Lefschetz and Nielsen coincidence numbers on nilmanifolds and solvmanifolds, II

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Abstract

McCord (1991) claimed that Nielsen coincidence numbers and Lefschetz coincidence numbers are related by the inequality \( N(f, g) \geq |L(f, g)| \) for all maps \( f, g : S_1 \rightarrow S_2 \) between compact orientable solvmanifolds of the same dimension. It was further claimed that \( N(f, g) = |L(f, g)| \) when \( S_2 \) is a nilmanifold. A mistake in that paper has been discovered. In this paper, that mistake is partially repaired. A new proof of the equality \( N(f, g) = |L(f, g)| \) for nilmanifolds is given, and a variety of conditions for maps on orientable solvmanifolds are established which imply the inequality \( N(f, g) \geq |L(f, g)| \). However, it still remains open whether \( N(f, g) \geq |L(f, g)| \) for all maps between orientable solvmanifolds.

Keywords: Nilmanifold; Solvmanifold; Nielsen number; Coincidences

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1. Introduction

The purpose of this work is to acknowledge an error in [10], and to offer at least a partial correction for that error. Since [10] served as the foundation for a series of papers [5,11,12], as well as being used in the work of other authors [3,4,7,8], I will also examine the consequences of that error and its correction for these other works. Since the present work is a correction to [10], I will assume that the reader is familiar with that paper and its notation.

The goal of the original paper was to determine the relationship between the Nielsen coincidence number \( N(f, g) \) and the Lefschetz coincidence number \( L(f, g) \) for pairs of
maps \( f, g : S_1 \to S_2 \) between compact orientable solvmanifolds of the same dimension. The assertion was made that \( N(f, g) \geq |L(f, g)| \), with equality if \( S_2 \) is a nilmanifold. While I still believe the statement to be correct, the proof given in [10] contains a mistake.

The idea of the proof was to use the Mostow fibrations

\[
\begin{align*}
N_1 & \to S_1 & & \to T_1 \\
\downarrow & & f & \downarrow g & \quad \bar{f} & \downarrow \bar{g} \\
N_2 & \to S_2 & & \to T_2
\end{align*}
\]

(in which the spaces \( N_i \) are nilmanifolds and the spaces \( T_i \) are tori) and prove the result by induction on the dimension. Of course, for induction to work, the dimensions must match: we need \( \dim(T_1) = \dim(T_2) \). But, if \( S_1 \) and \( S_2 \) are different solvmanifolds, there is no guarantee a priori that \( \dim(T_1) = \dim(T_2) \). If \( \dim(T_1) < \dim(T_2) \), a simple transversality argument suffices to show that \( N(f, g) = L(f, g) = 0 \). If \( \dim(T_1) > \dim(T_2) \), the goal was to show that either a new fibration \( N'_1 \to S_1 \to T'_1 \) with \( \dim(T'_1) = \dim(T_2) \) could be constructed so that \( f \) and \( g \) were still fibration preserving; or that \( N(f, g) = L(f, g) = 0 \). Lemma 3.2 of [10] was a technical result that formed a part of that construction. Daciberg Goncalves has constructed a simple counter-example to that lemma. His example, however, does not show that the dichotomy is false, nor that the inequality \( N(f, g) \geq |L(f, g)| \) is false. Most importantly, it has no effect on the inductive argument used when \( \dim(T_1) = \dim(T_2) \).

It does, however, mean that when \( \dim(T_1) > \dim(T_2) \) and no \( T'_1 \) with the requisite properties can be shown to exist, we have no results on the relationship between \( N(f, g) \) and \( L(f, g) \). Consequently, either a new argument be found for the case \( \dim(T_1) > \dim(T_2) \), or that the scope of the theorem be restricted to avoid that case. The correction offered in this paper has some aspects of both. For nilmanifolds, a new argument is given which avoids fibrations and inductive arguments altogether, and so hold for all spaces and maps. In fact, as detailed in the next section, the proof constructed for maps on nilmanifolds applies to a larger class of functions — those maps on exponential solvmanifolds which are covered (up to homotopy) by group homomorphisms of the covering solvable Lie groups. And while we cannot guarantee that all maps between exponential solvmanifolds have this property, it is known that all maps between nilmanifolds do. Hence, we have

**Theorem 1.1.** If \( N_1, N_2 \) are nilmanifolds of the same dimension, then \( N(f, g) = |L(f, g)| \) for any \( f, g : N_1 \to N_2 \).

This recovers the results of [10] for nilmanifolds, and allows the results of [6,8,11] for infranilmanifolds to stand unchanged as well.

The general case of orientable solvmanifolds, on the other hand, is still open. While it is very probable that \( N(f, g) \geq |L(f, g)| \) for all pairs of maps between orientable solvmanifolds of the same dimension, the gap in the proof cannot be filled at this point.
It can, however, be narrowed. First, Theorem 1.1 can be combined with the lifting results of [11], to prove the following partial result.

**Theorem 1.2.** Suppose \( S_1, S_2 \) are compact infrasolvmanifolds of the same dimension, and \( f, g: S_1 \to S_2 \) are maps such that \( L(f, g) \) is defined (i.e., either \( S_1 \) and \( S_2 \) are orientable, or one of \( f, g \) is a homeomorphism) If there exist finite regular covers \( p_1: \tilde{S}_1 \to S_1 \) and \( p_2: \tilde{S}_2 \to S_2 \) such that

1. \( \tilde{S}_1, \tilde{S}_2 \) are solvmanifolds;
2. \( \text{im}(f \# p_1^\#), \text{im}(g \# p_1^\#) \subseteq \text{im}(p_2^\#) \in \pi_1(S_2) \);
3. \( \dim(H_1(\tilde{S}_1; \mathbb{Q})) = \dim(H_1(\tilde{S}_2; \mathbb{Q})) \)

then \( N(f, g) \geq |L(f, g)| \).

In particular, if the domain and range of the maps are the same, we will see that all of these hypotheses are satisfied.

**Corollary 1.3.** Given \( f, g: S \to S \) with \( S \) a compact infrasolvmanifold, \( N(f, g) \geq |L(f, g)| \) whenever \( L(f, g) \) is defined. In particular, \( N(f) \geq |L(f)| \) for every \( f: S_1 \to S_2 \).

These results are proved by lifting the problem to the finite covers, then using the Mostow fibrations of those spaces to decompose them. Since the fibers will be nilmanifolds, Theorem 1.1 can be applied without any further decomposition of the manifolds. Alternatively, if the finite covers are themselves nilmanifolds, no Mostow fibrations are required.

**Corollary 1.4.** If \( N_2 \) is a compact orientable infranilmanifold and \( S_1 \) is a compact orientable infrasolvmanifold of the same dimension, then \( N(f, g) \geq |L(f, g)| \) for every \( f, g: S_1 \to N_2 \).

This is essentially a generalization of the case \( \dim(T_1) = \dim(T_2) \) of the original construction in [10]. While stronger than the original result in some ways (we can lift to a finite cover, and do not need to further decompose the fibers of the Mostow fibrations), it still requires some dimension-matching hypotheses. Since it is not known if these hypotheses are satisfied in general, we are still left with a gap. However, this gap can be narrowed somewhat by a weaker version of the flawed Lemma 3.2 of [10]. Namely,

**Theorem 1.5.** Suppose \( S_1, S_2 \) are compact solvmanifolds of the same dimension. Given \( f, g: S_1 \to S_2 \), if \( f \# - g \# : H_1(S_1; \mathbb{Q}) \to H_1(S_2; \mathbb{Q}) \) is not surjective, then \( N(f, g) = 0 \) and \( L(f, g) = 0 \) when defined.

The paper is organized as follows. In the next section, it is shown that \( N(f, g) = |L(f, g)| \) when \( S_1 \) and \( S_2 \) are covered by exponential solvable Lie groups and \( f \) and \( g \) lift to group homomorphisms. As noted above, Theorem 1.1 follows immediately from this. In Section 3, Theorems 1.2 and 1.5 and their corollaries are proved. Finally, in the last section, these results will be put in context. The effect on results which were based on [10] will be surveyed, and the current status of the problem will be summarized.
2. The Anosov theorem for exponential solvmanifolds

Throughout this section, we will consider the following setting, and adopt the following notation: Take $G_1$ and $G_2$ to be exponential, simply connected solvable Lie groups of the same dimension, and let $\Gamma_1 \subset G_1$, $\Gamma_2 \subset G_2$ be uniform discrete subgroups. Then $S_i = G_i/\Gamma_i$ is a compact solvmanifold with universal cover $G_i$ and fundamental group $\pi_i$. Denote the covering maps by $p_i : G_i \to S_i$. Suppose $\phi_1, \phi_2 : G_1 \to G_2$ are homomorphisms such that $\phi_1(\Gamma_1) \subset \Gamma_2$. Then $\phi_1$ and $\phi_2$ cover maps $f_1, f_2 : S_1 \to S_2$. We will identify $\pi_i = \pi_i(S_i, p_i(e_i))$ with $\pi_i$, and $\phi_i|\Gamma_1 : \Gamma_1 \to \Gamma_2$ with $f_i : \pi_1 \to \pi_2$.

The goal of this section is to prove the following

**Theorem 2.1.** Suppose $G_1, G_2$ are simply connected exponential solvable Lie groups, with $\Gamma_1$ a cocompact discrete subgroup of $G_1$ and $S_i = G_i/\Gamma_i$. If $f_1, f_2 : S_1 \to S_2$ are covered by homomorphisms $\phi_1, \phi_2 : G_1 \to G_2$ with $\phi_1(\Gamma_1) \subset \Gamma_2$, then $N(f_1, f_2) = \text{adj}(f_1, f_2)$.

The basic outline of the proof follows the approach of Brooks and Wong in [1]. Because we assume the maps $f_1, f_2$ are covered by homomorphisms, coincidence classes are covered by subgroups of $G_1$. Exploiting this group structure, we show that coincidence classes are connected, and have index 0 if they have positive dimension; index $\pm 1$ if the class consists of a single point. To show that all essential classes have the same index, it is useful to consider the map $\phi : G_1 \to G_2$ defined by $\phi(g) = \phi_1(g^{-1})\phi_2(g)$. Note that $\phi$ is not $\Gamma$-equivariant, so it does not define a map from $S_1$ to $S_2$. Nevertheless, the introduction of $\phi$ allows us to translate the coincidence index of $\phi_1$ and $\phi_2$ into the root index of $\phi$, and from there derive a formula for the coincidence index in terms of the adjoint action of $G_2$ on its Lie algebra. Our knowledge of that action for exponential solvmanifolds allows us to conclude that all coincidence classes have the same index.\footnote{As the authors of [1] acknowledge that there is a (subsequently corrected) mistake in their original paper, modeling a new result after [1] deserves some explanation—especially if that new result is itself being used to correct a mistake. In [1], the mistake occurs when the authors claim in Theorem 2.3 that all root classes have the same index. This mistake they are later able to correct, and the rest of their approach is valid. In this paper, the reduction of the root index to the behavior of the adjoint action provides a way to avoid [1, Theorem 2.3].}

Recall that coincidence classes in $\text{Coin}(f_1, f_2)$ have the form $p_1(\text{Coin}(\tilde{f}_1, \tilde{f}_2))$, as $\tilde{f}_1, \tilde{f}_2 : G_1 \to G_2$ range over all possible lifts of $f_1$ and $f_2$. Since $\phi_1$ and $\phi_2$ cover $f_1$ and $f_2$, we obtain all coincidence classes by taking $p_1(\text{Coin}(\phi_1 \gamma, \phi_2))$, as $\gamma$ ranges over $\Gamma_2$. Let $C_\gamma$ denote $\text{Coin}(\phi_1 \gamma, \phi_2)$ and $c_\gamma$ denote $p_1(C_\gamma)$. We begin to develop the proof of Theorem 2.1 by examining the structure of these coincidence sets.

**Lemma 2.2.** $C_0 = \text{Coin}(\phi_1, \phi_2)$ is a connected subgroup of $G_1$, and $\text{Coin}(f_1 \# f_2 \#)$ is a uniform subgroup of $C_0$.

**Proof.** It is trivial to check that $C_0$ is a closed subgroup of $G_1$.

Suppose $g \in C_0$, and let $\alpha(t)$ be the one-parameter subgroup of $G_1$ through $g$. Then $\phi_1 \alpha$ and $\phi_2 \alpha$ are one-parameter subgroups of $G_2$ (unless $\phi_1(g) = \phi_2(g) = e_2$, in which case $\alpha(t)$ is a nilpotent group). Therefore, $\phi_1 \alpha$ and $\phi_2 \alpha$ are also one-parameter subgroups of $G_2$ with $\phi_1(g) = \phi_2(g) = e_2$. Hence, $C_0 = \text{Coin}(\phi_1, \phi_2)$ is a connected subgroup of $G_1$, and $\text{Coin}(f_1 \# f_2 \#)$ is a uniform subgroup of $C_0$.\n


case $\phi_1 \alpha(t) = \phi_2 \alpha(t) = e_2$ for all $t$, with $\phi_1 \alpha(1) = \phi_2 \alpha(1)$. But since the exponential map is one-to-one, $\phi_1 \alpha(1)$ has a unique one-parameter subgroup through it, and $\phi_1 \alpha(t) = \phi_2 \alpha(t)$ for all $t$. That is, $\alpha \subset C_0$, and $C_0$ is connected.

It remains to show that $\text{Coin}(f_1\#, f_2\#)$ is a cocompact subgroup of $C_0$. Since $c_0$ is a coincidence class in $\text{Coin}(f_1, f_2)$, it is compact. But $c_0 = \text{Coin}(f_1\#, f_2\#)$, so $C_0/\text{Coin}(f_1\#, f_2\#)$ is compact, as required. $\Box$

**Lemma 2.3.** If $C_\gamma$ is nonempty for some $\gamma \in \Gamma_2$, then $C_\gamma = \phi^{-1}(\gamma)$, and is homeomorphic to $C_0$.

**Proof.** The first point is obvious. For the second, choose $g_1 \in C_\gamma$. Then right multiplication $R_{g_1}$ maps $C_0$ homeomorphically to $C_\gamma$. $\Box$

Note that, since $R_{g_1}$ does not map $\Gamma_1$ to itself, the homeomorphism from $C_0$ to $C_\gamma$ does not project down to a homeomorphism from $c_0$ to $c_\gamma$. However, it is still true that:

**Corollary 2.4.** All coincidence classes in $\text{Coin}(f_1, f_2)$ are compact connected submanifolds, and all have the same dimension.

We now turn to the calculation of the coincidence index.

**Lemma 2.5.** The following are equivalent:

(i) $\text{Coin}(f_1\#, f_2\#) = e$;
(ii) $\dim(C_0) = 0$;
(iii) $\phi$ is injective;
(iv) $C_0 = e_1$;
(v) $\text{Ind}(f_1, f_2; c_0) = \pm 1$;
(vi) $\text{Ind}(f_1, f_2; c_0) \neq 0$.

**Proof.** From the previous results, it follows immediately that (i) through (iv) are equivalent. It suffices to show (iv) $\Rightarrow$ (v) and (vi) $\Rightarrow$ (ii).

Suppose $C_0 = e_1$. Since $p_1$ is a local homeomorphism, $\text{Ind}(f_1, f_2; p_1(e_1)) = \text{Ind}(\phi_1, \phi_2; e_1)$. To show that $|\text{Ind}(\phi_1, \phi_2; e_1)| = 1$, it suffices to show that $\phi_1 \times \phi_2 : G_1 \to G_2 \times G_2$ is transverse to the diagonal $\Delta(G_2)$ at $(e_2, e_2)$. But this will be the case as long as there is no $v \in g_1$ with $D\phi_1(v) = D\phi_2(v)$. This is equivalent to there being no one-parameter subgroup $\alpha$ in $G_1$ with $\phi_1 \alpha = \phi_2 \alpha$, or $\dim(C_0) = 0$.

If $C_0$ is a Lie subgroup of positive dimension, then $T_{e_1}C_0$ is the kernel of $D = D\phi_1 - D\phi_2 : T_{e_1}G_1 \to T_{e_2}G_2$, so $\text{rk}(D) < n$, and $D$ is not onto. Choose a one-parameter subgroup $\omega$ in $G_2$ such that $\omega(0) \notin \text{Im}(D)$. Let $\phi_{1t} = \omega(t) \cdot \phi_1$. Then, for $t$ small but positive, $\phi_{1t}$ and $\phi_2$ are coincidence-free.

Since $\phi_{1t}$ is $\Gamma$-equivariant, it covers $f_{1t} : S_1 \to S_2$. The coincidence class $c_0$ then continues to $p_1(\text{Coin}(\phi_{1t}, \phi_2))$, i.e. to the empty set, so $\text{Ind}(f_1, f_2; c_0) = 0$. $\Box$

These statements are clearly also valid for any other coincidence class. Since all coincidence classes have the same dimension, it follows that either all classes are essential,
with coincidence index $\pm 1$; or that all are inessential. To complete the proof of Theorem 2.1, it remains only to show that all essential coincidence classes have the same index. To do so, we first show that, for each coincidence class $c_\gamma$, there is a $g \in G_2$ such that $\text{Ind}(f_1, f_2; c_\gamma) = \det(\text{Ad}_g)\text{Ind}(f_1, f_2; c_0)$.

**Lemma 2.6.** If $C_0 = e_1$, then for every $\gamma \in \Gamma_2$, 

$$\text{Ind}(f_1, f_2; c_\gamma) = \det(\text{Ad}_{\phi_1(g_1)})\text{Ind}(f_1, f_2; c_0),$$

where $g_1 \in G_1$ is the unique element of $C_\gamma$.

**Proof.** As noted above, $\text{Ind}(f_1, f_2; c_\gamma) = \text{Ind}(\phi_1 \gamma, \phi_2; C_\gamma)$. The map $\phi$ converts coincidences of $\phi_1$ and $\phi_2$ to roots, and this process is index-preserving: $\text{Ind}(\phi_1 \gamma, \phi_2; C_\gamma) = \text{Ind}(\phi; C_\gamma)$, where $\text{Ind}(\phi; C_\gamma)$ is the index of $\gamma$ as a root of $\phi$ (cf. the proof of Lemma 3.1 in [11]). Thus it suffices to show that $\text{Ind}(\phi; C_0) = \text{Ind}(\phi; C_\gamma)$.

To do so, define $\Phi_t : G_1 \to G_2$ by $\Phi_t(g) = \gamma^{-1} \phi(g \alpha(t))$, where $\alpha$ is the one-parameter subgroup through $g_1$. Note that $\Phi_0(g) = \gamma^{-1} \phi(g)$ and $\Phi_1(g) = \phi_2(g_1^{-1}) \phi(g) \phi_2(g_1)$. Further, $\Phi_t^{-1}(e_2) = \phi^{-1}(\phi_1(\alpha(t)) \gamma \phi_2(\alpha(-t)))$. In particular, $\Phi_0^{-1}(e_2) = \phi^{-1}(e_2) = C_0$. To apply the homotopy invariance of the index and conclude that $\text{Ind}(\Phi_0; C_\gamma) = \text{Ind}(\Phi_1; C_0)$, we need to know that the set 

$$\bigcup_{0 \leq t \leq 1} \Phi_t^{-1}(e_2)$$

is compact. This set can be described as the preimage under $\phi$ of the path 

$$\omega(t) = \phi_1(\alpha(t)) \gamma \phi_2(\alpha(-t))$$

from $e_2$ to $\gamma$. Since $\phi$ is injective, its preimage is compact.

Now, $\Phi_0$ simply translates the root problem from $\phi(g) = \gamma$ to $\gamma^{-1} \phi(g) = e_2$, so $\text{Ind}(\phi; C_\gamma) = \text{Ind}(\Phi_0; C_\gamma)$. It remains then to show that 

$$\text{Ind}(\Phi_1; C_0) = \det(\text{Ad}_{\phi_1(g_1)})\text{Ind}(\phi; C_0).$$

If $\kappa : G_2 \to G_2$ denotes conjugation by $\phi_2(g_1)$, then $\Phi_1 = \kappa \circ \phi$, and 

$$\begin{align*}
\text{Ind}(\Phi_1; C_0) &= \text{Ind}(\kappa \circ \phi; e_1) \\
&= \text{Ind}(\kappa; c_2)\text{Ind}(\phi; e_1) \\
&= \det(D\kappa_{e_2})\text{Ind}(\phi; e_1) \\
&= \det(\text{Ad}_{\phi_2(g_1)})\text{Ind}(\phi; e_1).
\end{align*}$$

At this point we have established the following dichotomy for functions satisfying the hypotheses of Theorem 2.1: either all coincidence classes have index 0 (in which case all are inessential, and $N(f_1, f_2) = L(f_1, f_2) = 0$); or all have index $\pm 1$, with the change in sign controlled by $\det(\text{Ad}_{g_2})$ for some $g_2 \in G_2$. However, since $G_2$ is exponential, $g_2 = \exp(X)$ for some $X \in L(G_2)$, and
\[ \det(Ad_{g_2}) = \det(Ad_{\exp X}) \]
\[ = \det(\exp(\text{ad}X)) \]
\[ = \exp(\text{tr}(\text{ad}X)). \]

Since \( \text{ad}(X) \) is a real matrix, \( \text{tr}(\text{ad}X) \) is a real number and \( \det(Ad_{g_2}) \) is positive. Thus all \( \text{Ind}(f_1, f_2; c_\gamma) \) have the same sign, so either all equal +1 or all equal −1. In either case, \( N(f_1, f_2) = |\mathcal{L}(f_1, f_2)|. \)

This completes the proof of Theorem 2.1. We now want to derive Theorem 1.1 from it. Since nilmanifolds are clearly exponential solvmanifolds, it suffices to show that

**Lemma 2.7.** If \( N_1, N_2 \) are nilmanifolds with universal covers \( G_1, G_2 \), then (up to homotopy) every \( f : N_1 \to N_2 \) is covered by a homomorphism \( \phi : G_1 \to G_2. \)

**Proof.** The proof is the same as that used in [2] for the special case of \( N_1 = N_2. \) Consider \( f_* : \Gamma_1 \to \Gamma_2 \) and embed \( \Gamma_i \) in \( G_i. \) Then there is an injection
\[ \text{id} \times f_* : \Gamma_1 \to \Gamma_1 \times \Gamma_2 \hookrightarrow G_1 \times G_2. \]

There is a Lie subgroup \( G \subset G_1 \times G_2 \) such that \( \text{im}(\text{id} \times f_*) \) is a uniform subgroup of \( G. \) Since every isomorphism of uniform subgroups extends to an isomorphism of nilpotent Lie groups, \( \text{id} \times f_* \) extends to an isomorphism \( \Phi : G_1 \to G. \) The required homomorphism \( \phi \) is then obtained by projection onto \( G_2 : \phi = \pi_2 \circ \Phi. \)

3. Fibrations and finite covers

In this section, we prove the remaining results from Section 1. In the previous section, we were able to work directly with the manifolds \( S_i \) and their universal covers \( G_i. \) The arguments presented did not require the Mostow decompositions, nor any other decompositions of the spaces. However, the utility of the argument was limited by the necessity of assuming that the Lie groups were exponential solvable Lie groups, and that the maps involved were group homomorphisms.

We now consider what can be said in the absence of those hypotheses. We will combine two types of structures: fibrations and finite covering spaces. Both have proven to be useful in analyzing the behavior of Nielsen numbers, and both are applicable to the problem studied in [10]. There, the calculation of Nielsen numbers on solvmanifolds was reduced to an examination of the situation
\[
\begin{align*}
N_1 & \to S_1 & \to & T_1 \\
\downarrow f & \downarrow g & \overline{f} & \downarrow \overline{g} \\
N_2 & \to S_2 & \to & T_2
\end{align*}
\]
with \( \dim(T_1) = \dim(T_2) \). \( \bar{f} \) and \( \bar{g} \) can be deformed to have exactly \( N(\bar{f}, \bar{g}) = |L(\bar{f}, \bar{g})| \) coincidence points. If these are labeled \( b_1, \ldots, b_N \), denote the fiber over \( b_i \) by \( N_{1i} \), the fiber over \( \bar{f}(b_i) \) by \( N_{2i} \), and the maps between these fibers by \( f_i, g_i : N_{1i} \rightarrow N_{2i} \). Then

\[
N(f, g) = \sum_{i=1}^{N} N(f_i, g_i),
\]

and if \( L(f, g) \) is defined (i.e., if \( S_1 \) and \( S_2 \) orientable, or if one of \( f \) or \( g \) is a homeomorphism), then

\[
L(f, g) = \text{Ind}(\bar{f}, \bar{g}; b_1) \sum_{i=1}^{N} L(f_i, g_i).
\]

Since \( \text{Ind}(\bar{f}, \bar{g}; b_1) = \pm 1 \) and \( N(f_i, g_i) = |L(f_i, g_i)| \) by Theorem 1.1, it follows that \( N(f, g) \geq |L(f, g)| \).

To extend this result, we employ the lifting techniques used in [11]. If \( S_1, S_2 \) are infrasolvmanifolds of the same dimension, then there are regular finite covers \( p_i : S_i \rightarrow S_i \) with the spaces \( S_i \) orientable solvmanifolds. Suppose \( f, g : S_1 \rightarrow S_2 \) have lifts \( \bar{f}, \bar{g} : \tilde{S}_1 \rightarrow \tilde{S}_2 \). We can use the lifting diagram

\[
\begin{array}{ccc}
\tilde{S}_1 & \xrightarrow{\bar{f}, \bar{g}} & \tilde{S}_2 \\
\downarrow p_1 & & \downarrow p_2 \\
S_1 & \xrightarrow{f, g} & S_2
\end{array}
\]

to relate the Nielsen and Lefschetz numbers of \( f \) and \( g \) to those of \( \bar{f} \) and \( \bar{g} \). To do so, let \( D_i \) be the covering group of \( p_i \), and \( d_i = |D_i| \). Then

\[
N(f, g) \geq \frac{1}{d_1} \sum_{\beta \in D_2} N(\bar{f}, \beta \bar{g})
\]

and, when defined,

\[
L(f, g) = \frac{1}{d_1} \sum_{\beta \in D_2} L(\bar{f}, \beta \bar{g})
\]

So, if it is known that \( N(\bar{f}, \beta \bar{g}) \geq |L(\bar{f}, \beta \bar{g})| \) for all maps from \( \tilde{S}_1 \) to \( \tilde{S}_2 \), then

\[
N(f, g) \geq \frac{1}{d_1} \sum_{\beta \in D_2} |L(\bar{f}, \beta \bar{g})|,
\]

and \( N(f, g) \geq |L(f, g)| \) if \( L(f, g) \) is defined.

This is essentially the proof of Theorem 1.2. Hypothesis (2) is exactly the lifting condition required to form \( \bar{f} \) and \( \bar{g} \). Once we have lifted the problem to \( \tilde{S}_1 \) and \( \tilde{S}_2 \), we need to know that the Mostow fibrations of \( \tilde{S}_1 \) and \( \tilde{S}_2 \) have tori of the same dimension as their bases. Hypothesis (3) guarantees this, since \( \pi_1(T_i) \) is by construction a finite quotient of \( \pi_1(S_i) \sim H_1(S_i; \mathbb{Z}) \). So \( \tilde{S}_1 \) and \( \tilde{S}_2 \) fit the requirements of the "old" construction, and \( N(\bar{f}, \beta \bar{g}) \geq |L(\bar{f}, \beta \bar{g})| \) for all \( f \) and \( g \). This completes the proof of Theorem 1.2.
Now, to derive Corollary 1.3 from Theorem 1.2, we need to show that the appropriate solvmanifolds $S_1, S_2$ can always be found when $S_1 = S_2$. By definition, an infrasolvmanifold has a finite regular cover by a solvmanifold. Let $\mathcal{C}(S)$ be the set of such finite covers of $S$. For every $\tilde{S}_2 \in \mathcal{C}(S)$, the covering map $p_2: \tilde{S}_2 \to S$ defines a subgroup $\Gamma(\tilde{S}_2) \subset \pi_1(S)$ by

$$\Gamma(\tilde{S}_2) = f^{-1}_#(\text{im}(p_2#)) \cap g^{-1}_#(\text{im}(p_2#)) \cap \text{im}(p_2#).$$

$\Gamma(\tilde{S}_2)$ is a subgroup of finite index in $\pi_1(S)$ and is normal in $\pi_1(S)$, so $\Gamma(\tilde{S}_2) = \pi_1(\tilde{S}_1)$ for some $\tilde{S}_1 \in \mathcal{C}(S)$.

That is, for every $\tilde{S}_2 \in \mathcal{C}(S)$, there is an $\tilde{S}_1 \in \mathcal{C}(S)$ such that $f$ and $g$ satisfy the lifting condition, and $\tilde{S}_1$ is a finite cover of $\tilde{S}_2$. Because $\tilde{S}_1$ is a finite cover of $\tilde{S}_2$, $\dim H_1(\tilde{S}_1; \mathbb{Q}) \geq \dim H_1(\tilde{S}_2; \mathbb{Q})$. Further, since $\dim H_1(\tilde{S}; \mathbb{Q}) \leq \dim(S)$ for all $\tilde{S} \in \mathcal{C}(S)$, there exists an $\tilde{S}_2$ such that $H_1(\tilde{S}_2; \mathbb{Q})$ is maximal. Clearly, for this choice of $\tilde{S}_2$, $\dim H_1(\tilde{S}_1; \mathbb{Q}) = \dim H_1(\tilde{S}_2; \mathbb{Q})$, and we have the solvmanifolds required to apply Theorem 1.2.

Corollary 1.4 follows from Theorems 1.1 and 1.2 and Corollary 3.3 of [10]. This last result (which depends on the valid half of Lemma 3.2) asserts that if $\tilde{N}_2$ is a nilmanifold and $\tilde{S}_1$ is not, then $N(\tilde{f}, \tilde{g}) = L(\tilde{f}, \tilde{g}) = 0$ for all $\tilde{f}$ and $\tilde{g}$.

Finally, we turn to the proof of Theorem 1.5. The basic idea here is very simple. Given Mostow fibrations

$$\begin{align*}
N_1 & \to S_1 \to T_1 \\
N_2 & \to S_2 \to T_2
\end{align*}$$

if $\tilde{f}$ and $\tilde{g}$ can be deformed to be coincidence-free, then $f$ and $g$ must likewise be coincidence free, so $N(f, g) = 0$ and $L(f, g) = 0$ when defined. If $\dim H_1(S_1; \mathbb{Q}) < \dim H_1(S_2; \mathbb{Q})$, then $\dim(T_1) < \dim(T_2)$, and a simple dimension count shows that, if $\tilde{f}$ and $\tilde{g}$ are deformed to be transverse, they will be coincidence-free. On the other hand, if $\dim H_1(S_1; \mathbb{Q}) \geq \dim H_1(S_2; \mathbb{Q})$, $\tilde{f}, \tilde{g}: T_1 \to T_2$ can be taken to be homomorphisms. Then $(\tilde{f} - \tilde{g})(T_1)$ is a subgroup of $T_2$ whose dimension equals $\text{rk}(f_1 - g_1)$. If $(f_1 - g_1)$ is not surjective, then there is a $v \in T_2 \setminus (\tilde{f} - \tilde{g})(T_1)$. Let $F: T_1 \times I \to T_2$ be defined by $F(x, t) = \tilde{f} + tv$. Then $\tilde{F}_1$ and $\tilde{g}$ are coincidence-free. $\tilde{F}$ lifts to a homotopy $F: S_1 \to S_2$ based at $f$, with $F_1$ and $g$ coincidence-free.

4. Summary

We conclude by taking stock of the situation, to clarify how this error and its partial correction affect the published results on Nielsen theory for infrasolvmanifolds, and to clarify what questions remain open.

We have been concerned with questions of the form:
If $f, g : S_1 \to S_2$ are maps between compact solvmanifolds of the same dimension, when is $N(f, g)$ computed (i.e., equal to) or estimated by (i.e., greater than) a formula expressed in terms of Lefschetz numbers? In particular, if $L(f, g)$ is defined, when is $N(f, g) \geq |L(f, g)|$, and when is $N(f, g) = |L(f, g)|$?

To clarify the status of this question, let us first remove the complication introduced by the fact that $L(f, g)$ is not always defined. The obstruction, of course, is that $L(f, g)$ is not defined for all maps $f$ and $g$ unless both $S_1$ and $S_2$ are orientable. But, if one or both of the spaces in nonorientable, we can take an orientable double cover, and lift the problem to there. As described in the previous section, if we can relate the Nielsen and Lefschetz numbers of the lifts to each other, then we can analyze the original problem.

We will assume then that $L(f, g)$ is defined, and restrict our attention to the relation between $N(f, g)$ and $L(f, g)$.

At this point, we have established that $N(f, g) = L(f, g)$ if

- $S_2$ is a nilmanifold;
- $S_1 = S_2$ is an exponential solvmanifold and $g = \text{id}$ [5];
- $S_1 = S_2$, $g = \text{id}$ and $f$ is homotopically periodic [6,7,12];

and that $N(f, g) \geq L(f, g)$ if

- $S_1 = S_2$ (in particular, if $g = \text{id}$);
- $S_2$ is an infranilmanifold.

These results emphasize the conditions on the spaces, more than conditions on the maps. Theorems 1.2, 1.5 and 2.1 all give more specialized hypotheses on the structure of $f$ and $g$ which guarantee that $N(f, g) \geq |L(f, g)|$. In particular, when $S_1 = S_2$, all of the results claimed in [10] remain valid. With the exception of [11], all of the published results (e.g., [4,5,7,8,12]) based on [10] are set in in the case $S_1 = S_2$, so those results are not compromised by the remaining gap in the theory.

The general question remains open, but on the basis of the existing results, it is quite reasonable to conjecture that

- $N(f, g) \geq |L(f, g)|$ for all maps on orientable infrasolvmanifolds;
- $N(f, g) = |L(f, g)|$ for all maps on exponential solvmanifolds.

Again, it is worth noting that the inequality will be established for all maps on infrasolvmanifolds once it is established for all maps on solvmanifolds. Indeed, since every infrasolvmanifold is finitely covered by a special solvmanifold (i.e., a solvmanifold $S$ of the form $G/\Gamma$, where $G$ is a connected, simply connected Lie group and $\Gamma$ is a discrete subgroup), it suffices to prove the result for special solvmanifolds.

Now, from the results presented here, we can formulate the following necessary conditions for a counter-example to exist:

Suppose $S_1$ and $S_2$ are compact orientable solvmanifolds and $f, g : S_1 \to S_2$ have $N(f, g) < |L(f, g)|$. Then, for every finite regular cover $p_2 : \tilde{S}_2 \to S_2$, the finite regular cover $p_1 : \tilde{S}_1 \to S_1$ with

$$\text{im}(p_{1\#}) = f_2^{-1}\left(\text{im}(p_{2\#})\right) \cap g_2^{-1}\left(\text{im}(p_{2\#})\right)$$

has $\dim H_1(\tilde{S}_1; \mathbb{Q}) > \dim H_1(\tilde{S}_2; \mathbb{Q})$ and $\text{rk}(f_{1\#} - g_{1\#}) = \dim H_1(\tilde{S}_2; \mathbb{Q})$. 
One approach to the problem would be to show that no such $S_1$ and $S_2$ exist. That is, to show that the hypotheses of either Theorem 1.2 or Theorem 1.5 are satisfied for all maps $f, g : S_1 \to S_2$. This, however, is false. There are solvmanifolds $S_1, S_2$ and maps $f, g : S_1 \to S_2$ which do not satisfy the hypotheses of either theorem. The simplest such example is obtained by taking groups $I_1 = \mathbb{Z}^n$ and $I_2 = \mathbb{Z} \times \mathbb{Z}^{n-1}$, where the action of $\mathbb{Z}$ on $\mathbb{Z}^{n-1}$ is generated by a single $A \in \text{SL}_n^+(\mathbb{Z})$. For each $\Gamma_i$, there is a unique solvmanifold $S_i$ with $\pi_1(S_i) = \Gamma_i$. Of course, $S_1$ is a torus. If $A$ is chosen to have eigenvalues on the unit circle, but no roots of unity as eigenvalues, then $S_2$ is not an exponential solvmanifold, and every finite cover $\tilde{S}_2$ has $\dim H_1(\tilde{S}_2; \mathbb{Q}) = 1$. That is, the dimension-matching hypothesis of Theorem 1.2 is never satisfied in this example. Likewise, Theorem 2.1 is inapplicable. Further, we can construct maps $f, g : S_1 \to S_2$ so that $f_1^* g_1^* : H\mathcal{I}_1(S_1) \to H\mathcal{I}_2(S_2)$ is surjective. For example, if $\mathbb{Z}^n$ has generators $v_1, \ldots, v_n$ and $\mathbb{Z} \times \mathbb{Z}^{n-1}$ has generators $w_1, w_2, \ldots, w_n$ (with $w_1$ acting on $w_2, \ldots, w_n$), then define $F, G : \mathbb{Z}^n \to \mathbb{Z} \times \mathbb{Z}^{n-1}$ by

$$F(v_i) = \begin{cases} 0, & i = 1, \\ w_i, & i > 1, \end{cases} \quad G(v_i) = \begin{cases} w_1, & i = 1, \\ 0, & i > 1. \end{cases}$$

Then $F$ and $G$ clearly extend to homomorphisms between the Lie groups $\mathbb{R}^n$ and $\mathbb{R} \times \mathbb{R}^{n-1}$, and so define maps $f, g : S_1 \to S_2$ such that $f_1^* = 0$ and $g_1^*$ is surjective. This example shows that Theorems 1.2, 1.5 and 2.1 do not encompass all pairs of maps between solvmanifolds. However, in this example, there is only a single fixed point, with index $\pm 1$ (the sign depends on the orientations chosen), so $N(f, g) = |L(f, g)|$. That is, this example shows that some new approach is needed to prove the conjectures, but is not a counter-example to the conjectures.

One possible approach to such a result is provided by the work of D. Witte. He has shown in [13] that, if $S_1, S_2$ are special solvmanifolds with universal covers $G_1, G_2$ and fundamental groups $\Gamma_1, \Gamma_2$, then for every $f : S_1 \to S_2$, there is a $\Gamma$-equivariant crossed homomorphism $\phi$ covering $f$. That is, there is a homomorphism $\sigma : G_1 \to \text{Aut}(G_2)$ which is trivial on $\Gamma_1$, such that

$$\phi(gg') = \sigma(g')(\phi(g))\phi(g').$$

Thus, given $f_1, f_2 : S_1 \to S_2$, there are crossed homomorphisms $\phi_1, \phi_2 : G_1 \to G_2$ covering them. That is, we are almost in the setting of Theorem 2.1, and it is reasonable to hope that the proof of Theorem 2.1 might extend to this more general setting. However, the difference between homomorphisms and crossed homomorphisms is just enough that I have not been been able to make this extension work. The interested reader is invited to give it a try.
References