quad \text{on } B_{\varepsilon_5} \setminus \{0\}, \ \forall\ |x| < \varepsilon_5, |\lambda - 1| < \varepsilon_5,$$

and for any $|x| < \varepsilon_5$, $|\lambda - 1| < \varepsilon_5$, $|y| < \varepsilon_5$ satisfying $u(y) = v^{x,\lambda}(y)$, $\nabla u(y) = \nabla v^{x,\lambda}(y)$, $u \geqslant v^{x,\lambda}$ on $B_{\varepsilon_5} \setminus \{0\}$, we have

$$\left(-\frac{\partial T}{\partial u_{ij}}\left(u(y), \nabla u(y), \theta \nabla^2 u(y) + (1-\theta)\nabla^2 v^{x,\lambda}(y)\right)\right) > 0 \quad \forall 0 \leqslant \theta \leqslant 1.$$

Remark 1.15. Taking $F(A^u) - 1$ as the operator T, the properties in Remark 1.14 are satisfied by the u and v in Theorem 1.6—see arguments towards the end of the proof of Lemma 2.1 in [13]. Therefore Theorem 1.6 is, in view of Remark 1.14, a consequence of Theorem 1.7.

The following follows from a classical result in [7]: Let E be a closed subset of B_2 of capacity 0—the standard capacity with respect to the Dirichlet integral, and let $u \in C^2(B_2 \setminus E)$ and $v \in C^2(B_2)$ satisfy

$$u > v \text{ and } \Delta u \leqslant 0 \leqslant \Delta v \text{ in } B_2 \setminus E.$$

Then

$$\liminf_{dist(x,E)\to 0} [u(x) - v(x)] > 0. \tag{39}$$

Theorems 1.1 and 1.7 can be viewed as an extension of this for $E = \{0\}$.

Question 1.1. *In Theorem* 1.7, *if we replace* $\{0\}$ *by some* E *with capacity* 0, *does* (39) *still hold*? *Maybe there is a notion of* T*-capacity for* (39) *to hold for zero* T*-capacity set* E?

A more concrete question is

Question 1.2. *Let* T *be as in Theorem* 1.7 *or* $F(A^u)$ *be as in Theorem* 1.1, *and let* $E = E^k \subset B_2$ *be an embedded closed smooth manifold of dimension k. What is the* $k^*(n, T)$ *for which* (39) *holds for all* $0 \leqslant k \leqslant k^*$*—with the hypotheses of Theorem* 1.7 *or Theorem* 1.1 *for* $\{0\}$ *being changed in an obvious way to that for* E^k? *What about for*

$$E^k = \left\{ (x_1, \dots, x_k, 0, \dots, 0) \mid \sum_{i=1}^{n} (x_i)^2 = 1 \right\}?$$

Another question is

Question 1.3. *For what classes of elliptic operators* $T(x, u, \nabla u, \nabla^2 u)$ *the conclusion of Theorem* 1.7 *holds*?

Concerning this question we will give in Corollaries 1.5 and 1.6 some operators with the property.

For a one variable function φ, we define, instead of $v^{x,\lambda}$,

$$v_\varphi^{x,\lambda}(y) = \varphi(\lambda) v(x + \lambda y), \quad v_\varphi^\lambda = v_\varphi^{0,\lambda}.$$

Theorem 1.8. *Let* $\Omega \subset \mathbb{R}^n$ *be a bounded open set containing the origin* 0, $n \geqslant 2$, *and let* φ *be a* C^1 *function defined in a neighborhood of* 1 *satisfying* $\varphi(1) = 1$ *and*

$\varphi'(1) > 0$. We assume that $u \in C^0(\overline{\Omega} \setminus \{0\})$, v is C^0 in some open neighborhood of $\overline{\Omega}$ and v is C^1 in a neighborhood of 0,

$$v > 0 \quad in\ \overline{\Omega}, \tag{40}$$

$$u > v \quad on\ \overline{\Omega} \setminus \{0\}, \tag{41}$$

$$\Delta u \leqslant 0 \quad in\ \Omega \setminus \{0\}. \tag{42}$$

Assume also that there exists some $\varepsilon_3 > 0$ such that for any $|x| < \varepsilon_3$ and $|\lambda - 1| < \varepsilon_3$,

$$\inf_{\Omega\setminus\{0\}} [u - v_\varphi^{x,\lambda}] = 0 \ implies\ \liminf_{|y|\to 0} [u - v_\varphi^{x,\lambda}](y) = 0. \tag{43}$$

Then (38) holds.

Theorem 1.9. *Under the hypotheses of Theorem 1.8, except changing $\varphi'(1) > 0$ to $\varphi'(1) < 0$. Then (38) holds if $\varphi'(1) < -1$. If $-1 \leqslant \varphi'(1) < 0$, either (38) holds, or*

$$\liminf_{|x|\to 0} [u(x) - v(x)] = 0 \tag{44}$$

and, for some $\varepsilon > 0$ and $V \in \mathbb{R}^n$,

$$\psi(v(x)) + V \cdot x \equiv 0, \quad |x| < \varepsilon, \tag{45}$$

where

$$\psi(s) := \int_{v(0)}^{s} \frac{v(0)}{t} \varphi^{-1}\left(\frac{v(0)}{t}\right) dt.$$

We give a corollary which concerns Question 1.3. Let $S \in C^1(\mathbb{R}^n \times \mathcal{S}^{n\times n})$ satisfy

$$\left(-\frac{\partial S}{\partial M_{ij}}(p, M)\right) > 0 \quad \forall (p, M) \in \mathbb{R}^n \times \mathcal{S}^{n\times n}, \tag{46}$$

and let, for $\beta \in \mathbb{R} \setminus \{0\}$,

$$T(t, p, M) := S\left(t^{-\frac{1+\beta}{\beta}} p, t^{-\frac{2+\beta}{\beta}} M\right) \quad (t, p, M) \in \mathbb{R}_+ \times \mathbb{R}^n \times \mathcal{S}^{n\times n}. \tag{47}$$

Corollary 1.5. *For $n \geqslant 2$, let S, β and T be as above. If $-1 < \beta < 0$, we further require that*

$$S(p, 0) \geqslant 0 \quad \forall p \in \mathbb{R}^n. \tag{48}$$

Assume that $u \in C^2(B_2 \setminus \{0\})$ and $v \in C^2(B_2)$ satisfy (34)–(37). *Then* (38) *holds.*

Clearly, the arguments in the proofs of Theorems 1.6–1.9 can be used to study some other problems. For instance, let

$$\Phi(v, x, \lambda; y) := \varphi(\lambda) v(x + \xi(\lambda) y) + \psi(\lambda).$$

We assume that φ, ψ and ξ are C^1 functions near 1 satisfying $\varphi(1) = \xi(1) = 1$, $\psi(1) = 0$,

$$\varphi' \geqslant 0, \psi' \geqslant 0, \varphi' + \psi' > 0 \quad \text{near } 1, \tag{49}$$

and

$$\varphi' \xi + \varphi \xi' \geqslant 0 \quad \text{near } 1. \tag{50}$$

Here is an extension of Theorem 1.8.

Theorem 1.10. *Let φ, ξ, ψ be as above, and let $\Omega \subset \mathbb{R}^n$ be a bounded open set containing the origin 0, $n \geqslant 2$. We assume that $u \in C^0(\overline{\Omega} \setminus \{0\})$, v is C^0 in some open neighborhood of $\overline{\Omega}$ and v is C^1 near the origin. Assume also that* (40)–(42) *hold, and there exists some $\varepsilon_4 > 0$ such that for any $|x| < \varepsilon_4$ and $|\lambda - 1| < \varepsilon_4$,*

$$\inf_{\Omega \setminus \{0\}} [u - \Phi(v, x, \lambda; \cdot)] = 0 \textit{ implies } \liminf_{|y| \to 0} [u(y) - \Phi(v, x, \lambda; y)] = 0. \tag{51}$$

Then (38) *holds.*

We now give some more operators T for which the conclusion of Theorem 1.7 holds. For S satisfying (46), we consider operators T satisfying one of the following.

(i) $T(t, p, M) := S(p, M)$.
(ii) There exists $\varepsilon > 0$ such that

$$\begin{aligned} \mathit{sign}\ T(\lambda t, \lambda p, \lambda M) = \mathit{sign}\, T(t, p, M) \quad & \forall (\lambda, t, p, M) \\ & \in (1 - \varepsilon, 1 + \varepsilon) \times \mathbb{R}_+ \times \mathbb{R}^n \times \mathcal{S}^{n \times n}. \end{aligned}$$

(iii) $T(t, p, M) := S\left(\frac{1}{t+1} p, \frac{1}{t+1} M\right)$, $(t, p, M) \in \mathbb{R}_+ \times \mathbb{R}^n \times \mathcal{S}^{n \times n}$.

Corollary 1.6. *For* $n \geqslant 2$, *let* $S \in C^1(\mathbb{R}^n \times \mathcal{S}^{n\times n})$ *satisfy* (46) *and let* $T \in C^1(\mathbb{R}_+ \times \mathbb{R}^n \times \mathcal{S}^{n\times n})$ *satisfy one of the above. Assume that* $u \in C^2(B_2 \setminus \{0\})$ *and* $v \in C^2(B_2)$ *satisfy* (34)–(37). *Then* (38) *holds.*

Corollary 1.6 follows from a more general

Corollary 1.7. *For* $n \geqslant 2$, *let* $T \in C^1(\mathbb{R}_+ \times \mathbb{R}^n \times \mathcal{S}^{n\times n})$ *satisfy* (32), *and let* $u \in C^2(B_2 \setminus \{0\})$ *and* $v \in C^2(B_2)$ *satisfy* (34)–(37). *Assume that for some* φ, ξ, ψ *as in Theorem* 1.10 *and for some* $\varepsilon > 0$,

$$T\left(\Phi(v,0,\lambda;\cdot), \nabla\Phi(v,0,\lambda;\cdot), \nabla^2\Phi(v,0,\lambda;\cdot)\right) \leqslant 0 \quad \text{in } B_\varepsilon \text{ for all } |\lambda - 1| < \varepsilon. \tag{52}$$

Then (38) *holds.*

The operators T in Corollary 1.6 satisfy the hypotheses of Corollary 1.7, see Section 9.

In some applications, see [17], assumption (41) in Theorem 1.10 needs to be weakened. For this purpose, we give

Theorem 1.11. *Let* $\Omega \subset \mathbb{R}^n$ *be a bounded open set containing the origin* 0, $n \geqslant 2$. *We assume that* $u \in C^0(\Omega \setminus \{0\})$, v *is* C^1 *in some open neighborhood of* $\overline{\Omega}$, v *satisfies* (40), u *satisfies* (42), *and*

$$u \geqslant v \quad \text{in } \Omega \setminus \{0\}.$$

Assume also that φ, ξ, ψ *are* C^1 *functions near* 1 *satisfying* $\varphi(1) = \xi(1) = 1$, $\psi(1) = 0$, $\varphi'(1) + \xi'(1) > 0$, *and*

$$\varphi'(1)v(y) + \xi'(1)\nabla v(y) \cdot y + \psi'(1) > 0 \quad \forall y \in \overline{\Omega},$$

and assume that there exists some $\varepsilon_4 > 0$ *such that* (51) *holds for any* $|x| < \varepsilon_4$ and $|\lambda - 1| < \varepsilon_4$. *Then either* (38) *holds or* $u = v = v(0)$ *near the origin.*

As mentioned earlier, we make, as in [16], delicate use of the following result.

Lemma 1.1 (*Li and Li [16]*). *For* $n \geqslant 2$, $B_1 \subset \mathbb{R}^n$, *let* $u \in L^1_{loc}(B_1 \setminus \{0\})$ *be the solution of*

$$\Delta u \leqslant 0 \quad \text{in } B_1 \setminus \{0\}$$

in the distribution sense. Assume $\exists\ a \in \mathbb{R}$ *and* $p \neq q \in \mathbb{R}^n$ *such that*

$$u(x) \geqslant \max\{a + p \cdot x - \delta(x), a + q \cdot x - \delta(x)\} \quad \forall x \in B_1 \setminus \{0\},$$

where $\delta(x) \geqslant 0$ *satisfies* $\lim_{x\to 0} \frac{\delta(x)}{|x|} = 0$. *Then*

$$\lim_{r\to 0} \inf_{B_r} u > a.$$

A slightly weaker version of Lemma 1.1 is the following Lemma 1.2.

Lemma 1.2 (*Li and Li [14]*). *For* $n \geqslant 2$, $R > 0$, *let* $u \in C^2(B_R \setminus \{0\})$ *satisfying* $\Delta u \leqslant 0$ *in* $B_R \setminus \{0\}$. *Assume that there exist* $w,\ v \in C^1(B_R)$ *satisfying*

$$w(0) = v(0), \quad \nabla w(0) \neq \nabla v(0),$$

and

$$u \geqslant w, \quad u \geqslant v \quad \text{in } B_R \setminus \{0\}.$$

Then

$$\liminf_{x\to 0} u(x) > w(0).$$

The way we use Lemma 1.1 is as follows. For some function u as in the lemma, we construct a family of C^1 functions $\{w^{(x)}\}$ satisfying

$$u \geqslant w^{(x)} \quad \text{in } B_1 \setminus \{0\}$$

and

$$w^{(x)}(0) = \liminf_{|y|\to 0} u(y).$$

An application of the lemma yields, for some $V \in \mathbb{R}^n$,

$$\nabla w^{(x)}(0) = V \quad \text{for all } x.$$

The above could contain much information.

To better illustrate the idea, we give the following

Proof of Corollary 1.6 in the case (i). For $|x|$ small, shift v by x to obtain $v(x+\cdot)$, which may not be $\leqslant u$. Lower the graph of $v(x+\cdot)$ and then move it up until one cannot move further without cutting through the graph of u. We have obtained

$$w^{(x)} := v(x+\cdot) + \bar{\lambda}(x),$$

which satisfies, for small x,

$$u \geqslant w^{(x)} \quad \text{in } B_1 \setminus \{0\}$$

and

$$\inf_{B_1\setminus\{0\}}\left[u - w^{(x)}\right] = 0.$$

By the smallness of x, the touching of the graphs of u and $w^{(x)}$ cannot occur on ∂B_1. The touching cannot occur in $B_1 \setminus \{0\}$ either, in view of the strong maximum principle. Thus we have

$$w^{(x)}(0) = \liminf_{|y|\to 0} u(y).$$

According to Lemma 1.1, $\nabla w^{(x)}(0) = \nabla v(x)$ is independent of x and, consequently, $v \equiv v(0) + \nabla v(0) \cdot x$ in B_ε for some $\varepsilon > 0$. Now we have $\Delta(u - v) \leqslant 0$ and $u - v > 0$ in $B_\varepsilon \setminus \{0\}$, and (38) follows. $\square$

The paper is organized as follows. In Section 2, we prove Theorem 1.7. In Section 3, we prove Theorems 1.8–1.11. In Section 4, we prove Theorem 1.1. In Section 5, we prove Theorems 1.2 and 1.3. In Section 6, we prove Theorem 1.4. In Section 7, we prove Theorem 1.5, Corollaries 1.2–1.4. In Section 8, we comment on the sharpness of Theorem 1.1. In Section 9, we prove Corollaries 1.5–1.7.

Theorems 1.1, 1.2, 1.4 and Corollary 1.4 were announced at the international conference in honor of Haim Brezis's 60th birthday in Paris, June 9–13, 2004.

2. Proof of Theorem 1.7

Proof of Theorem 1.7. We prove it by contradiction. Suppose the contrary of (38), then (44) holds.

We first give three lemmas. For $\varepsilon > 0$, let $\lambda_\varepsilon := 1 - \sqrt{\varepsilon}$.

Lemma 2.1. *There exists some* $\bar{\varepsilon} \in (0, 1)$ *such that*

$$v^{x,\lambda_\varepsilon}(y) < u(y) \quad \forall\ |x| < \varepsilon \leqslant \bar{\varepsilon},\ \ 0 < |y| \leqslant 1.$$

Proof. Let $\delta, \varepsilon_0 > 0$ be some small constants chosen later, we have, for $|x| < \varepsilon < \varepsilon_0$ and $0 < |y| < \delta$,

$$\begin{aligned} v^{x,\lambda_\varepsilon}(y) - u(y) &\leqslant \lambda_\varepsilon^{\frac{n-2}{2}}\, v(x + \lambda_\varepsilon y) - v(y) \\ &= \left[1 - \frac{n-2}{2}\sqrt{\varepsilon} + O(\varepsilon)\right][v(y) + O(|x - \sqrt{\varepsilon} y|] - v(y) \\ &= -\frac{n-2}{2}\sqrt{\varepsilon} v(y) + \sqrt{\varepsilon} O(\sqrt{\varepsilon} + \delta). \end{aligned}$$

Thus, for some small enough $\varepsilon_0, \delta > 0$,

$$v^{x,\lambda_\varepsilon}(y) < u(y) \quad \forall 0 < |x| < \varepsilon < \varepsilon_0, \ \forall 0 < |y| < \delta. \tag{53}$$

For the above ε_0 and δ,

$$v^{x,\lambda_\varepsilon}(y) = v(y) + O(\sqrt{\varepsilon}) \quad \forall \ |x| < \varepsilon < \varepsilon_0, \delta \leqslant |y| \leqslant 1.$$

Fix some small $\bar{\varepsilon} \in (0, \varepsilon_0)$ so that

$$O(\sqrt{\bar{\varepsilon}}) < \min_{\delta \leqslant |z| \leqslant 1} [u(z) - v(z)].$$

Then, for $|x| < \varepsilon < \bar{\varepsilon}$ and $\delta \leqslant |y| \leqslant 1$,

$$v^{x,\lambda_\varepsilon}(y) = v(y) + O(\sqrt{\varepsilon}) < v(y) + [u(y) - v(y)] = u(y). \tag{54}$$

Lemma 2.1 follows from (53) and (54). □

Lemma 2.2. *There exists* $\varepsilon_1 \in (0, 1)$ *such that*

$$v^{x,\lambda}(y) < u(y) \quad \forall 0 < \varepsilon < \varepsilon_1, 1 - \sqrt{\varepsilon} \leqslant \lambda \leqslant 1 + \sqrt{\varepsilon}, \ |x| < \varepsilon, |y| = 1. \tag{55}$$

Proof. Since $v^{0,1} = v$ and $\min_{|y|=1} [u(y) - v(y)] > 0$, (55) follows from the continuity of v. □

Lemma 2.3. *Under the contradiction hypothesis* (44), *there exists* $\varepsilon_2 \in (0, 1)$ *such that*

$$\sup_{0<|y| \leqslant 1} \left\{ v^{x,1+\frac{\sqrt{\varepsilon}}{2}}(y) - u(y) \right\} > 0 \quad \forall \ |x| < \varepsilon < \varepsilon_2.$$

Proof. For $|x| < \varepsilon < \varepsilon_2$, we have, using (44),

$$\begin{aligned}
&\limsup_{|y| \to 0} \left\{ v^{x,1+\frac{\sqrt{\varepsilon}}{2}}(y) - u(y) \right\} \\
&\quad = v^{x,1+\frac{\sqrt{\varepsilon}}{2}}(0) - v(0) = \left[1 + \frac{n-2}{2} \frac{\sqrt{\varepsilon}}{2} + O(\varepsilon) \right] [v(0) + O(\varepsilon)] - v(0) \\
&\quad = \frac{(n-2)\sqrt{\varepsilon}}{4} v(0) + O(\varepsilon) > 0,
\end{aligned}$$

provided that ε_2 is small. Lemma 2.3 is established. □

Now we complete the proof of Theorem 1.7. Let $\bar{\varepsilon}, \varepsilon_1$ and ε_2 be the constants in Lemmas 2.1–2.3, and let

$$\varepsilon := \frac{1}{8}\min\{\bar{\varepsilon}, \varepsilon_1, \varepsilon_2\}.$$

For $|x| < \varepsilon$, we know from Lemma 2.1 that

$$v^{x,1-\sqrt{\varepsilon}}(y) < u(y) \quad \forall 0 < |y| \leqslant 1.$$

Thus we can define, for $|x| < \varepsilon$,

$$\bar{\lambda}(x) := \sup\{\mu \geqslant 1 - \sqrt{\varepsilon} \mid v^{x,\lambda}(y) < u(y), \ \forall 0 < |y| \leqslant 1, \ \forall 1 - \sqrt{\varepsilon} \leqslant \lambda \leqslant \mu\}.$$

Clearly,

$$\bar{\lambda}(x) \geqslant 1 - \sqrt{\varepsilon} \quad \forall\ |x| < \varepsilon. \tag{56}$$

By Lemma 2.3,

$$\bar{\lambda}(x) \leqslant 1 + \frac{\sqrt{\varepsilon}}{2} \quad \forall\ |x| < \varepsilon. \tag{57}$$

By the definition of $\bar{\lambda}(x)$,

$$v^{x,\bar{\lambda}(x)}(y) \leqslant u(y) \quad \forall\ |x| < \varepsilon \quad \forall 0 < |y| \leqslant 1. \tag{58}$$

By Lemma 2.2, in view of (56) and (57),

$$v^{x,\bar{\lambda}(x)}(y) < u(y) \quad \forall\ |x| < \varepsilon \quad \forall\ |y| = 1. \tag{59}$$

By the invariance property of T and by (37),

$$T\left(v^{x,\bar{\lambda}(x)}, \nabla v^{x,\bar{\lambda}(x)}, \nabla^2 v^{x,\bar{\lambda}(x)}\right) \leqslant 0 \quad \text{in } B_{\frac{3}{2}} \quad \forall\ |x| < \varepsilon. \tag{60}$$

In view of (60), (37), (58) and (59), we apply the strong maximum principle to obtain

$$v^{x,\bar{\lambda}(x)}(y) < u(y) \quad \forall\ |x| < \varepsilon \quad \forall 0 < |y| \leqslant 1. \tag{61}$$

By (61) and the definition of $\bar{\lambda}(x)$,

$$\liminf_{y\to 0}\left[u(y)-v^{x,\bar{\lambda}(x)}(y)\right]=0 \quad \forall\ |x|<\varepsilon. \tag{62}$$

In view of (58), (62) and (36), we apply Lemma 1.2 as in [16] to obtain, for some constant vector $V\in\mathbb{R}^n$,

$$\nabla v^{x,\bar{\lambda}(x)}(0)=V \quad \forall\ |x|<\varepsilon. \tag{63}$$

Recall

$$v^{x,\bar{\lambda}(x)}(y)=\bar{\lambda}(x)^{\frac{n-2}{2}}v(x+\bar{\lambda}(x)y).$$

By (62),

$$\alpha:=\liminf_{y\to 0}\ u(y)=v^{x,\bar{\lambda}(x)}(0)=\bar{\lambda}(x)^{\frac{n-2}{2}}v(x) \quad \forall\ |x|<\varepsilon. \tag{64}$$

So, using (63) and (64),

$$V=\nabla v^{x,\bar{\lambda}(x)}(0)=\bar{\lambda}(x)^{\frac{n}{2}}\nabla v(x)=\alpha^{\frac{n}{n-2}}v(x)^{-\frac{n}{n-2}}\nabla v(x),$$

i.e.

$$\nabla\left\{\frac{n-2}{2}\alpha^{\frac{n}{n-2}}v(x)^{-\frac{2}{n-2}}+V\cdot x\right\}=0 \quad \forall\ |x|<\varepsilon.$$

This implies, for some constant vector $\widetilde{V}\in\mathbb{R}^n$,

$$v(x)\equiv v(0)[1-\widetilde{V}\cdot x]^{-\frac{n-2}{2}} \quad \forall\ |x|<\varepsilon.$$

It follows that

$$\Delta v(x)\geqslant 0 \quad \forall\ |x|<\varepsilon. \tag{65}$$

It is well known that (35), (36) and (65) imply (38), contradicting to (44). Theorem 1.7 is established. □

3. Proof of Theorems 1.8–1.11

Proof of Theorem 1.8. The proof is similar to that of Theorem 1.7. Suppose the contrary of (38), then (44) holds. We still use the notation $\lambda_\varepsilon := 1 - \sqrt{\varepsilon}$.

Instead of Lemma 2.1 we have

Lemma 3.1. *There exists some small $\bar{\varepsilon} > 0$ such that*

$$v_\varphi^{x,\lambda_\varepsilon}(y) < u(y) \quad \forall\ |x| < \varepsilon \leqslant \bar{\varepsilon},\ \ y \in \overline{\Omega} \setminus \{0\}.$$

Proof. Let $\delta, \varepsilon_0 > 0$ be some small constants chosen later, we have, for $|x| < \varepsilon < \varepsilon_0$ and $0 < |y| < \delta$,

$$\begin{aligned} v_\varphi^{x,\lambda_\varepsilon}(y) - u(y) &\leqslant \varphi(\lambda_\varepsilon) v(x + \lambda_\varepsilon y) - v(y) \\ &= [\varphi(1) - \varphi'(1)\sqrt{\varepsilon} + \circ(1)\sqrt{\varepsilon}][v(y) + O(|x - \sqrt{\varepsilon} y|] - v(y) \\ &= [-\varphi'(1) v(y) + \circ(1)]\sqrt{\varepsilon} + O(\delta\sqrt{\varepsilon}), \end{aligned}$$

where $\circ(1) \to 0$ as $\varepsilon \to 0$. Thus, for some small enough $\varepsilon_0, \delta > 0$,

$$v_\varphi^{x,\lambda_\varepsilon}(y) < u(y) \quad \forall 0 < |x| < \varepsilon < \varepsilon_0 \quad \forall 0 < |y| < \delta. \tag{66}$$

For the above ε_0 and δ,

$$v_\varphi^{x,\lambda_\varepsilon}(y) = \varphi(\lambda_\varepsilon) v(x + \lambda_\varepsilon y) = v(y) + O(\sqrt{\varepsilon}) \quad \forall\ |x| < \varepsilon < \varepsilon_0,\ y \in \overline{\Omega} \setminus B_\delta.$$

Fix some small $\bar{\varepsilon} \in (0, \varepsilon_0)$ so that

$$O(\sqrt{\bar{\varepsilon}}) < \inf_{\Omega \setminus B_\delta} [u(z) - v(z)].$$

Then, for $|x| < \varepsilon < \bar{\varepsilon}$ and $y \in \overline{\Omega} \setminus B_\delta$,

$$v_\varphi^{x,\lambda_\varepsilon}(y) = v(y) + O(\sqrt{\varepsilon}) < v(y) + [u(y) - v(y)] = u(y). \tag{67}$$

Lemma 3.1 follows from (66) and (67). □

Lemma 3.2. *Under the contradiction hypothesis* (44), *there exists* $\varepsilon_2 \in (0, 1)$ *such that*

$$\sup_{y \in \Omega \setminus \{0\}} \left\{ v_\varphi^{x, 1 + \frac{\sqrt{\varepsilon}}{2}}(y) - u(y) \right\} > 0 \quad \forall\ |x| < \varepsilon < \varepsilon_2.$$

Proof. For $|x| < \varepsilon < \varepsilon_2$, we have, using (44),

$$\limsup_{|y|\to 0} \left\{ v_\varphi^{x,1+\frac{\sqrt{\varepsilon}}{2}}(y) - u(y) \right\}$$
$$= v_\varphi^{x,1+\frac{\sqrt{\varepsilon}}{2}}(0) - v(0) = \left[\varphi(1) + \frac{\sqrt{\varepsilon}}{2}\varphi'(1) + \circ(\sqrt{\varepsilon}) \right] [v(0) + O(\varepsilon)] - v(0)$$
$$= \frac{\sqrt{\varepsilon}}{2}\varphi'(1)v(0) + \circ(\sqrt{\varepsilon}) > 0,$$

where we have used $\varphi'(1) > 0$ and ε_2 small. Lemma 3.2 is established. □

Now we complete the proof of Theorem 1.8. Let

$$0 < \varepsilon \leqslant \frac{1}{8}\{\bar{\varepsilon}, \varepsilon_1, \varepsilon_2, (\varepsilon_3)^2\} \tag{68}$$

such that

$$\frac{1}{2} \leqslant \varphi(\lambda) \leqslant 2, \quad \varphi'(\lambda) > \frac{1}{2}\varphi'(1) > 0 \quad \forall\ |\lambda - 1| \leqslant \sqrt{\varepsilon}. \tag{69}$$

For $|x| < \varepsilon$, we know from Lemma 3.1 that

$$v_\varphi^{x,1-\sqrt{\varepsilon}}(y) < u(y) \quad \forall y \in \overline{\Omega} \setminus \{0\}.$$

Thus we can define, for $|x| < \varepsilon$,

$$\bar{\lambda}(x) := \sup\{\mu \geqslant 1 - \sqrt{\varepsilon} \mid v_\varphi^{x,\lambda}(y) < u(y) \quad \forall y \in \overline{\Omega} \setminus \{0\} \quad \forall 1 - \sqrt{\varepsilon} \leqslant \lambda \leqslant \mu\}.$$

Clearly,

$$\bar{\lambda}(x) \geqslant 1 - \sqrt{\varepsilon} \quad \forall\ |x| < \varepsilon. \tag{70}$$

By Lemma 3.2,

$$\bar{\lambda}(x) \leqslant 1 + \frac{\sqrt{\varepsilon}}{2} \quad \forall\ |x| < \varepsilon. \tag{71}$$

By the definition of $\bar{\lambda}(x)$,

$$\inf_{\Omega \setminus \{0\}} \left[u - v_\varphi^{x,\bar{\lambda}(x)} \right] = 0 \quad \forall\ |x| < \varepsilon. \tag{72}$$

By (43), in view of (72),

$$\liminf_{y\to 0}\left[u(y)-v_{\varphi}^{x,\bar\lambda(x)}(y)\right]=0 \quad \forall\ |x|<\varepsilon. \tag{73}$$

In view of (72), (73) and (36), we apply Lemma 1.1 to obtain, for some constant vector $V\in\mathbb{R}^n$,

$$\nabla v_{\varphi}^{x,\bar\lambda(x)}(0)=V \quad \forall\ |x|<\varepsilon. \tag{74}$$

Recall

$$v_{\varphi}^{x,\bar\lambda(x)}(y)=\varphi\left(\bar\lambda(x)\right)v(x+\bar\lambda(x)y).$$

By (73) and (44),

$$\alpha := v(0)=\liminf_{y\to 0} u(y)=v_{\varphi}^{x,\bar\lambda(x)}(0)=\varphi(\bar\lambda(x))v(x) \quad \forall\ |x|<\varepsilon. \tag{75}$$

So, using (74) and (75),

$$V=\nabla v_{\varphi}^{x,\bar\lambda(x)}(0)=\bar\lambda(x)\varphi(\bar\lambda(x))\nabla v(x)=\frac{\alpha}{v(x)}\cdot\varphi^{-1}\left(\frac{\alpha}{v(x)}\right)\nabla v(x).$$

Let

$$\psi(s):=\int_{v(0)}^{s}\frac{\alpha}{t}\varphi^{-1}\left(\frac{\alpha}{t}\right)dt,$$

we have

$$V=\nabla_x\psi(v(x)) \quad \forall\ |x|<\varepsilon,$$

i.e.

$$\psi(v(x))+V\cdot x=0 \quad \forall\ |x|<\varepsilon. \tag{76}$$

Since φ is C^1 and $\varphi'>0$, we know that ψ is C^2,

$$\psi'(s)=\frac{\alpha}{s}\varphi^{-1}\left(\frac{\alpha}{s}\right)>0 \quad \text{and} \quad \psi''(s)=-\frac{\alpha}{s^2}\left\{\varphi^{-1}\left(\frac{\alpha}{s}\right)+\frac{\alpha}{s}(\varphi^{-1})'\left(\frac{\alpha}{s}\right)\right\}<0. \tag{77}$$

Note that we have used (69) in deriving the second inequality above.

Since $\psi \in C^2$ and $\psi' > 0$, we see from (76) that v is C^2 near the 0. Applying Δ to (76) leads to

$$\psi'(v(x))\Delta v(x) + \psi''(v(x))|\nabla v(x)|^2 = 0. \tag{78}$$

This implies that $\Delta v(x) \geqslant 0$ for x close to 0. This, together with (41) and (42), yields (38) which contradicts to the contradiction hypothesis (44). Impossible. Theorem 1.8 is established. $\square$

Now we give the

Proof of Theorem 1.9. The proof is similar to that of Theorem 1.8. We suppose that (44) holds, and we will derive a contradiction. We first give two lemmas whose proofs are almost identical to the proofs of Lemmas 3.1 and 3.2.

Lemma 3.3. *There exists some* $\bar{\varepsilon} > 0$ *such that*

$$v_\varphi^{x,1+\sqrt{\varepsilon}}(y) < u(y) \quad \forall\ |x| < \varepsilon \leqslant \bar{\varepsilon},\ \ y \in \overline{\Omega} \setminus \{0\}.$$

Lemma 3.4. *There exists* $\varepsilon_2 > 0$ *such that*

$$\sup_{y \in \Omega \setminus \{0\}} \left\{ v_\varphi^{x,1-\frac{\sqrt{\varepsilon}}{2}}(y) - u(y) \right\} > 0 \quad \forall\ |x| < \varepsilon < \varepsilon_2.$$

Let ε be defined by (68). For $|x| < \varepsilon$, we know from Lemma 3.3 that

$$v_\varphi^{x,1+\sqrt{\varepsilon}}(y) < u(y) \quad \forall y \in \overline{\Omega} \setminus \{0\}.$$

Thus we can define, for $|x| < \varepsilon$,

$$\bar{\lambda}(x) := \inf\{\mu \geqslant 1 + \sqrt{\varepsilon} \mid v_\varphi^{x,\lambda}(y) < u(y) \quad \forall y \in \Omega \setminus \{0\} \quad \forall\ \mu \leqslant \lambda \leqslant 1 + \sqrt{\varepsilon}\}.$$

Clearly,

$$\bar{\lambda}(x) \leqslant 1 + \sqrt{\varepsilon} \quad \forall\ |x| < \varepsilon.$$

By Lemma 3.4,

$$\bar{\lambda}(x) \geqslant 1 - \frac{\sqrt{\varepsilon}}{2} \quad \forall\ |x| < \varepsilon.$$

By the definition of $\bar{\lambda}(x)$,

$$v_\varphi^{x,\bar{\lambda}(x)}(y) \leqslant u(y) \quad \forall\ |x| < \varepsilon,\ \forall\ y \in \overline{\Omega} \setminus \{0\}.$$

The arguments between (72) and (76) yield (45). If $\varphi'(1) < -1$, then $\varphi^{-1}(1) + (\varphi^{-1})'(1) = 1 + \varphi'(1)^{-1} > 0$, and therefore, by (77), $\psi''(s) < 0$ for s close to $v(0)$. By (78), we still have $\Delta v \geqslant 0$ near the origin, and we obtain (38) as usual. Theorem 1.9 is established. □

Proof of Theorem 1.10. Suppose the contrary of (38), then (44) holds.

Lemma 3.5. *There exists some* $\bar{\varepsilon} > 0$ *such that*

$$\Phi(v, x, 1 - \sqrt{\varepsilon}; y) < u(y) \quad \forall\ |x| < \varepsilon \leqslant \bar{\varepsilon},\ y \in \overline{\Omega} \setminus \{0\}.$$

Proof. Use notation $\lambda_\varepsilon = 1 - \sqrt{\varepsilon}$. Let $\delta, \varepsilon_0 > 0$ be some small constants chosen later, we have, for $|x| < \varepsilon < \varepsilon_0$ and $0 < |y| < \delta$,

$$\begin{aligned}
&\Phi(v, x, \lambda_\varepsilon; y) - u(y) \\
&\quad \leqslant [\varphi(1) - \varphi'(1)\sqrt{\varepsilon}][v(y) + O(\delta\sqrt{\varepsilon})] - \psi'(1)\sqrt{\varepsilon} + \circ(\sqrt{\varepsilon}) - v(y) \\
&\quad = [-\varphi'(1)v(0) - \psi'(1)]\sqrt{\varepsilon} + \circ(\sqrt{\varepsilon}) + O(\delta\sqrt{\varepsilon}).
\end{aligned}$$

Thus, for some small enough $\varepsilon_0, \delta > 0$,

$$\Phi(v, x, \lambda_\varepsilon; y) < u(y) \quad \forall 0 < |x| < \varepsilon < \varepsilon_0 \quad \forall 0 < |y| < \delta.$$

For the above ε_0 and δ,

$$\Phi(v, x, \lambda_\varepsilon; y) = v(y) + O(\sqrt{\varepsilon}) \quad \forall\ |x| < \varepsilon < \varepsilon_0,\ y \in \overline{\Omega} \setminus B_\delta.$$

Lemma 3.5 follows from arguments in the proof of Lemma 3.1. □

Lemma 3.6. *Under the contradiction hypothesis* (44), *there exists* $\varepsilon_2 > 0$ *such that*

$$\sup_{y \in \Omega \setminus \{0\}} \left\{ \Phi\left(v, x, 1 + \frac{\sqrt{\varepsilon}}{2}; y\right) - u(y) \right\} > 0 \quad \forall\ |x| < \varepsilon < \varepsilon_2.$$

Proof. For $|x| < \varepsilon < \varepsilon_2$, we have, using (44),

$$\begin{aligned}
\limsup_{|y| \to 0} \left\{ \Phi\left(v, x, 1 + \frac{\sqrt{\varepsilon}}{2}; y\right) - u(y) \right\} &= \Phi\left(v, x, 1 + \frac{\sqrt{\varepsilon}}{2}; 0\right) - v(0) \\
&= [\varphi'(1)v(0) + \psi'(1)]\frac{\sqrt{\varepsilon}}{2} + \circ(\sqrt{\varepsilon}) > 0,
\end{aligned}$$

provided ε_2 is small. □

Now we complete the proof of Theorem 1.10. Let

$$0 < \varepsilon \leqslant \frac{1}{8}\min\{\bar{\varepsilon}, \varepsilon_1, \varepsilon_2, (\varepsilon_4)^2\}$$

such that (49) and (50) hold in $(1-2\varepsilon, 1+2\varepsilon)$.

For $|x| < \varepsilon$, we know from Lemma 3.5 that

$$\Phi(v, x, 1-\sqrt{\varepsilon}; y) < u(y) \quad \forall y \in \Omega \setminus \{0\}.$$

Thus we can define

$$\bar{\lambda}(x) := \sup\{\mu \geqslant 1-\sqrt{\varepsilon} \mid \Phi(v, x, \lambda; y) < u(y) \quad \forall y \in \Omega \setminus \{0\}, 1-\sqrt{\varepsilon} \leqslant \lambda \leqslant \mu\}.$$

It follows, using Lemma 3.6, that

$$|\bar{\lambda}(x) - 1| \leqslant \sqrt{\varepsilon} \quad \forall\ |x| < \varepsilon. \tag{79}$$

By the definition of $\bar{\lambda}(x)$,

$$\inf_{\Omega\setminus\{0\}} \Big[u - \Phi(v, x, \bar{\lambda}(x); \cdot)\Big] = 0. \tag{80}$$

By (51), in view of (80),

$$\liminf_{|y|\to 0} \Big[u(y) - \Phi(v, x, \bar{\lambda}(x); y)\Big] = 0 \quad \forall\ |x| < \varepsilon. \tag{81}$$

In view of (80), (81) and (42), we obtain, using Lemma 1.1, that for some constant vector $V \in \mathbb{R}^n$,

$$\nabla_y \Phi(v, x, \bar{\lambda}(x); y)\Big|_{y=0} = V \quad \forall\ |x| < \varepsilon,$$

i.e.

$$V = \varphi(\bar{\lambda}(x))\xi(\bar{\lambda}(x))\nabla v(x) \quad \forall\ |x| < \varepsilon. \tag{82}$$

We also know from (81) that

$$\liminf_{|y|\to 0} u(y) = \varphi(\bar{\lambda}(x))v(x) + \psi(\bar{\lambda}(x)) \quad \forall\ |x| < \varepsilon. \tag{83}$$

Note that (44) implies

$$\liminf_{|y|\to 0} u(y) = v(0) = \varphi(1)v(0) + \psi(1).$$

By (49) and (83), using the implicit function theorem, $\bar{\lambda}(x)$ depends C^1 on v, so $\bar{\lambda}$ is C^1, and $\bar{\lambda}(0) = 1$. By (82), we know that ∇v is C^1, so v is C^2. Applying *div* to (82) leads to

$$0 = (\varphi\xi)(\bar{\lambda}(x))\Delta v(x) + (\varphi\xi)'(\bar{\lambda}(x))\nabla\bar{\lambda}(x) \cdot \nabla v(x). \tag{84}$$

Applying ∇ to (83) gives

$$0 = \varphi(\bar{\lambda}(x))\nabla v(x) + \left[\varphi'(\bar{\lambda}(x))v(x) + \psi'(\bar{\lambda}(x))\right]\nabla\bar{\lambda}(x).$$

Taking inner product of the above with $\nabla\bar{\lambda}(x)$, we have

$$0 = \varphi(\bar{\lambda}(x))\nabla v(x) \cdot \bar{\lambda}(x) + \left[\varphi'(\bar{\lambda}(x))v(x) + \psi'(\bar{\lambda}(x))\right]|\nabla\bar{\lambda}(x)|^2. \tag{85}$$

This implies that $\nabla v(x) \cdot \bar{\lambda}(x) \leqslant 0$ and therefore, in view of (84), $\Delta v(x) \geqslant 0$ near the origin. This, together with $\Delta u(x) \leqslant 0$ and $u - v > 0$ for $0 < |x| < \varepsilon$, yields (38) violating the contradiction hypothesis (44). Impossible. Theorem 1.10 is established. $\square$

Proof of Theorem 1.11. We assume that (44) holds, otherwise we are done. For $\varepsilon > 0$, let $\lambda_\varepsilon := 1 - \sqrt{\varepsilon}$.

Lemma 3.7. *There exists some* $\bar{\varepsilon} > 0$ *such that*

$$\Phi(v, x, \lambda_\varepsilon; y) < u(y) \quad \forall\ |x| < \varepsilon \leqslant \bar{\varepsilon},\ \ y \in \Omega \setminus \{0\}.$$

Proof. Since v is C^1, we can find small $\bar{\varepsilon} > 0$ such that

$$\begin{aligned}\Phi(v, x, \lambda_\varepsilon; y) &= \left[1 - \varphi'(1)\sqrt{\varepsilon}\right]\left[v(y) - \xi'(1)\sqrt{\varepsilon}\nabla v(y) \cdot y\right] - \psi'(1)\sqrt{\varepsilon} + \circ(\sqrt{\varepsilon}) \\ &= v(y) - \left[\varphi'(1)v(y) + \xi'(1)\nabla v(y) \cdot y + \psi'(1)\right]\sqrt{\varepsilon} + \circ(\sqrt{\varepsilon}) \\ &< u(y) \quad \forall 0 < |x| < \varepsilon \leqslant \bar{\varepsilon},\ y \in \Omega \setminus \{0\}.\end{aligned}$$

Lemma 3.7 is established. $\square$

Lemma 3.8. *Under the contradiction hypothesis* (44), *there exists* $\varepsilon_2 > 0$ *such that*

$$\sup_{y\in\Omega\setminus\{0\}} \left\{\Phi\left(v, x, 1+\frac{\sqrt{\varepsilon}}{2}; y\right) - u(y)\right\} > 0 \quad \forall\ |x| < \varepsilon < \varepsilon_2.$$

Proof. The proof of Lemma 3.6 works here. □

Follow, with obvious modification, the proof of Theorem 1.10 from the line after the proof of Lemma 3.6 until "$\Delta v(x)\geqslant 0$ near the origin" towards the end. We know $\Delta u(x)\leqslant 0$ and $(u-v)(x)\geqslant 0$ for $0<|x|<\varepsilon$. By the mean value theorem, either $u-v>0$ on $B_\varepsilon\setminus\{0\}$ or $u-v\equiv 0$ on $B_\varepsilon\setminus\{0\}$. We know that $u-v>0$ on $B_\varepsilon\setminus\{0\}$ would imply (38) and would violate the hypothesis (44), so we must have $u-v\equiv 0$ on $B_\varepsilon\setminus\{0\}$. Thus $\Delta v(x)=0$ on B_ε. With this, we deduce from (84) and (85) that $|\nabla\bar\lambda|=0$ in B_ε, i.e., $\bar\lambda=\bar\lambda(0)=1$ in B_ε. Now we see from (83) that $v=v(0)$ in B_ε. Theorem 1.11 is established. □

4. Proof of Theorem 1.1

Proof of Theorem 1.1. Suppose the contrary of (11), then there exist some $\{x_j\}$ satisfying

$$|x_j|\to\infty \quad \text{as } j\to\infty,$$

$$|x_j|^{\frac{n-2}{2}}u(x_j)\to\infty \quad \text{as } j\to\infty. \tag{86}$$

Consider

$$v_j(x) := \left(\frac{|x_j|}{2} - |x-x_j|\right)^{\frac{n-2}{2}} u(x), \quad |x-x_j|\leqslant\frac{|x_j|}{2}.$$

Let $|\bar x_j - x_j| < \frac{|x_j|}{2}$ satisfy

$$v_j(\bar x_j) = \max_{|x-x_j|\leqslant\frac{|x_j|}{2}} v_j(x),$$

and let

$$2\sigma_j := \frac{|x_j|}{2} - |\bar x_j - x_j|.$$

Then

$$0 < 2\sigma_j \leqslant \frac{|x_j|}{2}. \tag{87}$$

We know

$$(2\sigma_j)^{\frac{n-2}{2}} u(\bar{x}_j) = v_j(\bar{x}_j) \geqslant v_j(x) \geqslant (\sigma_j)^{\frac{n-2}{2}} u(x) \quad \forall\ |x - \bar{x}_j| \leqslant \sigma_j.$$

Thus

$$u(\bar{x}_j) \geqslant 2^{\frac{2-n}{2}} u(x) \quad \forall\ |x - \bar{x}_j| \leqslant \sigma_j. \tag{88}$$

On the other hand, by (86),

$$(2\sigma_j)^{\frac{n-2}{2}} u(\bar{x}_j) = v_j(\bar{x}_j) \geqslant v_j(x_j) = \left(\frac{|x_j|}{2}\right)^{\frac{n-2}{2}} u(x_j) \to \infty. \tag{89}$$

Now, consider

$$w_j(y) := \frac{1}{u(\bar{x}_j)} u\left(\bar{x}_j + \frac{y}{u(\bar{x}_j)^{\frac{2}{n-2}}}\right), \quad y \in \Omega_j,$$

where

$$\Omega_j := \left\{ y \in \mathbb{R}^n \ \mid\ \bar{x}_j + \frac{y}{u(\bar{x}_j)^{\frac{2}{n-2}}} \ \in\ \mathbb{R}^n \setminus \overline{B}_1 \right\}.$$

By (88) and (89),

$$w_j(y) \leqslant 2^{\frac{n-2}{2}} \quad \forall\ |y| \leqslant R_j := \sigma_j u(\bar{x}_j)^{\frac{2}{n-2}} \to \infty. \tag{90}$$

Since $u(z) \geqslant \frac{1}{C} > 0$ for all $|z| = 1$, we have

$$w_j(y) \geqslant \frac{1}{C u(\bar{x}_j)} \quad \forall y \in \partial\Omega_j, \tag{91}$$

where

$$\partial\Omega_j = \left\{ y \in \mathbb{R}^n \ \mid\ \left| \bar{x}_j + \frac{y}{u(\bar{x}_j)^{\frac{2}{n-2}}} \right| = 1 \right\}.$$

For any $y \in \partial\Omega_j$,

$$\left| \frac{y}{u(\bar{x}_j)^{\frac{2}{n-2}}} \right| \geqslant |-\bar{x}_j| - \left| \bar{x}_j + \frac{y}{u(\bar{x}_j)^{\frac{2}{n-2}}} \right| = |\bar{x}_j| - 1 \geqslant \frac{1}{2}|\bar{x}_j|. \tag{92}$$

Thus, using (91) and (92),

$$\min_{y\in\partial\Omega_j} |y|^{n-2}w_j(y) \geqslant \min_{y\in\partial\Omega_j}\left\{\frac{1}{2}|\bar{x}_j|u(\bar{x}_j)^{\frac{2}{n-2}}\right\}^{n-2} w_j(y)$$

$$\geqslant \min_{y\in\partial\Omega_j}\left\{\left(\frac{1}{2}|\bar{x}_j|\right)^{n-2} u(\bar{x}_j)^2\right\}\frac{1}{Cu(\bar{x}_j)}$$

$$= \left(\frac{1}{2}\right)^{n-2}\frac{1}{C}|\bar{x}_j|^{n-2}u(\bar{x}_j). \tag{93}$$

Clearly

$$|\bar{x}_j| \geqslant \frac{1}{2}|x_j|. \tag{94}$$

We deduce from the above and (87) that

$$|\bar{x}_j| \geqslant 2\sigma_j. \tag{95}$$

Thus

$$|\bar{x}_j|^{n-2}u(\bar{x}_j) \geqslant \left(\frac{1}{2}|x_j|\right)^{\frac{n-2}{2}} |\bar{x}_j|^{\frac{n-2}{2}}u(\bar{x}_j) \geqslant \left(\frac{1}{2}|x_j|\right)^{\frac{n-2}{2}} (2\sigma_j)^{\frac{n-2}{2}}u(\bar{x}_j)$$

$$= |x_j|^{\frac{n-2}{2}}(R_j)^{\frac{n-2}{2}} \to \infty. \tag{96}$$

We deduce from (93) and (96) that

$$\lim_{j\to\infty}\min_{y\in\partial\Omega_j} |y|^{n-2}w_j(y) = \infty. \tag{97}$$

By (92) and (95),

$$|y| \geqslant \frac{1}{2}|\bar{x}_j|u(\bar{x}_j)^{\frac{2}{n-2}} \geqslant \sigma_j u(\bar{x}_j)^{\frac{2}{n-2}} = R_j \quad \forall y\in\partial\Omega_j. \tag{98}$$

By (9), (10) and the invariance of the equation, and by (90) and (97),

$$F(A^{w_j}) = 1, \quad A^{w_j}\in U, w_j > 0, \Delta w_j \leqslant 0 \text{ in } \Omega_j,$$

$$w_j(0) = 1,$$

$$w_j(y) \leqslant 2^{\frac{n-2}{2}} \quad \forall\ |y| \leqslant R_j,$$

$$\min_{y\in\partial\Omega_j}\left\{|y|^{n-2}w_j(y)\right\} \to \infty. \tag{99}$$

For all $|x| < \frac{R_j}{10}$, let

$$(w_j)_{x,\lambda}(y) := \left(\frac{\lambda}{|y-x|}\right)^{n-2} w_j\left(x + \frac{\lambda^2(y-x)}{|y-x|^2}\right)$$

and

$$\bar{\lambda}_j(x) = \sup\{\mu > 0 \mid (w_j)_{x,\lambda}(y) \leqslant w_j(y),\ \ \forall\ |y-x| \geqslant \lambda,\ y \in \overline{\Omega}_j, \forall 0 < \lambda < \mu\} > 0,$$

is well defined, see proof of lemma 2.1 in [18].

For $|x| < \frac{R_j}{10}$, $0 < \lambda \leqslant \frac{R_j}{4}$ and $y \in \partial\Omega_j$, we know, from (98) that

$$|y-x| \geqslant |y| - |x| \geqslant \frac{9}{10}|y| \geqslant \frac{9}{10} R_j,$$

and

$$\left| x + \frac{\lambda^2(y-x)}{|y-x|^2} \right| \leqslant |x| + \frac{\lambda^2}{|y-x|} \leqslant \frac{R_j}{10} + \left(\frac{R_j}{4}\right)^2 \left(\frac{10}{9R_j}\right) \leqslant \frac{R_j}{2}.$$

So

$$|y|^{n-2}(w_j)_{x,\lambda}(y) \leqslant 2^{\frac{n-2}{2}} \left(\frac{10}{9}\right)^{n-2} \lambda^{n-2} \quad \text{for } |x| < \frac{R_j}{10}, 0 < \lambda \leqslant \frac{R_j}{4}, y \in \partial\Omega_j.$$

Because of the last line in (99), there exist $r_j \to \infty$, $r_j \leqslant \frac{R_j}{4}$, such that

$$(w_j)_{x,\lambda} < w_j \quad \text{on } \partial\Omega_j \quad \text{for all } |\lambda| \leqslant r_j. \tag{100}$$

Namely, for all $|\lambda| \leqslant r_j$, no touching of $(w_j)_{x,\lambda}$ and w_j can occur on $\partial\Omega_j$.

Now we prove

$$\bar{\lambda}_j(x) \geqslant r_j. \tag{101}$$

Suppose the contrary, $\bar{\lambda}_j < r_j$, then, in view of (100), we can use the strong maximum principle and the Hopf Lemma as in the proof of lemma 2.1 in [13] to show

$$(w_j)_{x,\bar{\lambda}(x)} < w_j \quad \text{in } \Omega_j \setminus \overline{B_{\bar{\lambda}(x)}(x)}, \tag{102}$$

$$\frac{\partial}{\partial \nu}\left[w_j - (w_j)_{x,\bar{\lambda}(x)}\right]\Big|_{\partial B_{\bar{\lambda}(x)}(x)} > 0, \tag{103}$$

where $\frac{\partial}{\partial \nu}$ denotes differentiation in outer normal direction of $B_{\bar{\lambda}(x)}(x)$. Applying Theorem 1.6 to the Kelvin transformation of w_j and $(w_j)_{x,\bar{(x)}}$ which turn the singularity of w_j from ∞ to 0, we have

$$\liminf_{|y|\to\infty} |y|^{n-2}\left[w_j(y) - (w_j)_{x,\bar{\lambda}(x)}(y)\right] > 0. \tag{104}$$

As usual, (100), (102), (103) and (104) allow the moving sphere procedure to go beyond $\bar{\lambda}(x)$, contradicting to the definition of $\bar{\lambda}(x)$. We have established (101). Once we have (101), the argument in the proof of theorem 1.2 in [16] then leads to contradiction. Theorem 1.1 is established. □

5. Proof of Theorems 1.2 and 1.3

Proof of Theorem 1.2. By the positivity and the superharmonicity of u in $\mathbb{R}^n \setminus \{0\}$,

$$\liminf_{|y|\to 0} u(y) > 0, \quad \liminf_{|y|\to\infty} |y|^{n-2}u(y) > 0.$$

For all $|x| > 0$, we can prove as usual, see e.g. [18] or [13], that there exists $\lambda_0(x) \in (0, |x|)$ such that for all $0 < \lambda < \lambda_0(x)$,

$$u_{x,\lambda}(y) := \left(\frac{\lambda}{|y-x|}\right)^{n-2} u\left(x + \frac{\lambda^2(y-x)}{|y-x|^2}\right) \leqslant u(y) \quad \forall\ |y-x| \geqslant \lambda,\ |y| > 0.$$

Define

$$\bar{\lambda}(x) = \sup\{0 < \mu < |x| \mid u_{x,\lambda}(y) \leqslant u(y), \quad \forall\ |y-x| \geqslant \lambda,\ |y| \neq 0,\ 0 < \lambda < \mu\}.$$

We will prove

$$\bar{\lambda}(x) = |x| \quad \forall\ |x| > 0. \tag{105}$$

Suppose for some $|x| > 0$, $\bar{\lambda}(x) < |x|$, then we obtain, using the strong maximum principle and the Hopf Lemma as in section 2 of [13] and in view of (14),

$$u(y) - u_{x,\bar{\lambda}(x)}(y) > 0 \quad \forall\ |y-x| > \bar{\lambda}(x),\ |y| \neq 0, \tag{106}$$

and

$$\partial_\nu[u - u_{x,\bar{\lambda}(x)}]\Big|_{\partial B_{\bar{\lambda}(x)}} > 0, \tag{107}$$

where ∂_ν denotes the unit outer normal derivative.

By Theorem 1.6 with $v = u_{x,\lambda}$,

$$\liminf_{|y|\to 0} [u(y) - u_{x,\lambda}(y)] > 0. \tag{108}$$

Applying Theorem 1.6 with $u(y)$ replaced by $|y|^{2-n}u(\frac{y}{|y|^2})$ and $v(y)$ by $|y|^{2-n}u_{x,\lambda}(\frac{y}{|y|^2})$ leads to

$$\liminf_{|y|\to\infty} \left(|y|^{n-2}[u(y) - u_{x,\lambda}(y)]\right) > 0. \tag{109}$$

But this would violate the definition of $\bar{\lambda}(x)$, since (106), (107), (109) and (108) would allow the moving sphere procedure to continue beyond $\bar{\lambda}(x)$. Thus we have proved (105).

It follows that

$$u_{x,\lambda}(y) \leqslant u(y) \quad \forall 0 < \lambda < |x|, |y - x| \geqslant \lambda, y \neq 0. \tag{110}$$

For any unit vector $e \in \mathbb{R}^n$, for any $a > 0$, for any $y \in \mathbb{R}^n$ satisfying $(y - ae) \cdot e < 0$, and for any $R > a$, we have, by (110) with $x = Re$ and $\lambda = R - a$,

$$u(y) \geqslant u_{x,\lambda}(y) = \left(\frac{\lambda}{|y-x|}\right)^{n-2} u\left(x + \frac{\lambda^2(y-x)}{|y-x|^2}\right).$$

Sending R to infinity in the above leads to

$$u(y) \geqslant u(y - 2(y \cdot e - a)e).$$

This gives the radial symmetry of the u and

$$u(y) = u(y_1, y_2, \ldots, y_n) \geqslant u_a(y) := u(2a - y_1, y_2, \ldots, y_n) \quad \forall y_1 \leqslant a, a > 0.$$

Since $u = u_a$ on $y_1 = a$, we have $\frac{\partial(u-u_a)}{\partial y_1} \leqslant 0$ at $y = (a, 0, \ldots, 0)$, i.e. $u'(a) \leqslant 0$. Because u and u_a satisfy the same equation in $y_1 < a$, we have, by the Hopf Lemma, $\frac{\partial(u-u_a)}{\partial y_1} < 0$ at $y = (a, 0, \ldots, 0)$, i.e. $u'(a) < 0$. Theorem 1.2 is established. □

Proof of Theorem 1.3. As usual,

$$\liminf_{|y|\to 0} u(y) > 0$$

and, for all $0 < |x| < \frac{1}{2}$,

$$\bar{\lambda}(x) = \sup\{0 < \mu < |x| \mid u_{x,\lambda}(y) \leqslant u(y), \forall\ |y-x| \geqslant \lambda, 0 < |y| \leqslant 1, 0 < \lambda < \mu\} > 0$$

is well defined.

For $|y| = 1$, and $0 < \lambda < |x| < \frac{1}{2}$,

$$\left| \left\{ x + \frac{\lambda^2 (y-x)}{|y-x|^2} \right\} - x \right| \leqslant 4\lambda^2 \leqslant 4|x|^2.$$

So

$$\left| \left\{ x + \frac{\lambda^2 (y-x)}{|y-x|^2} \right\} - x \right| \leqslant \frac{|x|}{4} \quad \forall 0 < \lambda < |x| < \frac{1}{4}.$$

Thus, by Theorem 1.1′,

$$u\left(x + \frac{\lambda^2 (y-x)}{|y-x|^2} \right) \leqslant C|x|^{\frac{2-n}{2}},$$

and, for some $\varepsilon > 0$,

$$u_{x,\lambda}(y) \leqslant C\lambda^{n-2}|x|^{\frac{2-n}{2}} \leqslant C|x|^{\frac{n-2}{2}} < u(y) \quad \forall 0 < \lambda < |x| \leqslant \varepsilon, |y| = 1.$$

This means that no touching of $u_{x,\lambda}$ and u may occur on ∂B_1 in the moving sphere procedure. By the strong maximum principle as usual, the moving sphere procedure cannot stop due to touching of $u_{x,\lambda}$ and u in $B_1 \setminus \{0\}$. On the other hand, by Theorem 1.6, no touching of $u_{x,\lambda}$ and u at the origin may occur. Therefore, $\bar{\lambda}(x) = |x|$ for all $|x| \leqslant \varepsilon$. We have proved (15). Let $v(y) := |y|^{2-n} u(\frac{y}{|y|^2})$, (15) amounts to the following:

$$v(y) \leqslant v(y_\lambda) \quad \forall y \cdot e \geqslant \frac{1}{\varepsilon}, e \in \mathbb{R}^n, |e| = 1,$$

where $y_\lambda = y + 2(\lambda - x \cdot e)e$ is the reflection of y in the plane $x \cdot \tau = \lambda$. Now we can follow the proof of Corollary 6.2 in [1] to obtain (16). Theorem 1.3 is established. □

6. Proof of Theorem 1.4

Proof of Theorem 1.4. By (17) and the fact that $A^u \in U$, $\Delta u \leqslant 0$ in $B_2 \setminus \{0\}$. By Theorem 1.1′,

$$\sup_{0<|x|\leqslant 1} |x|^{\frac{n-2}{2}} u(x) < \infty. \tag{111}$$

Since $\Delta u \leqslant 0$ in $B_2 \setminus \{0\}$, we have

$$u(x) \geqslant \min_{\partial B_1} u > 0 \quad \forall 0 < |x| \leqslant 1. \tag{112}$$

Let

$$\xi(x) = \frac{n-2}{2} u(x)^{-\frac{2}{n-2}}, \quad 0 < |x| < 1,$$

we have, as in the proof of Lemma 6.5 in [13],

$$(D^2\xi) \geqslant \frac{1}{n-2} u^{\frac{2-2n}{n-2}} |\nabla u|^2 I, \quad B_1 \setminus \{0\}. \tag{113}$$

We know from (112) that

$$\xi(x) \leqslant C \quad \text{on } 0 < |x| \leqslant 1.$$

Here and throughout the rest of the proof of Theorem 1.4, $C > 1$ denotes some positive constant which may change its value from line to line. The constant C is allowed depend on u.

By the convexity of ξ—see (113),

$$|\nabla \xi(x)| \leqslant C \quad \forall 0 < |x| \leqslant \frac{1}{2}, \tag{114}$$

and ξ can be extended as a Lipschitz function in $B_{\frac{1}{2}}$.

Clearly $0 \leqslant \xi \leqslant C$ on $B_{\frac{1}{2}}$.

We divide into two cases:

Case 1: $\xi(0) > 0$,

Case 2: $\xi(0) = 0$.

In Case 1,

$$0 < \frac{1}{C} \leqslant \xi < C < \infty \quad \text{on } B_{\frac{1}{2}}. \tag{115}$$

By (115) and (114),

$$\frac{1}{C} \leqslant u \leqslant C \quad \text{and} \quad |\nabla u| \leqslant C \quad \text{on } B_{\frac{1}{2}}.$$

We arrive at the conclusion of Theorem 1.4.

We need to rule out the possibility of Case 2. In Case 2, we have, by (114),

$$0 < \xi(x) \leqslant C|x| \quad \forall 0 < |x| < \frac{1}{2},$$

i.e.

$$u(x) \geqslant \frac{1}{C}|x|^{-\frac{n-2}{2}} \quad \forall 0 < |x| < \frac{1}{2}.$$

This and (111) give

$$\frac{1}{C}|x|^{-\frac{n-2}{2}} \leqslant u(x) \leqslant C|x|^{-\frac{n-2}{2}} \quad \forall 0 < |x| < \frac{1}{2}. \tag{116}$$

Since

$$u = \left(\frac{2}{n-2}\right)^{-\frac{n-2}{2}} \xi^{-\frac{n-2}{2}},$$

we have, for some constant $a > 0$,

$$u^{\frac{2-2n}{n-2}}|\nabla u|^2 = (n-2)a\xi^{n-1}(\xi^{-\frac{n}{2}}|\nabla\xi|)^2 = (n-2)a\xi^{-1}|\nabla\xi|^2.$$

Thus, by (113),

$$(D^2\xi) \geqslant a\xi^{-1}|\nabla\xi|^2 I \quad \text{in } B_{\frac{1}{2}} \setminus \{0\}. \tag{117}$$

Fixing $e = (1, 0, \ldots, 0)$, and let

$$f(t) = \xi(te), \quad 0 < t < \frac{1}{2}.$$

Then

$$f'(t) = \xi_1(te), \quad f''(t) = \xi_{11}(te),$$

and, by (117),

$$f''(t) \geqslant a\xi(te)^{-1}|\nabla\xi(te)|^2 \geqslant a\xi(te)^{-1}|\xi_1(te)|^2 = af(t)^{-1}f'(t)^2, \quad 0 < t < \frac{1}{2}. \tag{118}$$

Claim. $f'(t) > 0$, $\forall 0 < t < \frac{1}{2}$.

Proof. For all $0 < t < \frac{1}{2}$, there exists some $0 < s < t$ such that

$$f'(s) = \frac{f(t) - f(0)}{t - 0} = \frac{f(t)}{t} > 0. \tag{119}$$

By (118), $f'' \geqslant 0$ on $(0, \frac{1}{2})$. So, since $s < t$, we have

$$f'(s) \leqslant f'(t). \tag{120}$$

The above claim follows from (120) and (119).

Because of the claim, we rewrite (118) as

$$\frac{f''}{f'} \geqslant \frac{af'}{f} \quad \text{on} \quad \left(0, \frac{1}{2}\right),$$

or

$$(\log f')' \geqslant (a \log f)' \quad \text{on} \quad \left(0, \frac{1}{2}\right).$$

For any $0 < s < t < \frac{1}{2}$, we deduce from the above that

$$\log f'(t) - \log f'(s) \geqslant a[\log f(t) - \log f(s)]. \tag{121}$$

By (116),

$$\frac{\tau}{C} \leqslant f(\tau) \leqslant C\tau \quad \forall 0 < \tau < \frac{1}{2}.$$

For all $0 < \tau < s$, there exists some $0 < \theta = \theta(\tau) < \tau$ such that

$$\frac{1}{C} \leqslant \frac{f(\tau)}{\tau} = \frac{f(\tau) - f(0)}{\tau - 0} = f'(\theta(\tau)). \tag{122}$$

Since $\theta(\tau) < \tau < s < \frac{1}{2}$, and since $f'' \geqslant 0$ on $(0, \frac{1}{2})$, we have

$$f'(\theta(\tau)) \leqslant f'(s). \tag{123}$$

Putting together (122) and (123), we have

$$\frac{1}{C} \leqslant f'(s) \quad \forall 0 < s < \frac{1}{2}. \tag{124}$$

By (121) and (124), for any $0 < s < t < \frac{1}{2}$, we have

$$\log f(t) - \log f(s) \leqslant \frac{1}{a}\{\log f'(t) - \log f'(s)\} \leqslant \frac{1}{a}\log f'(t) + \frac{1}{a}\log C.$$

Fixing $t = \frac{1}{4}$ in the above, we have

$$\log f(s) \geqslant -C \quad \forall 0 < s < \frac{1}{4},$$

i.e.

$$f(s) \geqslant e^{-C} \quad \forall 0 < s < \frac{1}{4}.$$

Sending s to 0 leads to

$$0 = f(0) \geqslant e^{-C},$$

impossible. We have ruled out the possibility of Case 2, and therefore have established Theorem 1.4. □

7. Proof of Theorem 1.5, Corollaries 1.2–1.4

Proof of Theorem 1.5. The proof makes use of arguments in the proof of theorem 2.7 in [19]. We mainly treat the possible singularity of ξ at the origin. We first assume in addition that $\xi \in C^2(B_2 \setminus \{0\})$ and $D^2\xi \in \overline{U}$ in $B_2 \setminus \{0\}$. In view of (4), we may assume that $D^2\xi \in U$ in $B_2 \setminus \{0\}$ since we may replace ξ by $\xi(x) + \varepsilon|x|^2$ for $\varepsilon > 0$ and then send ε to 0. Eq. (24) follows from subharmonicity of ξ in $B_2 \setminus \{0\}$ and the fact that $\sup_{B_2 \setminus \{0\}} \xi < \infty$. It is easy to see that we may assume without loss of generality that

$$1 \leqslant \xi \leqslant 2 \quad \text{in } B_2 \setminus \{0\}.$$

Fix some $C > 1$ such that for all $0 < |\bar{x}| < \frac{1}{4}$,

$$\xi(\bar{x}) + C\eta(x - \bar{x}) > \xi(x) \quad \forall\ |x - \bar{x}| = 1. \tag{125}$$

Here and throughout the proof we use C to denote some constant depending only on η which may vary from line to line. Consider, for $A \geqslant 0$,

$$\eta_A(x) := \xi(\bar{x}) + C\eta(x - \bar{x}) + A, \quad |x - \bar{x}| \leqslant 1.$$

Clearly

$$\eta_0(\bar{x}) = \xi(\bar{x}), \quad \eta_2(x) \geqslant \xi(x) \quad \forall\ |x - \bar{x}| \leqslant 1, x \neq 0.$$

It is easy to see that for some $0 \leqslant \overline{A} \leqslant 2$,

$$\eta_{\overline{A}}(x) \geqslant \xi(x), \quad |x - \bar{x}| \leqslant 1, x \neq 0, \tag{126}$$

and

$$\inf_{|x-\bar{x}| \leqslant 1, x \neq 0} [\eta_{\overline{A}}(x) - \xi(x)] = 0. \tag{127}$$

We must have

$$\eta_{\overline{A}}(x) > \xi(x), \quad |x - \bar{x}| \leqslant 1, x \neq 0, x \neq \bar{x}. \tag{128}$$

Indeed, by (125), $\eta_{\overline{A}}(x) > \xi(x)$ for all $|x - \bar{x}| = 1$. If for some $\hat{x} \neq 0, \hat{x} \neq \bar{x}$ and $|\hat{x} - \bar{x}| < 1$, $\eta_{\overline{A}}(\hat{x}) = \xi(\hat{x})$, then, in view of (126), $D^2\eta_{\overline{A}}(\hat{x}) \geqslant D^2\xi(\hat{x}) \in U$ which implies, in view of (4), $D^2\eta_{\overline{A}}(\hat{x}) \in U$, violating (19).

We know that

$$\eta_{\overline{A}}(x) > \xi(x), \quad 0 < |x| < |\bar{x}|.$$

Let $\varphi(\lambda) := \lambda$, $u = C - \xi$, $v = C - \eta_{\bar{A}}$, $a = \frac{1}{2}|\bar{x}|$, $\Omega = B_a$, where C is some constant satisfying $C > \eta_{\bar{A}}$ in $\overline{B}_a$.

Claim. There exists $\varepsilon_3 > 0$ such that (43) holds for the above.

Proof. Suppose the contrary, then for some $\varepsilon_3 > 0$ small and for some $|x| < \varepsilon_3$ and $|\lambda - 1| < \varepsilon_3$, we have

$$\min_{\partial B_a} [u - v_\varphi^{x,\lambda}] > \inf_{B_a \setminus \{0\}} [u - v_\varphi^{x,\lambda}] = 0$$

and

$$\liminf_{|y| \to 0} [u - v_\varphi^{x,\lambda}](y) > 0.$$

Then for some $0 < |\bar{y}| < a$,

$$[u - v_\varphi^{x,\lambda}](\bar{y}) = 0.$$

It follows that

$$D^2u(\bar{y}) \geqslant D^2 v_{\varphi}^{x,\lambda}(\bar{y})$$

i.e.

$$\lambda^2\varphi(\lambda)D^2\eta_{\overline{A}}(x+\lambda\bar{y}) \geqslant D^2\xi(\bar{y}) \in U.$$

It follows, using (4) and (5), $D^2\eta_{\overline{A}}(x+\lambda\bar{y}) \in U$, contradicting to (19). The Claim has been proved.

Now we apply Theorem 1.8 to obtain

$$\liminf_{|x|\to 0}\,[\eta_{\overline{A}}(x) - \xi(x)] > 0.$$

Thus, using also (127), (126) and (128), we have

$$\eta_{\overline{A}}(\bar{x}) = \xi(\bar{x}),$$

i.e. $\overline{A} = 0$. Then by (126),

$$\xi(x) \leqslant \xi(\bar{x}) + C\eta(x-\bar{x}) \quad \forall\ |x-\bar{x}| \leqslant 1,\ x \neq 0. \tag{129}$$

Since (129) holds for all $0 < |\bar{x}|, |x| < \frac{1}{4}$, switching the roles of $\bar{x}$ and x, we obtain

$$|\xi(x) - \xi(\bar{x})| \leqslant C[\eta(x-\bar{x}) + \eta(\bar{x}-x)], \quad \forall\ |x-\bar{x}| \leqslant 1 \quad \forall 0 < |\bar{x}|, |x| < \frac{1}{4}.$$

Now we complete the proof of Theorem 1.5: (24) still follows from (20) and the fact that ξ is bounded from above. Let $\{\xi_i\}$ be in $C^2(B_2 \setminus \{0\})$ such that (21)–(23) hold. We have proved (25) for $\{\xi_i\}$, with constant $C(\eta)$ independent of i. Sending i to ∞, we obtain (25) for ξ. Theorem 1.5 is established.

Proof of Corollary 1.3. Eq. (27) follows from the superharmonicity and the positivity of u in $B_2 \setminus \{0\}$. It is easy to see that (28) implies either (29) or (30). By a limit procedure, as in the proof of Theorem 1.5, we only need to establish (28) for the u_i. Now we drop the index i in the notation. Let $\xi = u^{-\frac{2}{n-2}}$, then $\xi \in L^\infty(B_2 \setminus \{0\})$,

$$\Delta\xi \geqslant 0, \quad D^2\xi \in \overline{U} \quad \text{in } B_2 \setminus \{0\}.$$

Estimate (28) follows from Theorem 1.5. □

Proof of Corollary 1.4. Let $U = U_k$ and $\eta(x) = |x|^\alpha$. Then it is known that η satisfies the properties in Corollary 1.3. Corollary 1.4 follows from Corollary 1.3. □

Proof of Corollary 1.2. Let $U = U_k$ and $\eta(x) = |x|^\alpha$. It follows from Theorem 1.5. □

8. Sharpness of Theorem 1.1

The two lemmas in this section give the sharpness of Theorem 1.1 as stated in Remark 1.3.

Lemma 8.1. *For $n \geqslant 3$, let*

$$u(x) = |x|^{\frac{2-n}{2}}, \quad x \in \mathbb{R}^n \setminus \{0\}.$$

Then

$$\lambda(A^u) \equiv \left\{-\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2}\right\} \quad on\ \mathbb{R}^n \setminus \{0\}. \tag{130}$$

Proof. We write $u(x)$ as $u(r)$ with $r = |x|$. We only need to verify (130) at $x = (r, 0, \ldots, 0), r > 0$. At the point, we have, as in the proof of theorem 1.6 in [16],

$$\nabla u(x) = (u'(r), 0, \ldots, 0), \quad \nabla^2 u(x) = diag\left(u''(r), \frac{u'(r)}{r}, \ldots, \frac{u'(r)}{r}\right),$$

and

$$A^u(x) = diag(\lambda_1^u(r), \lambda_2^u(r), \ldots, \lambda_n^u(r)),$$

where

$$\begin{cases} \lambda_1^u(r) = -\frac{2}{n-2} u^{-\frac{n+2}{n-2}} u'' + \frac{2(n-1)}{(n-2)^2} u^{-\frac{2n}{n-2}} (u')^2, \\ \lambda_2^u(r) = \cdots = \lambda_n^u(r) = -\frac{2}{n-2} u^{-\frac{n+2}{n-2}} \frac{u'}{r} - \frac{2}{(n-2)^2} u^{-\frac{2n}{n-2}} (u')^2. \end{cases}$$

With this we compute

$$u' = \frac{2-n}{2} r^{-\frac{n}{2}} = \frac{2-n}{2} u^{\frac{n}{n-2}}, \quad u'' = -\frac{n}{2} u^{\frac{2}{n-2}} u' = \frac{n(n-2)}{2} u^{\frac{n+2}{n-2}}$$

$$\lambda_1^u(r) = -\frac{n}{2} + \frac{n-1}{2} = -\frac{1}{2},$$

$$\lambda_2^u(r) = \cdots = \lambda_n(r) = -\frac{2}{n-2} u^{-\frac{n+2}{n-2}} \left(\frac{2-n}{2}\right) r^{-\frac{n+2}{2}} - \frac{2}{(n-2)^2} \left(\frac{n-2}{2}\right)^2 = \frac{1}{2}.$$

Lemma 8.1 is established. □

Lemma 8.2. *For* $\bar{\lambda} = (-1, 1, \ldots, 1) \in \mathbb{R}^n$, $n \geqslant 2$,

$$\begin{cases} \sigma_k(\bar{\lambda}) > 0, & \text{for } 1 \leqslant k < \frac{n}{2}, \\ \sigma_k(\bar{\lambda}) = 0, & \text{for } k = \frac{n}{2}, \\ \sigma_k(\bar{\lambda}) < 0, & \text{for } \frac{n}{2} < k \leqslant n. \end{cases} \tag{131}$$

It follows that $(-\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2})$ *belongs to* $\Gamma_k, \forall 1 \leqslant k < \frac{n}{2}$, *and* $(-\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2})$ *does not belong to* $\Gamma_k, \forall k \geqslant \frac{n}{2}$.

Proof of Lemma 8.2. For $n = 2$ or for $k \in \{1, n\}$, (131) is obvious. In the rest of the proof, we assume that $n \geqslant 3$ and $2 \leqslant k \leqslant n-1$. For $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n$,

$$\det(tI + diag(\lambda_1, \ldots, \lambda_n)) = t^n + \sigma_1(\lambda)t^{n-1} + \sigma_2(\lambda)t^{n-2} + \cdots + \sigma_{n-1}t + \sigma_n(\lambda).$$

Taking $\lambda = \bar{\lambda}$ and setting

$$f(t) := (t-1)(t+1)^{n-1} \equiv t^n + \sigma_1(\bar{\lambda})t^{n-1} + \cdots + \sigma_{n-1}(\bar{\lambda})t + \sigma_n(\bar{\lambda}).$$

Then

$$\frac{d^k}{dt^k} f(0) = k!\sigma_{n-k}(\bar{\lambda}), \quad 1 \leqslant k \leqslant n.$$

Rewriting

$$f(t) = (t-1)(t+1)^{n-1} = (t+1-2)(t+1)^{n-1} = (t+1)^n - 2(t+1)^{n-1}.$$

Since

$$\left.\frac{d^k}{dt^k}(t+1)^n\right|_{t=0} = n(n-1)\cdots(n-k+1),$$

$$\left.\frac{d^k}{dt^k}(t+1)^{n-1}\right|_{t=0} = (n-1)(n-2)\cdots(n-k+1)(n-k),$$

we have

$$\begin{aligned} \frac{d^k}{dt^k} f(0) &= n\{(n-1)(n-2)\cdots(n-k+1)\} \\ &\quad -2\{(n-1)(n-2)\cdots(n-k+1)\}(n-k) \\ &= \{(n-1)(n-2)\cdots(n-k+1)\}(2k-n). \end{aligned}$$

Since $(n-1)(n-2)\cdots(n-k+1) > 0$, Lemma 8.2 follows from the above. $\square$

9. Proof of Corollaries 1.5–1.7

We first give the

Proof of Corollary 1.7. Since

$$\Phi(v, x, \lambda; y) = \Phi(v, 0, \lambda; y + \xi(\lambda)^{-1}x),$$

it is easy to see from (52) and (35) that for some small $\varepsilon_4 > 0$,

$$T\left(\Phi(v, x, \lambda; \cdot), \nabla\Phi(v, x, \lambda; \cdot), \nabla^2\Phi(v, x, \lambda; \cdot)\right) \leqslant 0, \quad \text{in } B_{\varepsilon/2}, \ \forall\ |x| < \varepsilon_4, |\lambda - 1| < \varepsilon_4,$$

and

$$u > \Phi(v, x, \lambda; \cdot), \quad \text{on } \partial B_{\varepsilon/2}, \ \forall\ |x| < \varepsilon_4, |\lambda - 1| < \varepsilon_4.$$

Since

$$T(u, \nabla u, \nabla^2 u) \geqslant 0, \quad \text{in } B_{\varepsilon/2} \setminus \{0\},$$

and since the operator is elliptic, we can easily verify (51), with $\Omega = B_{\varepsilon/2}$, by a contradiction argument using the maximum principle on $B_{\varepsilon/2} \setminus B_\delta$ for some small $\delta > 0$. An application of Theorem 1.10 yields (38). □

Now we give

Proof of Corollary 1.6. We only need to verify that operators T satisfy the hypotheses of Corollary 1.7.

If T satisfies (i), we let $\varphi(\lambda) \equiv \xi(\lambda) \equiv 1$ and $\psi(\lambda) = \lambda - 1$. Then

$$\Phi(v, 0, \lambda; y) = v(y) + \lambda - 1,$$

and

$$T\left(\Phi(v, 0, \lambda; \cdot), \nabla\Phi(v, 0, \lambda; \cdot), \nabla^2\Phi(v, 0, \lambda; \cdot)\right)$$
$$= S\left(\nabla\Phi(v, 0, \lambda; \cdot), \nabla^2\Phi(v, 0, \lambda; \cdot)\right) = S(\nabla v, \nabla^2 v) = T(v, \nabla v, \nabla^2 v) \leqslant 0.$$

The hypotheses of Corollary 1.7 are satisfied.

If T satisfies (ii), we let $\varphi(\lambda) = \lambda$, $\xi(\lambda) \equiv 1$ and $\psi(\lambda) \equiv 0$. Then

$$\Phi(v, 0, \lambda; y) = \lambda v(y),$$

and therefore, for $|\lambda - 1| < \varepsilon$,

$$sign\ T\left(\Phi(v,0,\lambda;\cdot), \nabla\Phi(v,0,\lambda;\cdot), \nabla^2\Phi(v,0,\lambda;\cdot)\right) = sign\ T(v, \nabla v, \nabla^2 v).$$

The hypotheses of Corollary 1.7 are satisfied.

If T satisfies (iii), we let $\varphi(\lambda) = \lambda$, $\xi(\lambda) \equiv 1$ and $\psi(\lambda) = \lambda - 1$. Then

$$\Phi(v,0,\lambda;y) = \lambda v(y) + \lambda - 1$$

and

$$\begin{aligned} &T\left(\Phi(v,0,\lambda;\cdot), \nabla\Phi(v,0,\lambda;\cdot), \nabla^2\Phi(v,0,\lambda;\cdot)\right) \\ &= S\left(\frac{1}{\lambda v + \lambda}\cdot\lambda\nabla v, \frac{1}{\lambda v + \lambda}\cdot\lambda\nabla^2 v\right) = S\left(\frac{1}{v+1}\nabla v, \frac{1}{v+1}\nabla^2 v\right) \\ &= T(v, \nabla v, \nabla^2 v) \leqslant 0. \end{aligned}$$

The hypotheses of Corollary 1.7 are satisfied. □

Before proving Corollary 1.5, we give a lemma. For $\beta \in \mathbb{R}$, let

$$\varphi_\beta(\lambda) := \lambda^\beta, \quad v^\lambda_{\varphi_\beta}(y) := \varphi_\beta(\lambda)v(\lambda y) = \lambda^\beta v(\lambda y).$$

Lemma 9.1. *For $n \geqslant 1$ and $\beta \in \mathbb{R}\setminus\{0\}$, let $T \in C^0(\mathbb{R}_+ \times \mathbb{R}^n \times \mathcal{S}^{n\times n})$. Then*

$$T(v^\lambda_{\varphi_\beta}, \nabla v^\lambda_{\varphi_\beta}, \nabla^2 v^\lambda_{\varphi_\beta})(\cdot) \equiv T(v, \nabla v, \nabla^2 v)(\lambda\cdot) \quad in\ \mathbb{R}^n \tag{132}$$

holds for any positive function $v \in C^2(\mathbb{R}^n)$ and for any $\lambda > 0$ if and only if

$$T(t,p,M) \equiv S(t^{-\frac{1+\beta}{\beta}}p, t^{-\frac{2+\beta}{\beta}}M) \quad \forall (t,p,M) \in \mathbb{R}_+ \times \mathbb{R}^n \times \mathcal{S}^{n\times n} \tag{133}$$

for some $S \in C^0(\mathbb{R}^n \times \mathcal{S}^{n\times n})$.

Proof. Assuming (132), then for any positive C^2 function v and for all $\lambda > 0$, we know from (33) that

$$T\left(\lambda^\beta v(\lambda y), \lambda^{1+\beta}\nabla v(\lambda y), \lambda^{2+\beta}\nabla^2 v(\lambda y)\right) \equiv T\left(v(\lambda y), \nabla v(\lambda y), \nabla^2 v(\lambda y)\right),$$

i.e.

$$T(ts, t^{\frac{1+\beta}{\beta}}p, t^{\frac{2+\beta}{\beta}}M) = T(s,p,M) \quad \forall (t,s,p,M) \in \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^n \times \mathcal{S}^{n\times n}. \tag{134}$$

Taking $t = \frac{1}{s}$ in the above leads to (133), with $S(p,M) := T(1,p,M)$.

On the other hand, if (133) holds for some S, then

$$T(ts, t^{\frac{1+\beta}{\beta}} p, t^{\frac{2+\beta}{\beta}} M) = S\left((ts)^{-\frac{1+\beta}{\beta}}(t^{\frac{1+\beta}{\beta}} p), (ts)^{-\frac{2+\beta}{\beta}}(t^{\frac{2+\beta}{\beta}} M)\right)$$
$$= S\left(s^{-\frac{1+\beta}{\beta}} p, s^{-\frac{2+\beta}{\beta}} M\right) = T(s, p, M).$$

This implies (33). Lemma 9.1 is established. □

Now the

Proof of Corollary 1.5. Let

$$\varphi(\lambda) := \lambda^{\beta}, \quad v_{\varphi}^{x,\lambda}(y) := \varphi(\lambda)v(x + \lambda y), \quad \Omega := B_1.$$

For $\varepsilon_3 > 0$ small, we have, for any $|x| < \varepsilon_3$ and $|\lambda - 1| < \varepsilon_3$,

$$v_{\varphi}^{x,\lambda} < u \quad \text{on } \partial B_1$$

and, by Lemma 9.1,

$$T\left(v_{\varphi}^{x,\lambda}, \nabla v_{\varphi}^{x,\lambda}, \nabla^2 v_{\varphi}^{x,\lambda}\right)(\cdot) \equiv T\left(v, \nabla v, \nabla^2 v\right)(x + \lambda\cdot).$$

Thus (43) can be proved by a contradiction argument using the maximum principle since

$$T(u, \nabla u, \nabla^2 u) \geqslant 0 \geqslant T\left(v_{\varphi}^{x,\lambda}, \nabla v_{\varphi}^{x,\lambda}, \nabla^2 v_{\varphi}^{x,\lambda}\right) \quad \text{in } \Omega \setminus \{0\}.$$

If $0 < \beta < \infty$, then $\varphi'(1) > 0$, and we can apply Theorem 1.8 to obtain (38). If $-\infty < \beta < -1$, then $\varphi'(1) < -1$, and an application of Theorem 1.9 yields (38). If $\beta = -1$, then, by Theorem 1.9, either (38) holds or, for some $V \in \mathbb{R}^n$ and $\varepsilon > 0$,

$$v(x) - v(0) + V \cdot x \equiv 0, \quad |x| < \varepsilon.$$

The latter implies that $\Delta(u - v) \leqslant 0$ in $B_\varepsilon \setminus \{0\}$, and (38) follows as usual since $u - v > 0$ in $B_\varepsilon \setminus \{0\}$.

If $-1 < \beta < 0$, then by Theorem 1.9, either (38) holds or, for some $V \in \mathbb{R}^n$ and $\varepsilon > 0$,

$$\psi(v(x)) + V \cdot x = -\beta v(0)\left[\left(\frac{v(x)}{v(0)}\right)^{-\frac{1}{\beta}} - 1\right] + V \cdot x \equiv 0, \quad |x| < \varepsilon. \tag{135}$$

We deduce from (135) that, in B_ε,

$$v(x) \equiv v(0)\left[1+\frac{V\cdot x}{\beta v(0)}\right]^{-\beta}, \quad v^{-\frac{1+\beta}{\beta}}\nabla v \equiv -v(0)^{-\frac{1+\beta}{\beta}}V,$$

and

$$v^{-\frac{2+\beta}{\beta}}\nabla^2 v \equiv \left(1+\frac{1}{\beta}\right)v(0)^{-\frac{2+2\beta}{\beta}}V\otimes V.$$

It follow, using also (37) and (48), that

$$\begin{aligned}
0 &\geqslant T(v,\nabla v,\nabla^2 v) = S\left(v^{-\frac{1+\beta}{\beta}}\nabla v, v^{-\frac{2+\beta}{\beta}}\nabla^2 v\right)\\
&= S\left(v(0)^{-\frac{1+\beta}{\beta}}V, \left(1+\frac{1}{\beta}\right)v(0)^{-\frac{2+2\beta}{\beta}}V\otimes V\right)\\
&= S\left(v(0)^{-\frac{1+\beta}{\beta}}V, 0\right) + \int_0^1\left[\frac{d}{dt}S\left(v(0)^{-\frac{1+\beta}{\beta}}V, t\left(1+\frac{1}{\beta}\right)v(0)^{-\frac{2+2\beta}{\beta}}V\otimes V\right)\right]dt\\
&\geqslant \left(1+\frac{1}{\beta}\right)v(0)^{-\frac{2+2\beta}{\beta}}\int_0^1\left[\frac{\partial S}{\partial M_{ij}}\left(v(0)^{-\frac{1+\beta}{\beta}}V, t\left(1+\frac{1}{\beta}\right)v(0)^{-\frac{2+2\beta}{\beta}}V\otimes V\right)V_iV_j\right]dt.
\end{aligned}$$

Since $1+\frac{1}{\beta}<0$ and $\left(-\frac{\partial S}{\partial M_{ij}}\right)>0$, we see from above that $V=0$, i.e. $v\equiv v(0)$, in B_ε. We obtain (38) as usual. □

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