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Analytic solutions to the boundary layer problem over a stretching wall

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ABSTRACT

Analytic solutions to similarity boundary layer equations are given for boundary layer flows of Newtonian fluid over a stretching wall with power law stretching velocity. The existence of analytic solutions is proven. The Crane's solution is generalized and recurrence relations are obtained for the determination of coefficients of the exponential series.

1. Governing equations for boundary layers

The problem considered here is the steady boundary layer flow due to a moving flat surface in an otherwise quiescent Newtonian fluid medium moving at a speed of $U_w(x)$. In the absence of body force and an external pressure gradient, laminar boundary layer equations expressing conservation of mass and the momentum boundary layer equations for an incompressible fluid are written as

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \tag{1}$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\nu\frac{\partial^2 u}{\partial y^2}, \tag{2}$$

where (x, y) are the respective streamwise and plate-normal directions with (u, v) the corresponding velocities, and ν is the kinematic viscosity of the ambient a fluid which will be assumed constant. We consider the boundary-layer flow induced by a continuous surface stretching with velocity $U_w(x)$. The surface is assumed in general to be permeable and a lateral suction/injection with a certain velocity distribution $V_w(x)$ is applied. Accordingly, the boundary conditions are

$$u\,(x,0)=U_w(x), \qquad v\,(x,0)=V_w(x), \qquad \lim_{y\to\infty} u\,(x,y)=0. \tag{3}$$

The streamfunction ψ is formulated by

$$u=\frac{\partial\psi}{\partial y}, \qquad v=-\frac{\partial\psi}{\partial x}.$$

Eq. (2) reduces to

$$\frac{\partial\psi}{\partial y}\frac{\partial^2\psi}{\partial y\partial x}-\frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial y^2}=\nu\frac{\partial^2\psi}{\partial y}. \tag{4}$$

Assume the velocity of the plate is the form

$$U_w(x)=Ax^{\kappa}, \qquad V_w(x)=Bx^{(\kappa-1)/2}$$

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where A, B and κ are constants, $A > 0$. The case $B < 0$ corresponds to the suction and $B > 0$ to the injection of the fluid. If the wall is impermeable then $B = 0$. Under transformation

$$\psi = \sqrt{\frac{2\nu}{A(\kappa+1)}} A x^{\frac{\kappa+1}{2}} f(\eta), \qquad \eta = \sqrt{\frac{A(\kappa+1)}{2\nu}} y x^{\frac{\kappa-1}{2}}.$$

Eq. (4) can be written

$$f''' + ff'' - \frac{2\kappa}{\kappa+1} f'^2 = 0, \tag{5}$$

and the boundary conditions (3) become

$$f(0) = f_w, \qquad f'(0) = 1, \qquad \lim_{\eta\to\infty} f'(\eta) = 0, \tag{6}$$

where

$$f_w = -B\left[\nu A \frac{\kappa+1}{2}\right]^{-\frac{1}{2}}.$$

Now, the velocity components are given by

$$u(x, y) = Ax^{\kappa} f'(\eta),$$
$$v(x, y) = -\left(\frac{2\nu A}{\kappa+1}\right)^{1/2} x^{(\kappa-1)/2}\left[\frac{\kappa+1}{2} f(\eta) + \frac{\kappa-1}{2}\eta f'(\eta)\right].$$

We note that the same boundary value problem appears for the steady free convection flow over a vertical semi-infinite flat plate embedded in a fluid saturated porous medium of ambient temperature T_∞, and the temperature of the plate is $T_w = T_\infty + \bar{A}x^{\kappa}$. There is difference in the region of κ between the two physical problem. For flows in a porous medium, there is a physical meaning when $-1/2 < \kappa < +\infty$ (see [1]), and for boundary layer flows over a stretching wall $-\infty < \kappa < -1$, and $-1/2 < \kappa < +\infty$ [2].

Banks [2] has proved if the wall is impermeable then the boundary value problems (5)–(6) does not admit a similarity solution when $-1 < \kappa \le -1/2$. Numerical solutions were given in papers [2,1]. For some special cases of κ problems (5)–(6) are exactly solvable. These particular cases are $\kappa = 1$ and $\kappa = -1/3$. For the impermeable case with $\kappa = 1$ we refer to the exact solution by Crane [3] and for the permeable case [4]. For an impermeable case with $\kappa = -1/3$ the exact solution is in [2] and the exact analytic solution for the permeable case by Magyari and Keller [5].

In this paper our goal is to prove the existence of the exponential series solution to the nonlinear boundary value problems (5)–(6). Both for permeable and impermeable cases we give a method for the determination of the coefficients and parameters. Numerical results are also presented.

2. Exact solutions

The exact solutions for some special values of κ are known. These are $\kappa = 1$ and $\kappa = -1/3$.

2.1. $\kappa = 1$

The solution of the boundary-value problems (5)–(6) for the velocity $U_w(x) = Ax$, ($\kappa = 1$) of an impermeable surface, $V_w(X) = 0$, has been reported by Crane [3]. Thus, the stream function of Crane's problem has the form

$$\psi = \sqrt{\frac{\nu}{A}} A x f(\eta), \qquad \eta = \sqrt{\frac{A}{\nu}} y,$$

where $f(\eta)$ is the solution of the ordinary differential equation

$$f''' + ff'' - f'^2 = 0,$$

subject to the boundary conditions

$$f(0) = 0, \qquad f'(0) = 1, \qquad \lim_{\eta\to\infty} f'(\eta) = 0.$$

Crane's well known solution for $f(\eta)$ and for the corresponding velocity field reads

$$f(\eta) = 1 - e^{-\eta}, \tag{7}$$

and the velocity components are

$$u(x, y) = Axe^{-\eta},$$
$$v(x, y) = -(\nu A)^{1/2}(1 - e^{-\eta}).$$

For that solution one gets $f''(0) = -1$.

For the permeable case the solution has been given by Gupta and Gupta [4]

$$f(\eta) = f_w - \frac{1}{f_0}\left[1 - e^{f_0\eta}\right], \tag{8}$$

with

$$f_0 = -\frac{1}{2}\left[f_w + \sqrt{f_w^2 + 4}\right].$$

In this way the velocity field is obtained as

$$u(x, y) = Axe^{f_0\eta},$$
$$v(x, y) = -(\nu A)^{1/2}\left[f_w - \frac{1}{f_0}\left(1 - e^{f_0\eta}\right)\right],$$

and $f''(0) = f_0$.

2.2. $\kappa = -1/3$

The exact solution for $\kappa = -1/3$ and $f_w = 0$ can be given

$$f(\eta) = \sqrt{2}\tanh(\eta/\sqrt{2}), \qquad f''(0) = 0.$$

For that solution one obtains $f''(0) = 0$.

3. Existence of analytic solutions

The aim of this section is to show the existence of analytic solutions to the boundary value problems (5)–(6) and to determine the approximate local solution $f(\eta)$. We use the shooting method and replace the condition at infinity by one at $\eta = 0$. Therefore, (5)–(6) is converted into an initial value problem of (5) with initial conditions

$$f(0) = f_w, \qquad f'(0) = 1, \qquad f''(0) = \gamma. \tag{9}$$

We consider the nonlinear differential equation (5) as a system of certain differential equations, namely, the special Briot–Bouquet differential equations. For this type of differential equations, we refer to the book by Hille [6], and by Ince [7]. In order to establish the existence of a series representation of $f(\eta)$ we apply the following theorem:

Briot–Bouquet Theorem (*[8]*). *Let us assume that for the system of equations*

$$\left.\begin{aligned} \xi\frac{dz_1}{d\xi} &= u_1(\xi, z_1(\xi), z_2(\xi)), \\ \xi\frac{dz_2}{d\xi} &= u_2(\xi, z_1(\xi), z_2(\xi)), \end{aligned}\right\} \tag{10}$$

where functions u_1 and u_2 are holomorphic functions of ξ, $z_1(\xi)$, and $z_2(\xi)$ near the origin, moreover

$$u_1(0, 0, 0) = u_2(0, 0, 0) = 0,$$

then a holomorphic solution of (10) satisfying the initial conditions $z_1(0) = 0$, $z_2(0) = 0$ exists if none of the eigenvalues of the matrix

$$\begin{bmatrix} \left.\dfrac{\partial u_1}{\partial z_1}\right|_{(0,0,0)} & \left.\dfrac{\partial u_1}{\partial z_2}\right|_{(0,0,0)} \\ \left.\dfrac{\partial u_2}{\partial z_1}\right|_{(0,0,0)} & \left.\dfrac{\partial u_2}{\partial z_2}\right|_{(0,0,0)} \end{bmatrix}$$

is a positive integer.

The Briot–Bouquet theorem ensures the existence of formal solutions

$$z_1 = \sum_{k=1}^{\infty} a_k\xi^k, \qquad z_2 = \sum_{k=1}^{\infty} b_k\xi^k$$

to system (10), and also the convergence of formal solutions.

This theorem and the method presented here have been successfully applied to the determination of local analytic solutions of different nonlinear initial value problems (see [9–11]).

In view of the third of the boundary conditions (6), let us consider the initial value problems (5)–(9), and take its solution in the form

$$f(\eta) = \alpha\left(1 + S\left(ae^{-\alpha\eta}\right)\right), \quad \eta \in I, \tag{11}$$

where the function $S \in C^2(I)$ for some interval I and $\alpha > 0$. Let us introduce the new variable Z such as $Z = ae^{-\alpha\eta}$ and functions U and T as follows

$$U(Z) = Z\frac{\mathrm{d}S}{\mathrm{d}Z},$$
$$T(Z) = \frac{\mathrm{d}U}{\mathrm{d}Z}.$$

Then the differential equation (5) can be rewritten by the following system of differential equations

$$\left.\begin{aligned} \frac{\mathrm{d}S}{\mathrm{d}Z} &= \frac{U}{Z}, \\ \frac{\mathrm{d}U}{\mathrm{d}Z} &= T, \\ \frac{\mathrm{d}T}{\mathrm{d}Z} &= \frac{S}{Z}(1+T) - \frac{2\kappa}{\kappa+1}\frac{U^2}{Z^2}. \end{aligned}\right\} \tag{12}$$

One can restate the third order differential equation in (5) as a system of Briot–Bouquet differential equations

$$\left.\begin{aligned} u_1(Z, S(Z), U(Z), T(Z)) &= ZS'(Z), \\ u_2(Z, S(Z), U(Z), T(Z)) &= ZU'(Z), \\ u_3(Z, S(Z), U(Z), T(Z)) &= ZT'(Z), \end{aligned}\right\}$$

such as

$$\left.\begin{aligned} u_1(Z, S(Z), U(Z), T(Z)) &= U(Z), \\ u_2(Z, S(Z), U(Z), T(Z)) &= Z(1 + T(Z)), \\ u_3(Z, S(Z), U(Z), T(Z)) &= S\left(1 + T(Z)\right) - \frac{2\kappa}{\kappa+1}\frac{U^2(Z)}{Z}, \end{aligned}\right\} \tag{13}$$

with choosing

$$\left.\begin{aligned} S(0) &= 0, \\ U(0) &= 0, \\ T(0) &= 0, \end{aligned}\right\}$$

one gets

$$\left.\begin{aligned} u_1(0, 0, 0, 0) &= 0, \\ u_2(0, 0, 0, 0) &= 0, \\ u_3(0, 0, 0, 0) &= 0. \end{aligned}\right\}$$

Since the conditions in the Briot–Bouquet theorem are satisfied and the eigenvalues of the matrix

$$\begin{bmatrix} \frac{\partial u_1}{\partial S} & \frac{\partial u_1}{\partial U} & \frac{\partial u_1}{\partial T} \\ \frac{\partial u_2}{\partial S} & \frac{\partial u_2}{\partial U} & \frac{\partial u_2}{\partial T} \\ \frac{\partial u_3}{\partial S} & \frac{\partial u_3}{\partial U} & \frac{\partial u_3}{\partial T} \end{bmatrix}$$

at $(0, 0, 0, 0)$ are zero then referring to the theorem above we obtain the existence of unique analytic solutions S, U and T near zero.

We note that system (13) for [9] was created similarly as in [12].

4. The exponential series solution of boundary layer problems

Applying the results of the previous section in order to find the solution of Eq. (5) with boundary conditions (6), we assume

$$f(\eta) = \alpha\left(A_0 + \sum_{i=1}^{\infty} A_i a^i e^{-\alpha i\eta}\right), \tag{14}$$

where $\alpha > 0$, $A_0 = 1$, and A_i $(i = 1, 2, \ldots)$ are coefficients.

The conditions (6) yield the following equations:

$$\alpha\left(A_0+\sum_{i=1}^{\infty}A_i a^i\right)=f_w, \tag{15}$$

$$-\alpha^2\sum_{i=1}^{\infty}iA_i a^i=1. \tag{16}$$

It is evident that the third of the boundary conditions is automatically satisfied.

Substituting (14) into (5) we have

$$-\sum_{i=1}^{\infty}i^3A_iZ^i+\left(A_0+\sum_{i=1}^{\infty}A_iZ^i\right)\sum_{i=1}^{\infty}i^2A_iZ^i-\frac{2\kappa}{\kappa+1}\left(\sum_{i=1}^{\infty}iA_iZ^i\right)^2=0$$

or

$$-\sum_{i=1}^{\infty}i^3A_iZ^i+A_0\sum_{i=1}^{\infty}i^2A_iZ^i+\sum_{i=2}^{\infty}\sum_{k=1}^{i-1}k^2A_kA_{i-k}Z^i-\frac{2\kappa}{\kappa+1}\sum_{i=1}^{\infty}\sum_{k=1}^{i-1}k(i-k)A_kA_{i-k}Z^i=0.$$

Equating the coefficients of like powers of Z, we get recurrence relations for $A_2, A_3, \ldots$ and we obtain

$$A_2=-\frac{1}{4}A_1^2\frac{\kappa-1}{\kappa+1},$$

$$A_3=\frac{1}{72}A_1^3\frac{(\kappa-1)(3\kappa-5)}{(\kappa+1)^2},$$

$$A_4=-\frac{1}{864}A_1^4\frac{(\kappa-1)\left(6\kappa^2-19\kappa+17\right)}{(\kappa+1)^3},$$

$$A_5=\frac{1}{86400}A_1^5\frac{(\kappa-1)\left(93\kappa^3-464\kappa^2+783\kappa-484\right)}{(\kappa+1)^4},$$

$$A_6=-\frac{1}{2592000}A_1^6\frac{(\kappa-1)\left(432\kappa^4-2889\kappa^3+7461\kappa^2-8759\kappa+4139\right)}{(\kappa+1)^5},$$

$$A_7=\frac{1}{4572288000}A_1^7\frac{(\kappa-1)P_5(\kappa)}{(\kappa+1)^6},$$

$$P_5(\kappa)=115839\kappa^5-983892\kappa^4+3399550\kappa^3-6012140\kappa^2+5447171\kappa-2081728$$

$$A_8=-\frac{1}{64012032000}A_1^8\frac{(\kappa-1)P_6(\kappa)}{(\kappa+1)^7}$$

$$P_6(\kappa)=44854\kappa^6-2521077\kappa^5+10974320\kappa^4-25899165\kappa^3+35072231\kappa^2-25921218\kappa+8309255$$

$$A_9=\frac{1}{9217732608000}A_1^9\frac{(\kappa-1)P_7(\kappa)}{(\kappa+1)^8}$$

$$\begin{aligned}P_7(\kappa)\ =\ &5288733\kappa^7-64112391\kappa^6+337072482\kappa^5-997781298\kappa^4\\&+1799062257\kappa^3-1980424339\kappa^2+1236353168\kappa-341103412\end{aligned}$$

$$\vdots$$

The coefficients $A_2, A_3, A_4, \ldots$ are expressed as functions of κ. We note that our computations indicate that the coefficients obtained above can be written in the form

$$A_n=A_1^n\frac{\kappa-1}{(\kappa+1)^{n-1}}P_{n-2}(\kappa),$$

where P_{n-2} is a polynomial of κ of order $n-2$. When $\kappa=1$ is substituted then we get that each coefficient A_k, $k>1$ is equal to zero. This case results in Crane's solution (7) or (8) for impermeable or permeable cases, respectively.

The shear stress at the surface is given by $f''(0)$ and

$$f''(0)=\alpha^3\sum_{i=1}^{\infty}i^2A_ia^i.$$

From system (15)–(16) with coefficients $A_2, A_3, A_4, \ldots$ and with the choice of $A_1=1$ one can obtain the values of parameters a and α. Tables 1 and 2 representing the numerical results for some values of $\kappa\in(-\infty,-1)\cup(-1/2,+\infty)$, $f_w=0$ and $f_w=1$ on the base of the first 10 terms in the series.

Table 1
$f_w = 0$.

κ	a	α	$f''(0)$
−5	−0.7410	0.8687	−1.4033
−4	−0.7207	0.8575	−1.4417
−3	−0.6831	0.8364	−1.5156
−2	−0.5915	0.7826	−1.7166
−1.5	−0.4678	0.7040	−2.0337
−1/3	−1.5919	0.7957	−5.2423
−1/6	−1.5089	1.1868	−1.1407
−1/8	−1.4521	1.1859	−0.7543
−1/10	−1.4207	1.1786	−0.6684
0	−1.3186	1.1419	−0.6433
1/3	−1.1358	1.0628	−0.8300
1/2	−1.0864	1.0403	−0.8896
3/4	−1.0351	1.0166	−0.9540
1	−1	1	−1

Table 2
$f_w = 1$.

κ	a	α	$f''(0)$
−5	−0.3195	1.5663	−1.9547
−4	−0.3140	1.5615	−1.9883
−3	−0.3036	1.5523	−2.0538
−2	−0.2765	1.5279	−2.2376
−1.5	−0.2359	1.4889	−2.5620
−1/3	−0.5357	1.7320	−1.0014
−1/6	−0.4753	1.6890	−1.2179
−1/8	−0.4656	1.6819	−1.2553
−1/10	−0.4604	1.6781	−1.2757
0	−0.4433	1.6654	−1.3445
1/3	−0.4099	1.6400	−1.4878
1/2	−0.4000	1.6323	−1.5325
3/4	−0.3985	1.6240	−1.5820
1	−0.3820	1.6180	−1.6180

The radius of the convergence of the series can be found by applying the ratio test and the series converges absolutely for

$$\eta > -\frac{1}{\alpha}\left[\ln\left(\lim_{n\to\infty}\left|\frac{A_n}{A_{n+1}}\right|\right) - \ln|a|\right].$$

We note that the sequence of terms A_n/A_{n+1} converges very slowly, and for the determination of the convergence interval an alternative method was given by Samuel and Hall [13].

5. Conclusion

In this paper the existence of the exponential series solution to the boundary value problem describing the boundary layer flows of Newtonian fluids has been given. The Crane's solution is generalized for stretching walls with a power law stretching velocity. A method is given for the determination of the coefficients A_i $(i = 0, 1, 2, \ldots)$ and parameters a, α in (14). The shear stress at the surface and the radius of convergence are also discussed.

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