Advances in Mathematics **157**, 22–35 (2001)
doi:10.1006/aima.2000.1937, available online at http://www.idealibrary.com on IDEAL®

Well-posedness for the Navier–Stokes Equations

Herbert Koch[1]

Institut für Angewandte Mathematik, Universität Heidelberg, Heidelberg, Germany

and

Daniel Tataru[2]

Department of Mathematics, Northwestern University, Chicago, Illinois

Communicated by Charles Fefferman

Accepted May 22, 2000

1. INTRODUCTION

We study the incompressible Navier–Stokes equations in $\mathbb{R}^n \times \mathbb{R}^+$

$$\begin{cases} u_t + (u \cdot \nabla)\, u - \Delta u + \nabla p = 0 \\ \nabla \cdot u = 0 \\ u(0) = u_0, \end{cases} \tag{1}$$

where u is the velocity and p is the pressure. It is well known that the Navier–Stokes equations are locally well-posed for smooth enough initial data as long as one imposes appropriate boundary conditions on the pressure at ∞. For instance it is easy to see (see [9] for much more general results) that if $s > \frac{n}{2}$ then for any H^s initial data there exists a unique $C([0, t]; H^s(\mathbb{R}^n))$ local solution with a pressure $p \in C([0, t]; H^s(\mathbb{R}^n))$. In the sequel we consider solutions for less regular initial data. This has to be understood in the sense that the map from the initial data to the solution extends continuously to rougher function spaces.

The question we are interested in is the global well-posedness for small data and local well-posedness for large data, with respect to a certain space of initial data u_0. Kato [8] proved that this holds for initial data in $L^n(\mathbb{R}^n)$. Later Giga and Miyakawa [5] and Taylor [15] proved the same result for initial data in certain Morrey spaces. This was motivated to allow

[1] Research partially supported by the DFG and DAAD.

[2] Research partially supported by NSF grants DMS-9622942, INT-9815286 and an Alfred P. Sloan fellowship.

0001-8708/01 $35.00

vortex rings and vortex filaments for the initial data. A similar result has been obtained by Cannone [4] and Planchon [12] for data in the Besov spaces $B^{-1+n/p}_{p,\infty}(\mathbb{R}^n)$ $(1<p<\infty)$. See also the recent articles by Iftimie [7] and by Lions and Masmoudi [11].

Here we search for the largest function space such that local or global solutions exist. In order to make sense of the equation we want to have

$$u \in L^2_{loc}(\mathbb{R}^n \times [0, \infty)).$$

The Navier–Stokes equations are invariant with respect to scaling (here one considers u as velocity). Hence we want a scale and translation invariant version of L^2-boundedness:

$$\sup_{x,\, R>0} |B(x, R)|^{-1} \int_{B(x, R) \times [0, R^2]} |u|^2 \, dy \, dt < \infty. \tag{2}$$

Here $|A|$ denotes the Lebesgue measure of A. Then it is natural to choose as space of initial data the space of tempered distributions u_0 in $\mathbb{R}^n$ for which the caloric extension (i.e. convolution with the heat kernel) satisfies (2). This space is well known: it consists of functions which are the divergence of a vector field with components in BMO. Let us be more precise.

Let $\Phi(x) = \pi^{-n/2} e^{-|x|^2}$ and $\Phi_t(x) = t^{-n} \Phi(x/t)$.

DEFINITION 1.1. We say that the tempered distribution v is in BMO if

$$\|v\|_{BMO} := \sup_{x,\, R>0} \left(2\, |B(x, R)|^{-1} \int_{B(x, R)} \int_0^R t\, |\nabla(\Phi_t * v)|^2 \, dt \, dy \right)^{1/2} < \infty.$$

This is a Carleson measure characterization of BMO, which is equivalent to the standard definition, see Stein [13]. Examples of elements of BMO are functions in $L^\infty(\mathbb{R}^n)$ and $\ln |p|$ for all polynomials p.

Let w be the solution to the heat equation

$$w_t - \Delta w = 0$$

with initial data v. It is uniquely defined under mild restrictions on v and w by $w(t) = v * \Phi_{\sqrt{4t}}$. Therefore

$$\|v\|_{BMO} = \sup_{x,\, R} \left(|B(x, R)|^{-1} \int_{B(x, R)} \int_0^{R^2} |\nabla w|^2 \, dt \, dy \right)^{1/2}.$$

We define the BMO^{-1} norm by

$$\|v\|_{BMO^{-1}} := \sup_{x,\,R} \left(|B(x, R)|^{-1} \int_{B(x,\,R)} \int_0^{R^2} |w|^2 \, dt \, dy \right)^{1/2}.$$

Then BMO^{-1} is the space of tempered distributions for which the above norm is finite. The expression $\|.\|_{BMO}$ is not a norm: it gives zero when applied to constants. This problem vanishes with BMO^{-1}.

Clearly the divergence of a vector field with components in BMO is in BMO^{-1}. The following theorem asserts that the converse is also true.

THEOREM 1. *Let u be a tempered distribution. Then $u \in BMO^{-1}$ if and only if there exist $f^i \in BMO$ with $u = \sum \partial_i f^i$.*

This result is proved in the last section.

Let now $Q(x, R) = B(x, R) \times (0, R^2)$. The definition of BMO^{-1} motivates the introduction of the spaces X and Y of functions in $\mathbb{R}^n \times \mathbb{R}^+$ with norms

$$\|u\|_X = \sup_t t^{1/2} \|u(t)\|_{L^\infty(\mathbb{R}^n)} + \left(\sup_{x,\, R>0} |B(x, R)|^{-1} \int_{Q(x,\,R)} |u|^2 \, dy \, dt \right)^{1/2} \tag{3}$$

and

$$\|f\|_Y = \sup_t t \, \|f(t)\|_{L^\infty(\mathbb{R}^n)} + \sup_{x,\, R>0} |B(x, R)|^{-1} \int_{Q(x,\,R)} |f| \, dy \, dt. \tag{4}$$

Then our main result is:

THEOREM 2. *The Navier–Stokes equations* (1) *have a unique small global solution in X for all initial data u_0 with $\nabla \cdot u_0 = 0$ which are small in BMO^{-1}.*

We suppress the boundary condition ($p(t) \in BMO$ for all $t > 0$, for example) for the pressure at infinity in the formulation of the theorem. This boundary condition is implicitly chosen in our construction of the solution. It is not hard to see that weak solutions in X are solutions in our sense. Hence they are unique under the assumptions of Theorem 2.

In effect the proof gives more than that. We define the local BMO space BMO_R defined as BMO but where we only consider balls of size R and smaller. We define also the similar versions of X, Y, BMO^{-1} which we denote by X_R, Y_R, respectively BMO_R^{-1}. Let $v \in BMO_1$. Then we say that[3] $v \in \overline{VMO}$ if and only if

$$\|v\|_{BMO_R} \to 0 \qquad \text{as} \quad R \to 0.$$

[3] Note that $\overline{VMO}$ is larger than the usual VMO space as there is no condition on cylinders whose radius is away from 0.

Similarly we define $\overline{VMO}^{-1}$. Then

THEOREM 3. *There exists $\varepsilon > 0$ so that for all $R > 0$ the Navier–Stokes equations* (1) *have a unique small solution in X_R (up to $t = R^2$) whenever $\nabla \cdot u_0 = 0$ and $\|u_0\|_{BMO_R^{-1}} \leqslant \varepsilon$. In particular for all $u_0 \in \overline{VMO}^{-1}$ with $\nabla \cdot u_0 = 0$ there exists a unique small local solution.*

Remark 1.2. To compare our result with earlier results we shall verify that our function spaces BMO^{-1} and BMO_R^{-1} contain the other function spaces where local well-posedness has been proved before. This is done at the end of the last section. It is not hard to find sufficient conditions for tempered distributions to be in these function spaces. For instance $L^n_{unif} \subset BMO_1^{-1}$ and $\|u\|_{BMO_R^{-1}} \to 0$ as $R \to 0$ if $u \in L^n(\mathbb{R}^n)$. Here L^n_{unif} is the subspace of $L^1_{loc}(\mathbb{R}^n)$ of functions for which the norm on balls of radius 1 is uniformly bounded.

It seems hopeful that this regularity result allows to improve local criteria for regularity, which have been used by Caffarelli, Kohn and Nirenberg [3], Struwe [14] and Lin [10] to prove partial regularity.

There has been a strong interest in obtaining well-posedness under weak conditions and there is some evidence that BMO is the right space in several different problems. See Wu [16] for the relation between the regularity of the boundary and the mapping of the Riemann mapping theorem.

Is the smallness assumption in Theorem 2 and Theorem 3 necessary? There is an interesting open problem which exemplifies the difficulty. In two space dimensions there exists always a weak solution to the initial data

$$u_0(x) = \kappa \, |x|^{-2} \, (x_2, -x_1).$$

Heuristically one can see that the solution should solve the heat equation. Since the initial data is rotational (i.e. $u = (h(r)\, x_2, -h(r)\, x_1)$) and divergence free, the corresponding solution to the heat equation

$$u(x, t) = \kappa \int \Phi_{\sqrt{4t}}(x - y) \, |y|^{-2} \begin{pmatrix} y_2 \\ -y_1 \end{pmatrix} dy$$

remains rotational and divergence free. But then $(u \cdot \nabla)\, u$ is radial therefore it is a gradient. Hence u solves the Navier–Stokes equations as well as the heat equation.

This solution is unique according to Theorem 2 provided κ is small. We do not know whether it is unique if κ is large. See Ben-Artzi [1], Brezis [2] and Giga and Miyakawa [6] for approaches to Navier–Stokes equations in 2 dimensions based on vorticity.

We would like to thank M. Struwe for the observation in Remark 3.3.

2. PRELIMINARIES

We denote the heat semigroup by $S(t)$ without specifying the domain. The operator V is the parametrix for the inhomogeneous heat equation with 0 Cauchy data, i.e. $u = Vf$ iff

$$u_t - \Delta u = f, \qquad u(0) = 0.$$

Then

$$Vf(t) = \int_0^t S(s)\, f(t-s)\, ds$$

which, written in terms of the heat kernel, yields

$$Vf(x, t) = \int_{\mathbb{R}^n \times (0, t)} \frac{1}{(4\pi s)^{n/2}}\, e^{-|y|^2/4s} f(x-y, t-s)\, ds.$$

Then the solutions to the heat equation

$$u_t - \Delta u = f, \qquad u(\,.\,, 0) = u_0$$

are given by

$$u(x, t) = (S(t)\, u_0)(x) + Vf(x, t).$$

The Fourier transform of u is denoted either by $\hat{u}$ or $\mathscr{F}u$, the inverse by $\mathscr{F}^{-1}u$. If we take a symbol $m \in L^\infty$ then the corresponding multiplier

$$m(D_x)\, u = \mathscr{F}^{-1}(m\hat{u}) \tag{5}$$

is bounded in L^2. Here we are interested in the projection operator Π to divergence free vector fields, which is defined by its matrix valued Fourier multiplier

$$m(\eta) = \delta_{ij} - \frac{\eta_i \eta_j}{|\eta|^2}. \tag{6}$$

Then its symbol m satisfies the Mihlin-Hörmander condition

$$\sup_{\eta \neq 0} |\eta|^{|\alpha|}\, |\partial_\eta^\alpha m(\eta)| \leqslant C \tag{7}$$

for all multiindices α; hence, $m(D_x)$ is a singular integral operator. This implies with Φ as in Definition 1.1 that

$$|(\Pi\Phi)(x)| \leqslant c(1+|x|)^{-n};$$

hence scaling shows that the kernel function $k_t(x) = \Pi\Phi_{\sqrt{4t}}$ of $\Pi S(t)$ satisfies

$$|k_t(x)| \leqslant c(\sqrt{t}+|x|)^{-n}.$$

Similarly we get bounds for the kernel of $\Pi\nabla S(t)$,

$$|\Pi\nabla\Phi_{\sqrt{4t}}(x)| \leqslant c(\sqrt{t}+|x|)^{-n-1} \tag{8}$$

since $\int \nabla\Phi\, dx = 0$ and for the kernel of $\Pi(I-S(t))$,

$$|\Pi(\delta_0 - \Phi_{\sqrt{4t}})| \leqslant ct\, |x|^{-n-2} \tag{9}$$

because $\int(\delta_0 - \Phi_t)\, dx = 0$ and $\int x_i(\delta_0 - \Phi_t)\, dx = 0$.

For $a \in C^\infty$ supported in $B(0, 1)$ we define $S(-t)\, a(t^{1/2}D_x)$ in the obvious way. Its kernel function k_t is a Schwartz function and scaling implies

$$|k_t(x)| \leqslant c_N t^{-n/2} \left(1 + \frac{|x|}{\sqrt{t}}\right)^{-N} \tag{10}$$

for all $N \geqslant 1$.

For simplicity we will not be precise about the domain of operators. In the end it is not hard to verify that this does not cause difficulties.

3. PROOF OF THEOREM 2

We set up the problem so that we can use a fixed point argument. We can rewrite the Navier–Stokes equation as

$$u(x, t) = S(t)\, u_0(x) - (V\nabla\Pi N(u))(x, t), \qquad N(u) = u \otimes u. \tag{11}$$

For small initial data we want to solve this in X using a fixed point argument. Since N is quadratic, the small Lipschitz constant follows for small initial data if the nonlinearity has the correct mapping properties. Hence the result is a consequence of the following two lemmas:

LEMMA 3.1. *N maps X into Y.*

The proof is straightforward.

LEMMA 3.2. *$V\nabla\Pi$ maps Y into X.*

Proof. Step 1. Scaling and localization. We need to prove the pointwise estimate

$$|V\nabla\Pi f(x,t)| \leqslant ct^{-1/2}\,\|f\|_Y$$

and the L^2 estimate

$$\|V\nabla\Pi f\|^2_{L^2(Q(x,R))} \leqslant c\,|B(x,R)|\,\|f\|^2_Y.$$

Both estimates are scale invariant and translation invariant, therefore it suffices to show that

$$|V\nabla\Pi f(0,1)| \leqslant c\,\|f\|_{Y_1} \tag{12}$$

and

$$\|V\nabla\Pi f\|_{L^2(Q(0,1))} \leqslant c\,\|f\|_{Y_1}. \tag{13}$$

Let χ be the characteristic function of $B(0,2)\times[0,1]$. Then $f=\chi f+(1-\chi)f$. Clearly both components are still in Y. From (8) we see that the kernel K of $V\nabla\Pi$ satisfies

$$|K(x,t)| \leqslant c(\sqrt{t}+|x|)^{-n-1}. \tag{14}$$

Then

$$\|V\nabla\Pi(1-\chi)f\|_{L^\infty(Q(0,1))} \leqslant c\sup_{x\in\mathbb{R}^n}\int_{Q(x,1)}|f|\,dx\,dt$$

which is much stronger than what we actually need. Hence, it suffices to look now at χf; namely, without any restriction in generality, we can and do assume in the sequel that f is supported in $B(0,2)\times[0,1]$.

Step 2. The pointwise estimate. If f is supported in $B(0,2)\times[0,1]$ then the pointwise estimate (12) follows easily from the kernel bound (14). Indeed, for the part of f in $B(0,2)\times[0,\frac{1}{2}]$ we can use the L^1 bound on f combined with the boundedness of the kernel away from 0. For the part of f in $B(0,2)\times[\frac{1}{2},1]$ we can use the L^∞ bound on f combined with the integrability of the kernel at 0.

Step 3. Cutting off high frequencies. We shall in effect prove an estimate which is stronger than (13), namely

$$\int_0^1 \int_{\mathbb{R}^n} |\nabla Vf|^2 \, dx \, dt \leqslant \|f\|_Y \, \|f\|_{L^1(\mathbb{R}^n \times \mathbb{R})}. \tag{15}$$

Here we have dispensed with Π, which is a bounded operator in L^2 and which commutes with ∇V. Also we have removed the restriction on the support of f.

Let $a \in C_0^\infty$ satisfy $a(\xi) = 1$ if $|\xi| \leqslant 1$ and $a(\xi) = 0$ if $|\xi| \geqslant 2$. We consider the multipliers $A_t = a(t^{1/2} D_x)$ which cut off the frequencies larger than $t^{-1/2}$. Then, for $t \leqslant 1$,

$$\|(1 - A_t) \, g\|_{H^{-1}(\mathbb{R}^n)} \leqslant ct^{1/2} \, \|g\|_{L^2(\mathbb{R}^n)},$$

and the L^2 estimate

$$\|V\nabla(1 - A_t) \, f\|^2_{L^2(\mathbb{R}^n \times (0,1))} \leqslant c \int_0^1 \|(1 - A_t f)\|^2_{H^{-1}(\mathbb{R}^n)} \leqslant c \, \|f\|_Y \, \|f\|_{L^1(\mathbb{R}^n \times (0,1))}$$

follows immediately.

Step 4. The key estimate. It remains to look at $A_t f$. Let $k_t(x)$ be the kernel of $S(-t) \, A_t$, which is well defined since the range of A_t consists of functions with compactly supported Fourier transform. Then, for all $N \geqslant 1$, we have from (10),

$$|k_t(x)| \leqslant c_N t^{-n/2} \left(1 + \frac{|x|}{\sqrt{t}}\right)^{-N}.$$

In particular $\|k_t\|_{L^1(\mathbb{R}^n)} < \infty$ uniformly in t. Hence, (with the mild abuse of notation $(S(-t) \, A_t f)(x, t) = (S(-t) \, A_t f(., t))(x)$

$$\|S(-t) \, A_t f\|_Y \leqslant c \, \|f\|_Y, \qquad \|S(-t) \, A_t f\|_{L^1(\mathbb{R}^n)} \leqslant c \, \|f(t)\|_{L^1(\mathbb{R}^n)}.$$

Let $w(t) = S(-t) \, A_t f(t)$. Then $v(t) = \nabla V A_t f$ can be described by

$$v(t) = \nabla S(t) \int_0^t w(s) \, ds. \tag{16}$$

To conclude we need to prove the estimate

$$\|v\|^2_{L^2(\mathbb{R}^n \times (0,1))} \leqslant c \, \|w\|_Y \, \|w\|_{L^1(\mathbb{R}^n \times (0,1))}. \tag{17}$$

We compute

$$\|v\|^2_{L^2(\mathbb{R}^n\times(0,1))} = \int_0^1 \left\| \nabla S(t) \int_0^t w(s)\, ds \right\|^2_{L^2(\mathbb{R}^n)} dt$$

$$= -2 \int_0^1 \int_0^t \int_0^s \langle \Delta S(2t)\, w(s), w(\theta)\rangle_{L^2(\mathbb{R}^n)}\, d\theta\, ds\, dt$$

$$= -2 \int_0^1 \int_s^1 \int_0^s \langle \Delta S(2t)\, w(s), w(\theta)\rangle_{L^2(\mathbb{R}^n)}\, d\theta\, dt\, ds$$

$$= \int_0^1 \int_0^s \langle (S(2s)-S(2))\, w(s), w(\theta)\rangle_{L^2(\mathbb{R}^n)}\, d\theta\, ds$$

$$= \int_0^1 \langle w(s), (S(2s)-S(2)) \int_0^s w(\theta)\, d\theta\rangle_{L^2(\mathbb{R}^n)}\, ds$$

$$\leqslant \int_0^1 \|w(s)\|_{L^1(\mathbb{R}^n)} \left\| (S(2s)-S(2)) \int_0^s w(\theta)\, d\theta \right\|_{L^\infty(\mathbb{R}^n)} ds.$$

Here we use (16) for the first, integration by parts for the second, Fubini for the third, $\partial_t S(t) = \Delta S(t)$ and the fundamental theorem of calculus for the fourth and selfadjointness of $S(t)$ for the fifth equality.

If we could now prove the t independent bound

$$\left\| S(2t) \int_0^t w(\theta)\, d\theta \right\|_{L^\infty(\mathbb{R}^n)} \leqslant c\, \|w\|_Y, \tag{18}$$

then we get

$$\|v\|^2_{L^2(\mathbb{R}^n\times(0,1))} \leqslant c\, \|w\|_Y\, \|w\|_{L^1(\mathbb{R}^n\times(0,1))}$$

which implies (17).

Step 5. Estimate (18). To obtain (18) we start with

$$|B(x,R)|^{-1} \left\| \int_0^{R^2} w(\theta)\, d\theta \right\|_{L^1(B(x,R))}$$

$$\leqslant |B(x,R)|^{-1} \int_{Q(x,R)} |w(x,t)|\, dx\, dt \leqslant \|w\|_Y. \tag{19}$$

The operator $S(2t)$ has a kernel $k(x)$ which satisfies

$$k(x) = c_n t^{-n/2} e^{-|x|^2/8t};$$

therefore it acts as an averaging operator on the scale of $\sqrt{t}$. Hence if we use (19) on a lattice of cubes of size $\sqrt{t}$ then we get

$$\left\| S(2t) \int_0^t w(\theta)\, d\theta \right\|_{L^\infty(\mathbb{R}^n)} \leqslant c\, \|w\|_Y \sum_{q \in \mathbb{Z}^n} e^{-|q|^2} \leqslant c\, \|w\|_Y.$$

This implies (18) and completes the proof. ∎

Remark 3.3. There is a different argument leading to estimate (15), which has been pointed out to us by M. Struwe. We claim that

$$\|Vf\|_{L^\infty} \leqslant c\, \|f\|_Y \tag{20}$$

and

$$\|\nabla Vf\|_{L^2}^2 \leqslant \|f\|_{L^1}\, \|Vf\|_{L^\infty}. \tag{21}$$

Both inequalities together imply (15).

Estimate (20) is reduced to the estimate for $t=1$ and $x=0$ by scaling. There it is obvious. Estimate (21) follows from the standard energy inequality.

4. EQUIVALENT NORMS AND FUNCTION SPACES

Here we prove Theorem 1 and verify the imbeddings mentioned in the introduction. Its main part consists of Lemma 4.1 below, which is more or less an elementary alternative proof of the boundedness of singular integral operators in BMO, using the Carleson measure definition of BMO.

Here we can not make use of the boundedness of singular integral operators, since we need boundedness of singular integrals for the equivalence in Theorem 1.

Proof of Theorem 1. Suppose that $f^i \in BMO$, $1 \leqslant i \leqslant n$. Let v^i be the caloric extension. Then

$$|B(x, R)|^{-1} \int_{Q(x, R)} \sum_{i=1}^n |\partial_i v^i|^2\, dx\, dt \leqslant \sum \|f^i\|_{BMO}^2$$

by the definition of BMO. This implies $\nabla \cdot f \in BMO^{-1}$. The converse follows from the following

LEMMA 4.1. *Let $m \in C^\infty(\mathbb{R}^n \backslash \{0\})$ be homogeneous of degree zero. Then*

$$\|m(D_x)\, f\|_{BMO^{-1}} \leqslant c\, \|f\|_{BMO^{-1}}.$$

Indeed, suppose that $u \in BMO^{-1}$. Let $R_{ij} = \partial_i \partial_j \Delta^{-1}$ and

$$u_{ij} = R_{ij} u.$$

Then $u_{ij} \in BMO^{-1}$ and there exist functions f^i with $\partial_j f^i = u_{ij}$ since, by construction $\partial_k u_{ij} = \partial_i u_{kj}$. Now $f^i \in BMO$ by construction and $u = \sum \partial_i f^i$. This completes the proof of Theorem 1.

Proof of Lemma 4.1. Let $u \in BMO^{-1}$ and let v be the caloric extension of u. We need to prove, with $T = m(D_x)$, that

$$|B(x, R)|^{-1} \, \|Tv\|^2_{L^2(Q(x, R))} \leqslant c \, \|u\|^2_{BMO^{-1}}$$

which by rescaling and translation reduces to

$$\|Tv\|_{L^2(Q(0, 1))} \leqslant c \, \|u\|_{BMO^{-1}}.$$

We first claim that

$$|v(x, t)| \leqslant ct^{-1/2} \, \|u\|_{BMO^{-1}}. \tag{22}$$

By scaling and translating it suffices to prove this for $x = 0$ and $t = 1$. There the claim reduces to mere boundedness, which is a consequence of the fact that the heat kernel, evaluated at $t = 1$, lies in the Schwartz space.

For $0 < t \leqslant 1$ write

$$\begin{aligned} Tv(t) &= TS(t)\, u \\ &= T(S(t) - S(1))\, u + TS(1)\, u \\ &= T(S(t) - S(1))\, u + \int_1^\infty T\Delta S(s)\, u \, ds \\ &= T(1 - S(1-t))\, v(t) + \int_1^\infty T\Delta S(s/2)\, v(s/2)\, ds. \end{aligned}$$

The operators $T(1 - S(1-t))$ are bounded in L^2, and their kernels k_t satisfy by (9) the uniform bound

$$|k_t(x)| \leqslant c \, |x|^{-n-2}.$$

This implies that the first term above can be estimated as

$$\|T(1 - S(1-t))\, v(t)\|_{L^2(B(0, 1))} \leqslant c \sup_{x \in \mathbb{R}^n} \|v(t)\|_{L^2(B(x, 1))}$$

uniformly in $0<t<1$. On the other hand, for the second term above we can use (22) to get the stronger L^∞ bound

$$\left\| \int_1^\infty T\,\Delta S(s/2)\, v(s/2)\, ds \right\|_{L^\infty(\mathbb{R}^n)} \leqslant c \int_1^\infty s^{-1}\, \|v(s/2)\|_{L^\infty(\mathbb{R}^n)}\, ds$$

$$\leqslant c\, \|u\|_{BMO^{-1}}$$

which holds since the kernel $k(t)$ of $T\Delta S(t)$ satisfies

$$\|k(t)\|_{L^1(\mathbb{R}^n)} \leqslant ct^{-1}.$$

This completes the proof. ∎

Remark 4.2. This argument has to be modified for local spaces. Then we use

$$Tu = (R^{-2} - \Delta)^{-1}\, u$$

and

$$u = -\nabla(\nabla Tu) + R^{-2} Tu.$$

A modification of the proof above shows that

$$BMO_R^{-1} = \nabla \cdot (BMO_R)^n.$$

Other function spaces. Let $p>n$. The space $B_{p,\infty}^{-1+n/p}(\mathbb{R}^n)$ can be defined in terms of the caloric extension. The tempered distribution u lies in $B_{p,\infty}^{-1+n/p}$ iff its caloric extension v satisfies

$$\|v(t)\|_{L^p(\mathbb{R}^n)} \leqslant ct^{-(1-n/p)/2} \qquad \text{for} \quad 0<t\leqslant 1.$$

The norm can be defined to be the best constant. Let $R\leqslant 1$. Then

$$\left(|B(x,R)|^{-1} \int_{Q(x,R)} |v|^2\, dx\, dt \right)^{1/2}$$

$$\leqslant |B(x,R)|^{-n/p} \left(\int_0^{R^2} \|v(t)\|^2_{L^p(\mathbb{R}^n)}\, dt \right)^{1/2}$$

$$\leqslant \sqrt{p/n}\, |B(x,1)|^{-n/p} \sup_{0<t\leqslant R^2} t^{(1-n/p)/2}\, \|v(t)\|_{L^p(\mathbb{R}^n)}.$$

Moreover, by standard kernel estimates and Young's inequality

$$(4\pi t)^{(1-n/p)/2} \|v(t)\|_{L^p} \leqslant \|u\|_{L^n(\mathbb{R}^n)}$$

hence

$$L^n(\mathbb{R}^n) \subset B^{-1+n/p}_{p,\infty}(\mathbb{R}^n) \subset BMO^{-1}. \tag{23}$$

The Morrey spaces M^p_q $(1 \leqslant q \leqslant n)$ are defined as subspace of $L^1_{loc}(\mathbb{R}^n)$ of functions u for which

$$\sup_{x,R} R^{n/p} \left(|B(x,R)|^{-1} \int_{B(x,R)} |u|^q \, dx \right)^{1/q} < \infty \qquad \text{for} \quad R \leqslant 1.$$

Clearly $M^p_p = L^p_{unif}(\mathbb{R}^n)$ and $u \in M^n_1$ iff $u \in L^1_{unif}$ and $|u| \in B^{-1}_{\infty,\infty}$. It follows from the analysis of Taylor [15] and the previous results that $M^n_q \subset BMO^{-1}_1$ for $1 < q \leqslant n$. Hence his spaces are included in ours. It is clear that his smallness assumption does not imply ours.

REFERENCES

1. M. Ben-Artzi, Global solutions of two-dimensional Navier–Stokes and Euler equations, *Arch. Rational Mech. Anal.* **128** (1994), 329–358.
2. H. Brezis, Remarks on the preceding paper "Global Solutions of Two-Dimensional Navier–Stokes and Euler Equations," *Arch. Rational Mech. Anal.* **128** (1994), 359–360.
3. L. Caffarelli, R. Kohn, and L. Nirenberg, Partial regularity of suitable weak solutions of the Navier–Stokes equations, *Comm. Pure Appl. Math.* **35** (1982), 771–831.
4. M. Cannone, A generalization of a theorem by Kato on Navier–Stokes equations, *Rev. Mat. Iberoam.* **13** (1997), 515–541.
5. Y. Giga and T. Miyakawa, Navier–Stokes flow in $\mathbb{R}^3$ with measures as initial vorticity and Morrey spaces, *Comm. Partial Differential Equations* **14** (1989), 577–618.
6. Y. Giga, T. Miyakawa, and H. Osada, Two-dimensional Navier–Stokes flow with measures as initial vorticity, *Arch. Rational Mech. Anal.* **104** (1988), 223–250.
7. D. Iftimie, The resolution of the Navier–Stokes equations in anisotropic spaces, *Rev. Mat. Iberoam.* **15** (1999), 1–36.
8. T. Kato, Strong L^p-solutions of the Navier–Stokes equation in $\mathbb{R}^m$, with applications to weak solutions, *Math. Z.* **187** (1984), 471–480.
9. T. Kato and G. Ponce, Commutator estimates and the Euler and Navier–Stokes equations, *Comm. Pure Appl. Math.* **41** (1988), 891–907.
10. F. Lin, A new proof of the Caffarelli–Kohn–Nirenberg theorem, *Comm. Pure Appl. Math.* **51** (1998), 241–257.
11. P.-L. Lions and N. Masmoudi, Unicité des solutions faibles de Navier–Stokes dans $L^N(\Omega)$, *C. R. Acad. Sci. Paris Ser. I Math.* **327** (1998), 491–496.
12. F. Planchon, Global strong solutions in Sobolev or Lebesgue spaces to the incompressible Navier–Stokes equations in $\mathbf{R}^3$, *Ann. Inst. Henri Poincare, Anal. Non Lineaire* **13** (1996), 319–336.

13. E. M. Stein, "Harmonic Analysis," Princeton Mathematical Series, Vol. 43, Princeton University Press, Princeton, 1993.
14. M. Struwe, On partial regularity results for the Navier–Stokes equations, *Comm. Pure Appl. Math.* **41** (1988), 437–458.
15. M. Taylor, Analysis on Morrey spaces and applications to Navier–Stokes equation, *Comm. Partial Differential Equations* **17** (1992), 1407–1456.
16. S. Wu, Analytic dependence of Riemann mappings for bounded domains and minimal surfaces, *Comm. Pure Appl. Math.* **46** (1993), 1303–1326.