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Simplicial complexes and minimal free resolution of monomial algebras

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ABSTRACT

This paper is concerned with the combinatorial description of the graded minimal free resolution of certain monomial algebras which includes toric rings. We explicitly describe how the graded minimal free resolution of those algebras is related to the combinatorics of some simplicial complexes. Our description may be interpreted as an algorithmic procedure to partially compute this resolution.

0. Introduction

Let I be an ideal in a polynomial ring R over a field $\Bbbk$. The $\Bbbk$-algebra R/I is said to be monomial, if the algebraic set $\mathcal{V}(I)$ is parameterized by monomials.

Let R/I be a monomial algebra. Since monomial algebras are semigroup algebras, one can consider a semigroup S to study R/I. This approach makes it possible to define a particular S-grading on the monomial algebra R/I which allows defining the S-graded minimal free resolution of R/I as an R-module, under some reasonable hypothesis on S (see Section 1). This graded minimal free resolution of R/I has been explored by many authors with remarkable success (see e.g. [1] and the references therein).

The study of the graded minimal free resolution of the monomial algebra R/I from a semigroup viewpoint facilitates the use of methods based on the knowledge of the combinatorics of the semigroup. This paper is focused on this direction.

In this paper, we consider the simplicial complexes introduced by Eliahou in his Ph.D. Thesis [2] and we show how their reduced jth homology vector spaces over $\Bbbk$ are related with the jth module of syzygies appearing in the graded minimal free resolution of certain monomial algebras (see Corollary 4). Of course, this is not a very surprising theoretical result. A similar result was given by Briales et al. in [3], although they used different simplicial complexes. In fact, we prove that the reduced homology of both simplicial complexes are isomorphic (Theorem 3) and then, we use the results in [3] to obtain Corollary 4. Therefore, in this part, our main contribution should be regarded as showing the utility of Eliahou's simplicial complexes for studying monomial algebras.

It is convenient to note that in some cases Eliahou's simplicial complexes behave better than those used in [3], and in some other cases the latter are easier to handle. For instance, when the minimal syzygies are concentrated in small S-degrees, Eliahou's simplicial complexes seem to be the right choice. This is the case when I generated by its indispensable binomials (see, e.g. [4,5]), which is of special interest in Algebraic Statistics and includes generic lattice ideals ([6]) and Lawrence type semigroup ideals ([7,8]).

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The second part of the paper (Section 4) is devoted to the explicit computation of minimal systems of generators of the jth module of syzygies N_j of a monomial algebra with associated semigroup S. The main problem one encounters in known combinatorial algorithms is determining *a priori* the S-degrees in which the minimal generators of the syzygies are concentrated. The information provided by the simplicial complexes used by Briales et al. is clearly insufficient (see [3, 9]). Therefore, we propose a different approach (Theorem 8, Proposition 16 and Corollary 19): we fix one S-degree m and compute a subset of minimal generators of N_j by constructing Eliahou's simplicial complex associated to m and by choosing suitable bases for certain $\Bbbk$-vector spaces. This result does not solve the general problem, but opens new doors to the combinatorial description of monomial (semigroup) algebras. In particular, an algorithmic procedure to compute a chain complex of free R-modules contained in minimal free resolution of a monomial algebra R/I is described.

1. Preliminaries

In this section, we summarize some definitions and results that are useful for the understanding of this work. We refer to the readers to [10–13] for the details and much more.

Let S denote a commutative semigroup with zero element $0 \in S$. Let $G(S)$ be a commutative group with a semigroup homomorphism $\iota : S \to G(S)$ such that every homomorphism from S to a group factors in a unique way through ι. The commutative group $G(S)$ exists and is unique up to isomorphism; it is called the associated commutative group of S. Further, $G(S)$ is finitely generated when S is. The map ι is injective if, and only if, S is cancellative, that is to say, if $m+n = m+n'$, m, n and $n' \in S$, implies $n = n'$, in this case, $G(S)$ is the smallest group containing S.

For the purpose of this paper, we will assume that S is combinatorially finite,[1] i.e., there are only finitely many ways to write $m \in S$ as a sum $m = m_1 + \cdots + m_q$, with $m_i \in S \setminus \{0\}$. Equivalently, S is combinatorially finite if, and only if, $S \cap (-S) = \{0\}$ (see Proposition 1.1 in [14]). Notice that this property guarantees that $m' \prec_S m \iff m - m' \in S$ is a well defined partial order on S.

From now on, S will denote a finitely generated, combinatorially finite, cancellative and commutative semigroup with zero element. We write $\Bbbk[S]$ for the $\Bbbk$-vector space

$$\Bbbk[S] = \bigoplus_{m \in S} \Bbbk \chi^m$$

endowed with a multiplication which is $\Bbbk$-linear and such that $\chi^m \cdot \chi^n := \chi^{m+n}$, m and $n \in S$. Thus $\Bbbk[S]$ has a natural $\Bbbk$-algebra structure and we will refer to it as the semigroup algebra of S.

In addition, we will fix a system of nonzero generators $n_1, \ldots, n_r$ for S. Thus, $\Bbbk[S]$ may be regarded as the monomial $\Bbbk$-algebra generated by $\chi^{n_1}, \ldots, \chi^{n_r}$.

Moreover, this choice of generators induces a natural S-grading on $R = \Bbbk[x_1, \ldots, x_r]$, by assigning weight n_i to x_i, $i = 1, \ldots, r$, that is to say,

$$R = \bigoplus_{m \in S} R_m,$$

where R_m is the vector subspace of R generated by all the monomials $\mathbf{x}^\alpha := x_1^{a_1} \cdots x_r^{a_r}$ with $\sum_{i=1}^r a_i n_i = m$ and $\alpha = (a_1, \ldots, a_n)$. Since S is combinatorially finite, the vector spaces R_m are finite dimensional (see Proposition 1.2 in [14]). We will denote by $\mathfrak{m}$ the irrelevant ideal of R, that is to say, $\mathfrak{m} = \bigoplus_{m \in S \setminus \{0\}} R_m = (x_1, \ldots, x_r)$.

1.1. Minimal resolution

The surjective $\Bbbk$-algebra morphism

$$\varphi_0 : R \longrightarrow \Bbbk[S];\ x_i \longmapsto \chi^{n_i}$$

is S-graded, thus, the ideal $I_S := \ker(\varphi_0)$ is a S-homogeneous ideal called the ideal of S. Notice that I_S is a toric ideal (in the sense of [7] chapter 4) generated by

$$\left\{ \mathbf{x}^\alpha - \mathbf{x}^\beta : \sum_{i=1}^r a_i n_i = \sum_{i=1}^r b_i n_i \in S \right\}.$$

Now, by using the S-graded Nakayama's lemma recursively (see Proposition 1.4 in [14]), we may construct S-graded $\Bbbk$-algebra homomorphism

$$\varphi_{j+1} : R^{s_{j+1}} \longrightarrow R^{s_j},$$

[1] Some authors call this property "positivity" or "reducedness".

corresponding to a choice of a minimal set of S-homogeneous generators for each module of syzygies $N_j := \ker(\varphi_j)$, notice that $N_0 = I_S$. Thus, we obtain a minimal free S-graded resolution for the R-module $\Bbbk[S]$ of the form

$$\ldots \longrightarrow R^{s_{j+1}} \xrightarrow{\varphi_{j+1}} R^{s_j} \longrightarrow \ldots \longrightarrow R^{s_2} \xrightarrow{\varphi_2} R^{s_1} \xrightarrow{\varphi_1} R \xrightarrow{\varphi_0} \Bbbk[S] \longrightarrow 0,$$

where $s_{j+1} := \sum_{m\in S} \dim_{\Bbbk} V_j(m)$, with $V_j(m) := (N_j)_m/(\mathfrak{m}N_j)_m$, is the so-called $(j+1)$th Betti number. Observe that the dimension of $V_j(m)$ is the number of generators of degree m in a minimal system of generators of the jth module of syzygies N_j (i.e. the multigraded Betti number $s_{j,m}$), so, by the Noetherian property of R, s_{j+1} is finite. Moreover, the Auslander–Buchsbaum formula assures that $s_j = 0$ for $j > p = r - \mathrm{depth}_R \Bbbk[S]$ and $s_p \neq 0$. (cf. Theorem 1.3.3 in [11]).

1.2. *Simplicial homology*

Let K be a finite simplicial complex on $[n] := \{1, \ldots, n\}$. For each integer i, let $\mathcal{F}_i(K)$ be the set of i-dimensional faces of K, and let $\Bbbk^{\mathcal{F}_i(K)}$ be a $\Bbbk$-vector space whose basis element $\mathbf{e}_F$ correspond to i-faces $F \in \mathcal{F}_i(K)$.

The reduced chain complex of K over $\Bbbk$ is the complex $\widetilde{C}_\bullet(K)$:

$$0 \to \Bbbk^{\mathcal{F}_{n-1}(K)} \xrightarrow{\partial_{n-1}} \ldots \longrightarrow \Bbbk^{\mathcal{F}_i(K)} \xrightarrow{\partial_i} \Bbbk^{\mathcal{F}_{i-1}(K)} \longrightarrow \ldots \xrightarrow{\partial_0} \Bbbk^{\mathcal{F}_{-1}(K)} \to 0.$$

The boundary maps ∂_i are defined by setting $\mathrm{sing}(j, F) = (-1)^{r-1}$ if j is the rth element of the set $F \subseteq [n]$, written in increasing order, and

$$\partial_i(\mathbf{e}_F) = \sum_{\substack{j \in F \\ \#F = i+1}} \mathrm{sing}(j, F)\mathbf{e}_{F\setminus j}.$$

For each integer i, the $\Bbbk$-vector space

$$\widetilde{H}_i(K) = \ker(\partial_i)/\mathrm{im}(\partial_{i+1})$$

in homological degree i is the ith reduced homology of K. Elements of $\widetilde{Z}_i(K) := \ker(\partial_i)$ are called i-cycles and elements of $\widetilde{B}_i(K) := \mathrm{im}(\partial_{i+1})$ are called i-boundaries.

2. Simplicial complexes and minimal syzygies

In this section, we will consider two different simplicial complexes associated with S and compare their homologies. The first simplicial complex was introduced by Eliahou in [2] and the second one is used in [3] to describe the minimal free resolution of $\Bbbk[S]$.

For any $m \in S$, let $C_m = \{\mathbf{x}^\alpha = x_1^{a_1} \cdots x_r^{a_r} \mid \sum_{i=1}^r a_i n_i = m\}$ and define the abstract simplicial complex on the vertex set C_m,

$$\nabla_m = \{F \subseteq C_m \mid \gcd(F) \neq 1\},$$

where $\gcd(F)$ denotes the greatest common divisor of the monomials in F. Notice that ∇_m has finitely many vertices because S is combinatorially finite.

For any $m \in S$, we consider the abstract simplicial complex on the vertex set $[r]$,

$$\Delta_m = \{F \subseteq [r] \mid m - n_F \in S\},$$

where $n_F = \sum_{i\in F} n_i$.

Now, we are going to compare $\widetilde{H}_\bullet(\nabla_m)$ with $\widetilde{H}_\bullet(\Delta_m)$. To facilitate our work, we recall the so-called "Nerve Lemma".

Definition 1. A cover of a simplicial complex K is a family of subcomplexes $\mathcal{K} = \{K_\alpha \mid \alpha \in A\}$ with $K = \bigcup_{\alpha\in A} K_\alpha$.

We say that the cover $\mathcal{K}$ satisfies the Leray property if each non-empty finite intersection $K_{\alpha_1} \cap \ldots \cap K_{\alpha_q}$ is acyclic.

Definition 2. Let $\mathcal{K} = \{K_\alpha \mid \alpha \in A\}$ be a cover of a simplicial complex K. The nerve of $\mathcal{K}$, denoted by $N_{\mathcal{K}}$, is the simplicial complex having vertices A and with $\{\alpha_1, \ldots, \alpha_q\}$ being a simplex if $\bigcap_{i=1}^q K_{\alpha_i} \neq \varnothing$.

Nerve Lemma. *Assume that $\mathcal{K} = \{K_\alpha \mid \alpha \in A\}$ is a cover of a simplicial complex K. If $\mathcal{K}$ satisfies the Leray property, then*

$$H_j(N_{\mathcal{K}}) \cong H_j(K),$$

for all $j \geq 0$.

Proof. See Theorem 7.26 in [15]. □

Theorem 3. $\widetilde{H}_j(\nabla_m) \cong \widetilde{H}_j(\Delta_m)$, *for all $j \geq 0$ and $m \in S$.*

Proof. For each $\mathbf{x}^\alpha \in C_m$, define the simplicial complex $K_\alpha = \mathcal{P}(\text{supp}(\mathbf{x}^\alpha))$ to be the full subcomplex of Δ_m with vertices $\text{supp}(\mathbf{x}^\alpha)$. Set $\mathcal{K}^m = \{K_\alpha : \mathbf{x}^\alpha \in C_m\}$.

On the one hand, we have that $F \in \Delta_m$, i.e., $m - n_F \in S$ if, and only, if, there exists $\mathbf{x}^\alpha \in C_m$ with $\text{supp}(\mathbf{x}^\alpha) \supseteq F$, therefore, $\mathcal{K}^m$ is a cover of Δ_m.

Moreover, $\bigcap_{i=1}^q K_{\alpha_i} \neq \varnothing$ if, and only if, $\gcd(\mathbf{x}^{\alpha_1}, \ldots, \mathbf{x}^{\alpha_q}) \neq 1$, so ∇_m is the nerve of $\mathcal{K}^m$. Finally, since the cover $\mathcal{K}^m$ of Δ_m satisfies the Leray property, because $\bigcap_{i=1}^q K_{\alpha_i} \neq \varnothing$ is a full simplex, by the Nerve Lemma, we may conclude the existence of the desired isomorphism. □

The above theorem has been proved independently by Charalambous and Thoma (see Theorem 3.2 in [16]).

By Theorem 2.1 in [3], one has that $V_j(m) \cong \widetilde{H}_j(\Delta_m)$, for all $m \in S$. Therefore, we have the following straightforward consequence:

Corollary 4. *$\widetilde{H}_j(\nabla_m) \cong V_j(m)$, for all $j \geq 0$ and $m \in S$.*

Notice that the above corollary assures that the multigraded Betti number $s_{j+1,m}$ equals the rank of the j-reduced homology group $\widetilde{H}_j(\nabla_m)$ of the simplicial complex ∇_m, for every $m \in S$.

Furthermore, we emphasize that Corollary 2.2, 2.3 in [3] may be written in terms of the complexes $\nabla_\bullet$, by simply using Theorem 3; and thus, they give corresponding necessary and sufficient combinatorial conditions for $\Bbbk[S]$ to be Cohen–Macaulay or Gorenstein. For instance, $\Bbbk[S]$ is Cohen–Macaulay if, and only if, $\widetilde{H}_{r-d}(\nabla_m) = 0$, for every $m \in S$, where $d = \text{rank}(G(S))$. In this case, the Cohen–Macaulay type of $\Bbbk[S]$ is

$$s_{r-d} = \sum_{m \in S} \dim \widetilde{H}_{r-d-1}(\nabla_m).$$

3. On the computation of $\widetilde{H}_j(\nabla_m)$.

A key point to our results in the next section lies in the assumption that we are able to compute (and fix) a particular basis for the $\Bbbk$-vector space $\widetilde{Z}_j(\nabla_m)$, for each $j \geq -1$ and $m \in S$.

To do this we consider the reduced chain complex $\widetilde{C}_\bullet(\nabla_m)$, $m \in S$, as defined in Section 1.2 and order the faces according to a (fixed) criterion, e.g. by choosing a monomial term order $\prec$ on R. Indeed, $\prec$ induces a well ordering on the j-dimensional faces: $F < F'$ if, and only if, the leading coefficient of $\sum_{\mathbf{x}^\alpha \in F'} \mathbf{x}^\alpha - \sum_{\mathbf{x}^\beta \in F} \mathbf{x}^\beta$ is positive.

Thus, by ordering all the j-dimensional faces decreasingly according to the chosen criterion a basis $\mathcal{B}_j = \{F_1^{(j)}, \ldots, F_{d_j}^{(j)}\}$ of $\Bbbk^{\mathcal{F}_j(\nabla_m)}$ is fixed, for each $j \geq 0$ and $m \in S$.

Let $A_j \in \mathbb{Z}^{d_{j-1} \times d_j}$ be the matrix of ∂_j with respect to $\mathcal{B}_j$ and $\mathcal{B}_{j-1}$, $j \geq 0$. By performing Gaussian elimination on A_j two invertible matrices P_j and Q_j are obtained such that

$$P_j^{-1} A_j Q_j = \left(\begin{array}{c|c} I_{r_j} & 0 \\ \hline 0 & 0 \end{array}\right) \in \mathbb{Z}^{d_{j-1} \times d_j},$$

where I_{r_j} is the identity matrix of order $r_j = \text{rank}(A_j)$, $j \geq 0$. Then, the first r_j columns of P_j are the coordinates with respect to $\mathcal{B}_{j-1}$ of a basis of $\widetilde{B}_{j-1}(\nabla_m) = \text{im}(\partial_j)$ and the last $d_j - r_j$ columns of Q_j are the coordinates with respect to $\mathcal{B}_j$ of a basis of $\widetilde{Z}_j(\nabla_m) = \ker(\partial_j)$, for each $j \geq 0$ and $m \in S$.

Now, since $\widetilde{B}_j(\nabla_m) \subseteq \widetilde{Z}_j(\nabla_m)$, by using the bases obtained above and elementary linear algebra, we can extend the basis of $\widetilde{B}_j(\nabla_m)$ to a basis of $\widetilde{Z}_j(\nabla_m)$, for each $j \geq 0$ and $m \in S$.

Therefore, we may construct a $\Bbbk$-basis

$$\left\{\widehat{\mathbf{h}}_1^{(j)}, \ldots, \widehat{\mathbf{h}}_{t'_j}^{(j)}, \widehat{\mathbf{b}}_1^{(j)}, \ldots, \widehat{\mathbf{b}}_{t''_j}^{(j)}\right\} \tag{1}$$

of $\widetilde{Z}_j(\nabla_m)$ such that

(a) $\widehat{\mathbf{h}}_i^{(j)} = \sum_{k=1}^{d_j} q_{ki}^{(j)} \partial_{j+1}\left(F_k^{(j+1)}\right)$, where $q_{ki}^{(j)}$ is the (k, i)th entry of Q_j, $i = 1, \ldots, t'_j$.

(b) the classes of $\widehat{\mathbf{b}}_1^{(j)}, \ldots, \widehat{\mathbf{b}}_{t''_j}^{(j)}$ modulo $\widetilde{B}_j(\nabla_m)$ form a $\Bbbk$-basis of $\widetilde{H}_j(\nabla_m)$,

for each $j \geq 0$ and $m \in S$.

Remark 5. Since $A_0 = (1\ 1\ \ldots\ 1) \in \mathbb{Z}^{1 \times d_0}$, we may assume that the corresponding basis for $\widetilde{Z}_0(\nabla_m)$ is $\big\{\{\mathbf{x}^{\beta_1}\} - \{\mathbf{x}^\alpha\}, \ldots, \{\mathbf{x}^{\beta_{d_0}}\} - \{\mathbf{x}^\alpha\}\big\}$, with $\mathbf{x}^\alpha \succ \mathbf{x}^{\beta_1} \succ \ldots \succ \mathbf{x}^{\beta_{d_0}}$. So, $\widehat{\mathbf{b}}_i^{(0)} = \{\mathbf{x}^{\beta_{k_i}}\} - \{\mathbf{x}^\alpha\}$ for some $k_i \in \{1, \ldots, d_0\}$.

Notice that this general construction can be also applied to compute a $\Bbbk$-basis of $\widetilde{H}_j(\Delta_m)$. In any case, the computation of $\widetilde{H}_j(\nabla_m)$ and $\widetilde{H}_j(\Delta_m)$ is equally difficult (see [9] for a different approach on the computation of $\widetilde{H}_j(\Delta_m)$).

4. Computing syzygies from combinatorics

In this section, we will explicitly describe the isomorphisms whose existence we have proved in Corollary 4.

We will start by giving an isomorphism $\widetilde{H}_0(\nabla_m) \overset{\sigma_0}{\cong} V_0(m)$. As the reader can note, the construction of σ_0 follows from the definition of $\widetilde{H}_0(\nabla_m) = \widetilde{Z}_0(\nabla_m)/\widetilde{B}_0(\nabla_m)$ and $V_0(m) = (N_0)_m/(\mathfrak{m}N_0)_m$. However, we will give the construction by taking in mind the general case in order to introduce the notation of this section.

First of all, consider the $\Bbbk$-linear map

$$\psi_0 : \Bbbk^{\mathcal{F}_0(\nabla_m)} \longrightarrow R;\ \{\mathbf{x}^\alpha\} \longmapsto \mathbf{x}^\alpha. \tag{2}$$

This map induces an isomorphism from $\widetilde{Z}_0(\nabla_m)$ to $(N_0)_m$. More precisely,

$$\widetilde{Z}_0(\nabla_m) \longrightarrow (N_0)_m;\ \widehat{b} := \{\mathbf{x}^\alpha\} - \{\mathbf{x}^\beta\} \longmapsto b := \mathbf{x}^\alpha - \mathbf{x}^\beta,$$

recall that $\widetilde{Z}_0(\nabla_m)$ is generated by $\{\mathbf{x}^\alpha\} - \{\mathbf{x}^\beta\}$, with $\mathbf{x}^\alpha$ and $\mathbf{x}^\beta \in C_m$ (see Remark 5), and $(N_0)_m$ is generated by binomials of S-degree equals m (see Section 1.1).

Therefore, we have a surjective map $\overline{\psi}_0$ given by the composition

$$\widetilde{Z}_0(\nabla_m) \longrightarrow (N_0)_m \longrightarrow V_0(m) = (N_0)_m/(\mathfrak{m}N_0)_m.$$

Lemma 6. $\widetilde{B}_0(\nabla_m) \subseteq \ker \overline{\psi}_0$.

Proof. Since $\widehat{f} \in \widetilde{B}_0(\nabla_m) = \mathrm{im}(\partial_1)$, there exist $\{\mathbf{x}^{\alpha_j}, \mathbf{x}^{\beta_j}\} \in \nabla_m$ and $\mu_j \in \Bbbk$, such that

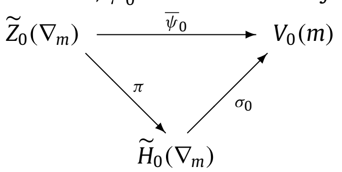
$$\partial_1\left(\sum_j \mu_j\{\mathbf{x}^{\alpha_j}, \mathbf{x}^{\beta_j}\}\right) = \sum_j \mu_j\left(\{\mathbf{x}^{\beta_j}\} - \{\mathbf{x}^{\alpha_j}\}\right) = \widehat{f}.$$

So,

$$f = \overline{\psi}_0(\widehat{f}) = \sum_j \mu_j\left(\mathbf{x}^{\beta_j} - \mathbf{x}^{\alpha_j}\right) = \sum_j \mu_j \mathbf{x}^{\gamma_j}\left(\underbrace{\mathbf{x}^{\beta'_j} - \mathbf{x}^{\alpha'_j}}_{\in N_0}\right),$$

with $\mathbf{x}^{\gamma_j} = \gcd(\mathbf{x}^{\alpha_j}, \mathbf{x}^{\beta_j})$, $\mathbf{x}^{\alpha'_j} = \mathbf{x}^{\alpha_j}/\mathbf{x}^{\gamma_j}$ and $\mathbf{x}^{\beta'_j} = \mathbf{x}^{\beta_j}/\mathbf{x}^{\gamma_j}$. Moreover, $\mathbf{x}^{\gamma_j} \neq 1$, because $\{\mathbf{x}^{\alpha_j}, \mathbf{x}^{\beta_j}\}$ is an edge of ∇_m. Thus, $f \in (\mathfrak{m}N_0)_m$ as claimed. □

By Lemma 6, $\overline{\psi}_0$ factors canonically through $\widetilde{H}_0(\nabla_m)$:



Notice, that σ_0 is an isomorphism because it is surjective and, by Corollary 4, $\dim \widetilde{H}_0(\nabla_m) = \dim V_0(m)$.

Now we will show a combinatorial method to compute some minimal binomial generators of I_S from a given binomial in I_S. But first, we will introduce an important property of the complexes ∇_m which claims that $\nabla_{m'}$ can be easily computed from ∇_m, for every $m' \prec_S m$, i.e., if $m - m' \in S$.

Lemma 7. *Let m and $m' \in S$. If $m' \prec_S m$, then*

$$\nabla_{m'} \cong \{F \in \nabla_m \mid \mathbf{x}^\beta \textit{ properly divides } \gcd(F)\},$$

for any (fixed) monomial $\mathbf{x}^\beta \in C_{m-m'}$.

Proof. Let $\mathbf{x}^\beta$ a monomial in $C_{m-m'}$. If $F' = \{\mathbf{x}^{\alpha'_1}, \ldots, \mathbf{x}^{\alpha'_t}\} \in \nabla_{m'}$, then $F = \{\mathbf{x}^{\alpha'_1+\beta}, \ldots, \mathbf{x}^{\alpha'_t+\beta}\} \in \nabla_m$. Conversely, consider $F = \{\mathbf{x}^{\alpha_1}, \ldots, \mathbf{x}^{\alpha_t}\} \in \nabla_m$ such that $\mathbf{x}^\beta$ divides $\gcd(F)$ and $\gcd(F) \neq \mathbf{x}^\beta$. Since $\mathbf{x}^\beta$ divides $\mathbf{x}^{\alpha_i}$, $i = 1, \ldots, t$, and $\gcd(F) \neq \mathbf{x}^\beta$, we conclude that $F' = \{\mathbf{x}^{\alpha_1-\beta}, \ldots, \mathbf{x}^{\alpha_t-\beta}\}$ is a face of $\nabla_{m'}$. □

Theorem 8. *Let $m \in S$ and let ∇_m be given. For each $\mathbf{x}^\alpha - \mathbf{x}^\beta \in (I_S)_m$, there exist a uniquely determined subset $\mathcal{B} = \{b_1, \ldots, b_t\}$ of a minimal system of binomial generators of I_S and uniquely determined elements $f_1, \ldots, f_t \in R$, such that*

(a) $\mathbf{x}^\alpha - \mathbf{x}^\beta = \sum_{i=1}^t f_i b_i$,
(b) $\gcd(\mathbf{x}^\alpha, \mathbf{x}^\beta)$ *divides* f_i, $i = 1, \ldots, t$.

Proof. We divide the proof into two steps.

STEP 1. Write

$$\mathbf{x}^\alpha - \mathbf{x}^\beta = \mathbf{x}^\gamma(\mathbf{x}^{\alpha'} - \mathbf{x}^{\beta'}),$$

where $\mathbf{x}^\gamma = \gcd(\mathbf{x}^\alpha, \mathbf{x}^\beta)$, $\mathbf{x}^{\alpha'} = \mathbf{x}^\alpha/\mathbf{x}^\gamma$ and $\mathbf{x}^{\beta'} = \mathbf{x}^\beta/\mathbf{x}^\gamma$. Notice that $\{\mathbf{x}^\alpha\}$ and $\{\mathbf{x}^\beta\}$ are adjacent in ∇_m when $\mathbf{x}^\gamma \neq 1$, and that $\{\mathbf{x}^{\alpha'}\}$ and $\{\mathbf{x}^{\beta'}\}$ are never adjacent in $\nabla_{m'}$, where m' is the S-degree of $\mathbf{x}^{\alpha'}$ (and $\mathbf{x}^{\beta'}$, of course). Moreover, $m' \prec_S m$, when $\mathbf{x}^\gamma \neq 1$. In this case, we consider the simplicial complex $\nabla_{m'}$ (computed from ∇_m by using Lemma 7) and the binomial $\mathbf{x}^{\alpha'} - \mathbf{x}^{\beta'} \in (I_S)_{m'}$.

STEP 2. For simplicity, by Step 1, we may assume that $\{\mathbf{x}^\alpha\}$ and $\{\mathbf{x}^\beta\}$ are not adjacent.

Let $\{\widehat{h}_1, \ldots, \widehat{h}_{t'}, \widehat{b}_1, \ldots, \widehat{b}_{t''}\}$ be a $\Bbbk$-basis of $\widetilde{Z}_0(\nabla_m)$ constructed as in Section 3. Then, $\widehat{h}_j = \sum_{k=1}^{d_1} q_{kj}^{(0)} \partial_1(F_k^{(1)}) \in \widetilde{B}_0(\nabla_m)$, for every j, where $\mathcal{F}_1(\nabla_m) = \{F_1^{(1)}, \ldots, F_{d_1}^{(1)}\}$, and the classes of $\widehat{b}_1, \ldots, \widehat{b}_{t''}$ modulo $\widetilde{B}_0(\nabla_m)$ form a basis of $\widetilde{H}_0(\nabla_m)$.

Since $\{\mathbf{x}^\alpha\} - \{\mathbf{x}^\beta\} \in \widetilde{Z}_0(\nabla_m)$ and $\{\widehat{h}_1, \ldots, \widehat{h}_{t'}, \widehat{b}_1, \ldots, \widehat{b}_{t''}\}$ is a $\Bbbk$-basis of $\widetilde{Z}_0(\nabla_m)$ there exist uniquely determined $\lambda_i, \mu_j \in \Bbbk$, $i = 1, \ldots, t'$ and $j = 1, \ldots, t''$, such that

$$\{\mathbf{x}^\alpha\} - \{\mathbf{x}^\beta\} = \sum_i \lambda_i \widehat{b}_i + \sum_j \mu_j \widehat{h}_j = \sum_i \lambda_i \widehat{b}_i + \sum_k \left(\sum_j \mu_j q_{kj}^{(0)}\right) \widehat{g}_k$$

where $\widehat{g}_k := \partial_1(F_k^{(1)}) = \{\mathbf{x}^{\alpha_k}\} - \{\mathbf{x}^{\beta_k}\} \in \widetilde{Z}_0(\nabla_m)$, $k = 1, \ldots, d_1$.

By Remark 5, we have that $\widehat{b}_i$, is a pure difference of vertices in ∇_m for every $i = 1, \ldots, t''$. Therefore,

$$\mathbf{x}^\alpha - \mathbf{x}^\beta = \psi_0(\{\mathbf{x}^\alpha\} - \{\mathbf{x}^\beta\}) = \sum_i \lambda_i b_i + \sum_k \nu_k g_k, \tag{3}$$

where the b_i's are binomials in R and $\nu_k = \sum_j \mu_j q_{kj}^{(0)} \in \Bbbk$, $k = 1, \ldots, d_1$.

If $\nu_k = 0$, for every k, we are done. Otherwise, we repeat this procedure (starting from Step 1) for each $g_k = \mathbf{x}^{\alpha_k} - \mathbf{x}^{\beta_k}$ with $\nu_k \neq 0$. Since $\gcd(\mathbf{x}^{\alpha_k}, \mathbf{x}^{\beta_k}) \neq 1$, the S-degree of the binomial produced in Step 1 will be strictly smaller than the degree of g_k, so, we may guarantee that this process ends in finitely many iterations.[2]

Finally, notice that we have considered a particular basis for each $\widetilde{Z}_0(\nabla_\bullet)$ appearing in. Thus, our computation depends on the choice of these bases. However, no other choice has been made. Thus, by assuming fixed basis for each $\widetilde{Z}_0(\nabla_\bullet)$ (see Section 3), we may guarantee that b_i's and f_i's are uniquely obtained. □

Observe that the proof of Theorem 8 may be considered as an algorithm which effectively computes a subset of a minimal set of binomial generators of I_S starting from any binomial of S-degree m.

Remark 9. In Theorem 8, binomials of the same S-degree in the sets $\mathcal{B}$'s corresponding to different binomials in $(I_S)_m$ are equal, because they only depend on the chosen bases of $\widetilde{Z}_0(\nabla_\bullet)$ which are fixed. So, in particular, the union of all such sets $\mathcal{B}$'s is a subset, say $\{b_1, \ldots, b_s\}$, of a minimal system of binomial generators of I_S, that is to say, there is no $\Bbbk$-linear dependence among $b_1, \ldots, b_s$.

Thus, an extensive use of Theorem 8 uniquely determines a subset, $\{b_1, \ldots, b_s\}$, of a minimal system of binomial generators of I_S and, for each $\mathbf{x}^\alpha - \mathbf{x}^\beta \in (I_S)_m$, uniquely defines $f_1, \ldots, f_s \in R$ such that

(a) $\mathbf{x}^\alpha - \mathbf{x}^\beta = \sum_{i=1}^s f_i b_i$,
(b) $\gcd(\mathbf{x}^\alpha, \mathbf{x}^\beta)$ divides f_i, $i = 1, \ldots, s$,

by taking $f_i = 0$ if necessary.

We show with an example how to compute some minimal binomial generators by using the above theorem.

Example 10. Let $S \subset \mathbb{Z}^2$ be the semigroup generated by the columns of the following matrix

$$\begin{pmatrix} 4 & 5 & 7 & 8 \\ 1 & 1 & 1 & 1 \end{pmatrix}.$$

The binomial $g = x_2{}^2x_3{}^6 - x_1{}^3x_4{}^5$ is clearly in $I_S \subset \Bbbk[x_1, \ldots, x_4]$ and its S-degree is $m = (52, 8) = 2\,(5, 1) + 6\,(7, 1) \in S$.

The vertex set[3] of ∇_m is

$$C_m = \left\{x_2{}^2x_3{}^6,\ x_2{}^3x_3{}^3x_4{}^2,\ x_2{}^4x_4{}^4,\ x_1x_2x_3{}^5x_4,\ x_1x_2{}^2x_3{}^2x_4{}^3,\ x_1{}^2x_3{}^4x_4{}^2,\ x_1{}^2x_2x_3x_4{}^4,\ x_1{}^3x_4{}^5\right\}.$$

2 It is convenient to recall that, by Lemma 7, we do not need to compute the new complexes $\nabla_{m'_k}$, for any k.

3 One can compute this set by solving diophantine equations (see [17]).

The simplicial complex ∇_m is clearly connected, so

$$\left\{\{\mathbf{x}^\alpha\} - \{x_2{}^3x_3{}^3x_4{}^2\} \mid \mathbf{x}^\alpha \in C_m \setminus \{x_2{}^3x_3{}^3x_4{}^2\}\right\}$$

is a $\Bbbk$-basis of $\widetilde{B}_0(\nabla_m) = \widetilde{Z}_0(\nabla_m)$ and

$$\begin{aligned}\widehat{g} &= \{x_2{}^2x_3{}^6\} - \{x_1{}^3x_4{}^5\}\\ &= \left(\{x_2{}^2x_3{}^6\} - \{x_2{}^3x_3{}^3x_4{}^2\}\right) - \left(\{x_1{}^3x_4{}^5\} - \{x_2{}^3x_3{}^3x_4{}^2\}\right).\end{aligned}$$

Since $x_2{}^2x_3{}^6 - x_2{}^3x_3{}^3x_4{}^2 = x_2{}^2x_3{}^3\left(x_3{}^3 - x_2x_4{}^2\right)$ and $x_1{}^3x_4{}^5 - x_2{}^3x_3{}^3x_4{}^2 = x_4{}^2\left(x_1{}^3x_4{}^3 - x_2{}^3x_3{}^3\right)$. The binomials that we are interested in now are

$$g_1 = x_3{}^3 - x_2x_4{}^2 \quad \text{and} \quad g_2 = x_1{}^3x_4{}^3 - x_2{}^3x_3{}^3$$

which have S-degree $m_1 = (21, 3)$ and $m_2 = (36, 6) \in S$, respectively.

By Lemma 7, $C_{m_1} = \{x_3{}^3,\ x_2x_4{}^2\}$ and ∇_{m_1} is disconnected. Thus, we conclude that $b_1 = g_1$ is minimal binomial generator of I_S. On the other hand, by Lemma 7 again, we have that

$$C_{m_2} = \{x_2{}^3x_3{}^3,\ x_2{}^4x_4{}^2,\ x_1x_2{}^2x_3{}^2x_4,\ x_1{}^2x_3{}^4, x_1{}^2x_2x_3x_4{}^2,\ x_1{}^3x_4{}^3\}$$

and that ∇_{m_2} is connected. A $\Bbbk$-basis of $\widetilde{B}_0(\nabla_{m_2}) = \widetilde{Z}_0(\nabla_{m_2})$ is

$$\left\{\{\mathbf{x}^\alpha\} - \{x_1x_2{}^2x_3{}^2x_4\} \mid \mathbf{x}^\alpha \in C_{m_2} \setminus \{x_1x_2{}^2x_3{}^2x_4\}\right\}$$

and

$$\begin{aligned}\widehat{g}_2 &= \{x_2{}^3x_3{}^3\} - \{x_1{}^3x_4{}^3\}\\ &= \left(\{x_2{}^3x_3{}^3\} - \{x_1x_2{}^2x_3{}^2x_4\}\right) - \left(\{x_1{}^3x_4{}^3\} - \{x_1x_2{}^2x_3{}^2x_4\}\right).\end{aligned}$$

Since $x_2{}^3x_3{}^3 - x_1x_2{}^2x_3{}^2x_4 = x_2{}^2x_3{}^2\left(x_2x_3 - x_1x_4\right)$ and $x_1{}^3x_4{}^3 - x_1x_2{}^2x_3{}^2x_4 = x_1x_4\left(x_1{}^2x_4{}^2 - x_2{}^2x_3{}^2\right)$ we have to consider now the binomials $g_{21} = x_2x_3 - x_1x_4$ and $g_{22} = x_1{}^2x_4{}^2 - x_2{}^2x_3{}^2$. By using Lemma 7 in order to compute the corresponding simplicial complexes, it is easy to see that $b_2 = g_{21} \in (I_S)_{(12,2)}$ is a minimal binomial generator and that $g_{22} = x_1{}^2x_4{}^2 - x_2{}^2x_3{}^2 \in (I_S)_{(24,4)}$ is not, because

$$C_{m_3} = \{x_1{}^2x_4{}^2, x_1x_2x_3x_4, x_2{}^2x_3{}^2\},$$

with $m_3 = (24, 2)$, and ∇_{m_3} is connected. A $\Bbbk$-basis of $\widetilde{B}_0(\nabla_{m_2}) = \widetilde{Z}_0(\nabla_{m_2})$ is $\left\{\{x_1{}^2x_4{}^2\} - \{x_1x_2x_3x_4\}, \{x_2{}^2x_3{}^2\} - \{x_1x_2x_3x_4\}\right\}$ and then

$$\begin{aligned}\widehat{g}_{22} &= \{x_1{}^2x_4{}^2\} - \{x_2{}^2x_3{}^2\}\\ &= \left(\{x_1{}^2x_4{}^2\} - \{x_1x_2x_3x_4\}\right) - \left(\{x_2{}^2x_3{}^2\} - \{x_1x_2x_3x_4\}\right).\end{aligned}$$

Thus, $g_{22} = x_1x_4(x_1x_4 - x_2x_3) - x_2x_3(x_2x_3 - x_1x_4) = (-x_1x_4 - x_2x_3)(x_2x_3 - x_1x_4)$.

Summarizing, we have obtained two minimal binomial generators, $b_1 = x_3{}^3 - x_2x_4{}^2$ and $b_2 = x_2x_3 - x_1x_4$ of I_S and two polynomials $f_1 = x_2{}^2x_3{}^3$, and $f_2 = x_4^2\left(x_2{}^2x_3{}^2 + x_1x_2x_3x_4 + x_1{}^2x_4{}^2\right)$ such that

$$x_2{}^2x_3{}^6 - x_1{}^3x_4{}^5 = f_1b_1 + f_2b_2.$$

4.1. First syzygies of semigroup ideals

Let $m \in S$ and let ∇_m be given. Let $\{b_1, \ldots, b_{s_1}\}$ denote the subset of a minimal system of binomial generators of I_S obtained by applying Theorem 8 to each $\mathbf{x}^\alpha - \mathbf{x}^\beta \in (I_S)_m$ with $\gcd(\mathbf{x}^\alpha, \mathbf{x}^\beta) \neq 1$ (see Remark 9) and define the $\Bbbk$-linear map

$$\psi_1 : \Bbbk^{\mathcal{F}_1(\nabla_m)} \longrightarrow R^{s_1};\ \{\mathbf{x}^\alpha, \mathbf{x}^\beta\} \longmapsto \mathbf{f} := \begin{pmatrix} f_1\\ \vdots\\ f_{s_1}\end{pmatrix}$$

where f_i is zero and/or given by Theorem 8 from $\mathbf{x}^\alpha - \mathbf{x}^\beta \in (I_S)_m$. Since we are working with fixed bases for $\widetilde{Z}_j(\nabla_\bullet),\ j \geq 0$, we may assure that ψ_1 is well defined (see Remark 9 again).

Example 11. For instance, in Example 10, we have obtained that

$$\psi_1\left(\left\{{x_2}^2{x_3}^6,\ {x_1}^3{x_4}^5\right\}\right) = \begin{pmatrix} {x_2}^2{x_3}^3 \\ {x_2}^2{x_3}^2{x_4}^2 + x_1x_2x_3{x_4}^2 + {x_1}^2{x_4}^4 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \in R^{s_1}.$$

Remark 12. In the following, we will write $[f_1, \ldots, f_s] \in R^s$ instead of to use column-vector notation.

Lemma 13. *The map ψ_1 makes the following diagram commute*

$$\begin{array}{ccc} \Bbbk^{\mathcal{F}_1(\nabla_m)} & \xrightarrow{\ \psi_1\ } & R^{s_1} \\ \Big\downarrow{\scriptstyle \partial_1} & & \Big\downarrow{\scriptstyle \varphi_1} \\ \Bbbk^{\mathcal{F}_0(\nabla_m)} & \xrightarrow{\ \psi_0\ } & R, \end{array} \tag{4}$$

where the bottom row map is defined as in (2) and $\varphi_1(\mathbf{e}_i) = b_i,\ i = 1, \ldots, s_1$.

Proof. Consider $\widehat{\mathbf{f}} = \sum_{j=1}^{d_1} \lambda_j\{\mathbf{x}^{\alpha_j}, \mathbf{x}^{\beta_j}\} \in \Bbbk^{\mathcal{F}_1(\nabla_m)}$. Then,

$$\begin{aligned} \varphi_1 \circ \psi_1(\widehat{\mathbf{f}}) &= \varphi_1\left(\sum_{j=1}^{d_1} \lambda_j[f_{1j}, \ldots, f_{s_1j}]\right) = \sum_{j=1}^{d_1} \lambda_j \sum_{i=1}^{s_1} f_{ij}b_i \\ &= \sum_{j=1}^{d_1} \lambda_j(\mathbf{x}^{\alpha_j} - \mathbf{x}^{\beta_j}) = \psi_0\left(\sum_{j=1}^{d_1} \lambda_j\left(\{\mathbf{x}^{\alpha_j}\} - \{\mathbf{x}^{\beta_j}\}\right)\right) \\ &= \psi_0 \circ \partial_1\left(\sum_{j=1}^{d_1} \lambda_j\left\{\mathbf{x}^{\alpha_j}, \mathbf{x}^{\beta_j}\right\}\right) = \psi_0 \circ \partial_1(\widehat{\mathbf{f}}). \quad \square \end{aligned}$$

Furthermore, one can see that ψ_1 sends 1-cycles to 1-syzygies. Indeed, if $z = \sum_i \lambda_i\{\mathbf{x}^{\alpha_i}, \mathbf{x}^{\alpha_{i+1}}\} \in \widetilde{Z}_1(\nabla_m)$, then $\partial_1(z) = 0$. Thus, if $\psi_1(z) = \sum_i \lambda_i[f_{i1}, \ldots, f_{is_1}]$, we have that

$$0 = \varphi_1\left(\sum_i \lambda_i[f_{i1}, \ldots, f_{is_1}]\right) = \sum_i \lambda_i \sum_j f_{ij}b_j = \sum_j\left(\sum_i \lambda_i f_{ij}\right)b_j. \tag{5}$$

Thus, $[\sum_i \lambda_i f_{i1}, \ldots, \sum_i \lambda_i f_{is_1}]$ is a syzygy, as claimed. The converse is also true in the following sense:

Lemma 14. *The map $\psi_1 : \widetilde{Z}_1(\nabla_m) \longrightarrow (N_1)_m$ is surjective.*

Proof. Let $\mathbf{f} = [f_1, \ldots, f_{s_1}] \in (N_1)_m$. Thus, if $b_i = \mathbf{x}^{\alpha_i} - \mathbf{x}^{\beta_i}$ and $f_i = \sum_j \lambda_{ij}\mathbf{x}^{\gamma_{ij}},\ i = 1, \ldots, s_1$, it follows that

$$\begin{aligned} 0 &= \sum_i f_ib_i = \sum_i\left(\sum_j \lambda_{ij}\mathbf{x}^{\gamma_{ij}}\right)\left(\mathbf{x}^{\alpha_i} - \mathbf{x}^{\beta_j}\right) \\ &= \sum_{i,j} \lambda_{ij}\left(\mathbf{x}^{\gamma_{ij}}\mathbf{x}^{\alpha_i} - \mathbf{x}^{\gamma_{ij}}\mathbf{x}^{\beta_j}\right). \end{aligned}$$

By taking $\widehat{\mathbf{f}} = \sum_{i,j} \lambda_{ij}\{\mathbf{x}^{\gamma_{ij}+\beta_i}, \mathbf{x}^{\gamma_{ij}+\alpha_i}\} \in \widetilde{Z}_1(\nabla_m)$, we conclude that $\mathbf{f} = \psi_1(\widehat{\mathbf{f}})$. $\square$

Thus, we have a surjective map $\overline{\psi}_1$ which is nothing but the composition

$$\widetilde{Z}_1(\nabla_m) \longrightarrow (N_1)_m \longrightarrow V_1(m) = (N_1)_m/(\mathfrak{m}N_1)_m.$$

Lemma 15. $\widetilde{B}_1(\nabla_m) \subseteq \ker \overline{\psi}_1$.

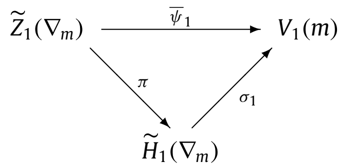
Proof. Since $\widetilde{B}_1(\nabla_m) = \operatorname{im}(\partial_2)$ and ∂_2 is $\Bbbk$-linear, it suffices to prove that $\partial_2(F) \in \ker \overline{\psi}_1$ for any 2-dimensional face F of ∇_m.

Let $F = \{\mathbf{x}^{\alpha_1}, \mathbf{x}^{\alpha_2}, \mathbf{x}^{\alpha_3}\}$ be a 2-dimensional face of ∇_m. Then $\mathbf{x}^{\gamma} = \gcd(F) \neq 1$. Thus, by Theorem 8, there exist $[f_{i1}, \dots, f_{is_1}] \in (N_1)_m,\ i = 1, 2, 3$, such that $\mathbf{x}^{\gamma}$ divides f_{ij} and

$$\begin{aligned}
\overline{\psi}_1(\partial_2(F)) &= \overline{\psi}_1\left(\{\mathbf{x}^{\alpha_2}, \mathbf{x}^{\alpha_3}\} - \{\mathbf{x}^{\alpha_1}, \mathbf{x}^{\alpha_3}\} + \{\mathbf{x}^{\alpha_1}, \mathbf{x}^{\alpha_2}\}\right) \\
&= [f_{11}, \dots, f_{1s_1}] - [f_{21}, \dots, f_{2s_1}] + [f_{31}, \dots, f_{3s_1}] \\
&= \mathbf{x}^{\gamma}\left([f'_{11}, \dots, f'_{1s_1}] - [f'_{21}, \dots, f'_{2s_1}] + [f'_{31}, \dots, f'_{3s_1}]\right) \\
&= \mathbf{x}^{\gamma}[f'_{11} - f'_{21} + f'_{31}, \dots, f'_{1s_1} - f'_{2s_1} + f'_{3s_1}].
\end{aligned}$$

Therefore, $\overline{\psi}_1(\partial_2(F)) \in (\mathfrak{m}N_1)_m$, as claimed. □

By Lemma 15, $\overline{\psi}_1$ factors canonically through $\widetilde{H}_1(\nabla_m)$:



As before, σ_1 is an isomorphism because it is surjective and, by Corollary 4, $\dim \widetilde{H}_1(\nabla_m) = \dim V_1(m)$.

Proposition 16. *Let $m \in S$ and let ∇_m be given. For each $\mathbf{g} := [g_1, \dots, g_{s_1}] \in (N_1)_m$, there exists a uniquely determined subset $\mathcal{B} = \{\mathbf{b}_1^{(1)}, \dots, \mathbf{b}_t^{(1)}\}$ of a minimal system of generators of the first module of syzygies of $\Bbbk[S]$ and uniquely determined elements $f_1, \dots, f_t \in R$ such that*

(a) $\mathbf{g} = \sum_{j=1}^{t} f_j \mathbf{b}_j^{(1)}$,
(b) $\gcd(g_1, \dots, g_{s_1})$ *divides* $f_j,\ j = 1, \dots, t$.

Proof. Write

$$\mathbf{g} = h\,\mathbf{g}',$$

where $h := \gcd(g_1, \dots, g_{s_1})$. Notice that when $h \neq 1$, the S-degree of $\mathbf{g}'$ is strictly smaller than the S-degree of $\mathbf{g}$. In this case, we consider the simplicial complex $\nabla_{m'}$, where m' is the S-degree of $\mathbf{g}'$ (recall that $\nabla_{m'}$ can be computed by using Lemma 7) and $\mathbf{g}' = [g'_1, \dots, g'_{s_1}] \in (N_1)_{m'}$.

For simplicity, we assume that $\gcd(g_1, \dots, g_{s_1}) = 1$, i.e. $\mathbf{g} = \mathbf{g}'$.

Let $\{\widehat{\mathbf{h}}_1, \dots, \widehat{\mathbf{h}}_{t'}, \widehat{\mathbf{b}}_1, \dots, \widehat{\mathbf{b}}_{t''}\}$ be a $\Bbbk$-basis of $\widetilde{Z}_1(\nabla_m)$ constructed as in Section 3. Then $\widehat{\mathbf{h}}_j = \sum_{k=1}^{d_2} q_{kj}^{(1)} \partial_2\left(F_k^{(2)}\right) \in \widetilde{B}_1(\nabla_m)$, for every j, and the classes of $\widehat{\mathbf{b}}_1, \dots, \widehat{\mathbf{b}}_{t''}$ form a basis of $\widetilde{H}_1(\nabla_m)$.

Let $\widehat{\mathbf{g}} \in \widetilde{Z}_1(\nabla_m)$ be the representative of $\psi_1^{-1}(\mathbf{g})$ constructed as in the proof of Lemma 14. Since $\{\widehat{\mathbf{h}}_1, \dots, \widehat{\mathbf{h}}_{t'}, \widehat{\mathbf{b}}_1, \dots, \widehat{\mathbf{b}}_{t''}\}$ is a $\Bbbk$-basis of $\widetilde{Z}_1(\nabla_m)$ there exist uniquely determined λ_i and $\mu_j \in \Bbbk,\ i = 1, \dots, t'$ and $j = 1, \dots, t''$, such that

$$\widehat{\mathbf{g}} = \sum_i \lambda_i \widehat{\mathbf{b}}_i + \sum_j \mu_j \widehat{\mathbf{h}}_j = \sum_i \lambda_i \widehat{\mathbf{b}}_i + \sum_k \left(\sum_j \mu_j q_{kj}^{(1)}\right) \widehat{\mathbf{g}}_k$$

where $\widehat{\mathbf{g}}_k := \partial_2\left(F_k^{(2)}\right)$, $k = 1, \dots d_2$. Therefore

$$\mathbf{g} = \psi_1(\widehat{\mathbf{g}}) = \sum_i \lambda_i \mathbf{b}_i + \sum_k \nu_k \mathbf{g}_k,$$

with $\nu_k = \sum_j \mu_j q_{kj}^{(1)} \in \Bbbk,\ k = 1, \dots, d_2$.

If $\nu_k = 0$, for every k, we are done. Otherwise, we repeat this procedure for $\mathbf{g}_k$ with $\nu_k \neq 0$. Since $\gcd(F_k^{(2)}) \neq 1$ divides $\gcd(\mathbf{g}_k)$, we may assure that this process ends for S-degree reasons.

The uniqueness follows from the same argument as in the proof of Theorem 8. □

Remark 17. Similarly to the case of Theorem 8, we may assure that the subsets $\mathcal{B}$'s are contained in the same minimal system of generators of the first module of syzygies of $\Bbbk[S]$ (see Remark 9). Therefore their union is a subset of a minimal system of generators of the first module of syzygies of $\Bbbk[S]$.

Let us illustrate the above theorem with an example.

Example 18. Let $\mathcal{B} = \{x_2x_3 - x_1x_4,\ x_3^3 - x_2x_4^2,\ x_1x_3^2 - x_2^2x_4,\ x_2^3 - x_1^2x_3\}$ be a minimal generating set of the semigroup ideal I_S of Example 10. Consider $m = (45, 7) \in S$ and

$$\mathbf{g} = [x_2{x_3}^4 + x_1x_2{x_4}^3, -{x_2}^2{x_3}^2 - {x_1}^2{x_4}^2, x_2{x_3}^2x_4 + x_1x_3{x_4}^2, 0] \in (N_1)_m.$$

Let $\widehat{\mathbf{g}} \in \widetilde{Z}_1(\nabla_{(45,7)})$, such that $\mathbf{g} = \psi_1(\widehat{\mathbf{g}})$, be defined as in the proof of Lemma 14:

$$\widehat{\mathbf{g}} = \left\{{x_2}^2{x_3}^5,\ x_1x_2{x_3}^4x_4\right\} + \left\{x_1{x_2}^2x_3{x_4}^3,\ {x_1}^2x_2{x_4}^4\right\} - \left\{{x_2}^2{x_3}^2{x_4}^2,\ x_2{x_3}^5\right\} - \left\{{x_1}^2{x_3}^3{x_4}^2,\ {x_1}^2x_2{x_4}^4\right\} + \cdots$$

As in the proof of Proposition 16, we fix a particular basis of $\widetilde{Z}_1(\nabla_{(45,7)})$ in such way we may write

$$\widehat{\mathbf{g}} = \left(\left\{{x_2}^2{x_3}^5, x_1x_2{x_3}^4x_4\right\} - \left\{{x_2}^2{x_3}^2{x_4}^2, x_2{x_3}^5\right\} + \left\{x_1{x_2}^2x_3{x_4}^2, {x_1}^2{x_3}^3x_4\right\}\right) + \left(\left\{x_1{x_2}^2x_3{x_4}^3, {x_1}^2x_2{x_4}^4\right\}\right.$$
$$\left. - \left\{{x_1}^2{x_3}^3{x_4}^2, {x_1}^2x_2{x_4}^4\right\} + \left\{x_1x_2{x_3}^4x_4, {x_2}^3{x_3}^2{x_4}^2\right\}\right) + \cdots$$

The image by ψ_1 of the first parenthesis is

$$x_3[-x_2{x_3}^3, {x_2}^2x_3, -x_1{x_4}^2, 0] \in N_1.$$

Then by taking $\mathbf{g}_1 = [-x_2{x_3}^3, {x_2}^2x_3, -x_1{x_4}^2, 0] \in (N_1)_{(38,6)}$ we may repeat the above process again. By proceeding similarly with all the other parenthesis, we finally get:

$$\mathbf{g} = x_2{x_3}^2[-{x_3}^2, x_2, -x_4, 0] + x_1{x_4}^2[-x_2x_4, x_1, -x_3, 0].$$

Thus, we have obtained two minimal syzygies in S-degrees $(25, 4)$ and $(26, 4)$, respectively.

4.2. *i*-syzygies of a semigroup ideal

Let $m \in S$ and let ∇_m be given. Let us suppose that

- We are able to compute a $\Bbbk$-linear map $\psi_{i-1} : \Bbbk^{\mathcal{F}_{i-1}(\nabla_m)} \longrightarrow R^{s_{i-1}}$ such that $\psi_i : \widetilde{Z}_{i-1}(\nabla_m) \to (N_{i-1})_m$ is a well defined surjective $\Bbbk$-linear map and $\widetilde{B}_{i-1}(\nabla_m) \subseteq \ker \overline{\psi}_{i-1}$, where $\overline{\psi}_{i-1}$ is the composition $\widetilde{Z}_{i-1}(\nabla_m) \longrightarrow (N_{i-1})_m \longrightarrow V_{i-1}(m) = (N_{i-1})_m/(\mathfrak{m}N_{i-1})_m$.
- For each $\mathbf{g} := [g_1, \ldots, g_{s_{i-1}}] \in (N_{i-1})_m$, we are able to compute a unique subset $\{\mathbf{b}_1^{(i-1)}, \ldots, \mathbf{b}_t^{(i-1)}\}$ of a minimal system of generators of the $(i-1)$th module of syzygies of $\Bbbk[S]$ and unique $f_1, \ldots, f_t \in R$ such that $\mathbf{g} = \sum_{j=1}^t f_j\mathbf{b}_j^{(i-1)}$ and $\gcd(g_1, \ldots, g_{s_{i-1}})$ divides $f_j,\ j = 1, \ldots, t$.

Similarly to the previous case, we assume that a subset, $\{\mathbf{b}_1^{(i-1)}, \ldots, \mathbf{b}_{s_i}^{(i-1)}\}$, of minimal generators of the ith module of syzygies of $\Bbbk[S]$ is obtained from the above hypothetical computation.

Then, we may define the new $\Bbbk$-linear map

$$\psi_i : \Bbbk^{\mathcal{F}_i(\nabla_m)} \longrightarrow R^{s_i};\ F := \{\mathbf{x}^{\alpha_0}, \ldots, \mathbf{x}^{\alpha_i}\} \longmapsto \mathbf{f} := [f_1, \ldots, f_{s_i}], \tag{6}$$

where f_i is given by the above hypothetic computation from $\psi_{i-1} \circ \partial_i(F) \in (N_{i-1})_m$.

Thus, the map ψ_i makes the following diagram commute

$$\begin{array}{ccc} \Bbbk^{\mathcal{F}_i(\nabla_m)} & \xrightarrow{\ \psi_i\ } & R^{s_i} \\ \downarrow{\scriptstyle \partial_i} & & \downarrow{\scriptstyle \varphi_i} \\ \Bbbk^{\mathcal{F}_{i-1}(\nabla_m)} & \xrightarrow{\ \psi_{i-1}\ } & R^{s_{i-1}}, \end{array} \tag{7}$$

where $\varphi_1(\mathbf{e}_k) = b_k,\ k = 1, \ldots, s_{i-1}$.

As in (5), it is easy to see that ψ_i sends i-cycles to i-syzygies. Besides $\psi_i : \widetilde{Z}_i(\nabla_m) \to (N_i)_m$ is surjective. Indeed, given $\mathbf{g} := [g_1, \ldots, g_{s_i}] \in (N_i)_m$ with $g_j = \sum_k \lambda_{jk}\mathbf{x}^{\delta_{jk}} \in R$, one obtains that

$$0 = \sum_j g_j\mathbf{b}_j^{(i-1)} = \sum_j \sum_k \lambda_{jk}\mathbf{x}^{\delta_{jk}}\mathbf{b}_j^{(i-1)},$$

where $\mathbf{x}^{\delta_{jk}}\mathbf{b}_j^{(i-1)} \in (\mathfrak{m}N_{i-1})_m$. Now, let us consider

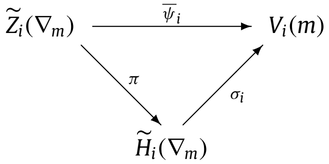
$$\widehat{\mathbf{x}^{\delta_{jk}}\mathbf{b}_j^{(i-1)}} = {\psi_{i-1}}^{-1}(\mathbf{x}^{\delta_{jk}}\mathbf{b}_j^{(i-1)}) \in \widetilde{B}_{i-1}(\nabla_m).$$

This cycle can be written as $\widehat{\mathbf{x}^{\delta_{jk}}\mathbf{b}_j^{(i-1)}} = \sum_l \mu_{jkl}\partial_i(\underbrace{F_l^{(i)}}_{\in \mathcal{F}_i(\nabla_m)})$. Therefore, we conclude that

$$\widehat{\mathbf{g}} = \sum_j \sum_k \lambda_{jk} \sum_l \mu_{jkl} F_l^{(i)} \in \widetilde{Z}_i(\nabla_m) \tag{8}$$

satisfies $\mathbf{g} = \psi_i(\widehat{\mathbf{g}})$, as claimed.

Finally, one can prove that $\widetilde{B}_i(\nabla_m) \subseteq \ker \overline{\psi}_i$ with the same arguments as in Lemma 15. So, we have surjection $\overline{\psi}_i : \widetilde{Z}_i(\nabla_m) \longrightarrow (N_i)_m \longrightarrow V_i(m)$, which factors canonically through $\widetilde{H}_i(\nabla_m)$:



This defines an isomorphism σ_i as desired.

The last ingredient in our construction is the following result which guarantees that we will be able to define ψ_{i+1} in similar terms as we assumed to be possible for ψ_i. This will complete our main objective: to give an explicit description of the isomorphism in Corollary 4.

Corollary 19. *Let $m \in S$ and let ∇_m be given. For each $\mathbf{g} := [g_1, \dots, g_{s_i}] \in (N_i)_m$, there exits a uniquely determined subset $\{\mathbf{b}_1^{(i)}, \dots, \mathbf{b}_t^{(i)}\}$ of a minimal system of generators of the ith module of syzygies of $\Bbbk[S]$ and uniquely determined elements $f_1, \dots, f_t \in R$ such that*

(a) $\mathbf{g} = \sum_{j=1}^{t} f_j \mathbf{b}_j^{(i)}$,
(b) $\gcd(g_1, \dots, g_{s_i})$ *divides* $f_j,\ j = 1, \dots, t$.

Proof. This proof is a natural generalization of the proofs of Theorem 8 and Proposition 16.

Write $\mathbf{g} = h\,\mathbf{g}'$, where $h := \gcd(g_1, \dots, g_{s_i})$. The S-degree$(\mathbf{g}') \prec_S S$-degree$(\mathbf{g})$, when $h \neq 1$. For the sake of notation, we suppose that $\gcd(\mathbf{g}) = 1$, i.e., $\mathbf{g} = \mathbf{g}'$.

Let $\{\widehat{\mathbf{h}}_1, \dots, \widehat{\mathbf{h}}_{t'},\ \widehat{\mathbf{b}}_1, \dots, \widehat{\mathbf{b}}_{t''}\}$ be a $\Bbbk$-basis of $\widetilde{Z}_i(\nabla_m)$ constructed as in Section 3. Then $\widehat{\mathbf{h}}_j = \sum_{k=1}^{d_i} q_{kj}^{(i)} \partial_{i+1}\big(F_k^{(i+1)}\big) \in \widetilde{B}_i(\nabla_m)$, for every j, and the classes of $\widehat{\mathbf{b}}_1, \dots, \widehat{\mathbf{b}}_{t''}$ form a basis of $\widetilde{H}_i(\nabla_m)$.

Let $\widehat{\mathbf{g}} \in \widetilde{Z}_i(\nabla_m)$ be the representative of $\psi_i^{-1}(\mathbf{g})$ constructed as (8). Since $\{\widehat{\mathbf{h}}_1, \dots, \widehat{\mathbf{h}}_{t'},\ \widehat{\mathbf{b}}_1, \dots, \widehat{\mathbf{b}}_{t''}\}$ is a $\Bbbk$-basis of $\widetilde{Z}_i(\nabla_m)$ there exist uniquely determined λ_i and $\mu_j \in \Bbbk$, $i = 1, \dots, t'$ and $j = 1, \dots, t''$, such that $\widehat{\mathbf{g}} = \sum_i \lambda_i \widehat{\mathbf{b}}_i + \sum_k \big(\sum_j \mu_j q_{kj}^{(i)}\big)\widehat{\mathbf{g}}_k$, with $\widehat{\mathbf{g}}_k := \partial_{i+1}(F_k^{(i+1)})$. Therefore $\mathbf{g} = \sum_i \lambda_i \mathbf{b}_i + \sum_k \nu_k \mathbf{g}_k$, with $\nu_k = \sum_j \mu_j q_{kj}^{(i)}$. If $\nu_k = 0$, for every k, we are done. Otherwise, we repeat this procedure for $\mathbf{g}_k$ with $\mu_k \neq 0$. This process ends for S-degree reasons.

The uniqueness follows from the same argument as in the proof of Theorem 8. □

By the above corollary, we can conclude that, starting from any i-syzygy $\mathbf{g}$ of I_S, our combinatorial algorithm computes a subset $\mathcal{B}'$ of a minimal generating set $\mathcal{B}$ of N_i and the polynomial coefficients of $\mathbf{g}$ with respect $\mathcal{B}'$ (and therefore with respect to $\mathcal{B}$) without knowing other i-syzygies. It is very important to note that $\mathbf{g}$ is not relevant by itself. Given its S-degree m, we can effectively produce i-syzygies of $\Bbbk[S]$ in the S-degree m and subsets of minimal generators of $N_j,\ j \leq i$. All the construction lies in the simplicial complex ∇_m.

Finally, let us see how our algorithm produces part of the minimal free resolution of a semigroup algebra $\Bbbk[S]$ starting from one S-degree. In fact, in the next example, we will get the whole resolution.

Example 20. Let S be the semigroup in Example 10 and consider $m = (60, 10) \in S$. The set of vertices of the simplicial complex ∇_m is

$$\begin{aligned} C_m = \{&x_2{}^5x_3{}^5,\ x_2{}^6x_3{}^2x_4{}^2, x_1x_2{}^4x_3{}^4x_4,\ x_1x_2{}^5x_3x_4{}^3, x_1{}^2x_2{}^2x_3{}^6,\ x_1{}^2x_2{}^3x_3{}^3x_4{}^2, x_1{}^2x_2{}^4x_4{}^4,\ x_1{}^3x_2x_3{}^5x_4,\\ &x_1{}^3x_2{}^2x_3{}^2x_4{}^3,\ x_1{}^4x_3{}^4x_4{}^2, x_1{}^4x_2x_3x_4{}^4, x_1{}^5x_4{}^5\}. \end{aligned}$$

We are going to “capture” syzygies of $\Bbbk[S]$ by using the method described in this section. To do that we choose the following 3-dimensional face of ∇_m

$$F = \{\underbrace{x_1x_2{}^4x_3{}^4x_4}_{A},\ \underbrace{x_1{}^2x_2{}^2x_3{}^6}_{B},\ \underbrace{x_1{}^2x_2{}^3x_3{}^3x_4{}^2}_{C},\ \underbrace{x_1{}^3x_2x_3{}^5x_4}_{D}\}.$$

By considering the 1-dimensional faces of F, we are able to construct four $\Bbbk$-linearly independent minimal binomial generators of I_S (see Remark 9) that is, 0-syzygies of $\Bbbk[S]$

$$\begin{aligned}
\{A, B\} &\longrightarrow\ b_1 = {x_2}^2 x_4 - x_1 {x_3}^2\\
\{A, C\} &\longrightarrow\ b_2 = x_2 x_3 - x_1 x_4\\
\{A, D\} &\longrightarrow\ b_3 = {x_2}^3 - x_1 x_3 x_4\\
\{B, C\} &\longrightarrow\ b_4 = {x_3}^3 - x_2 {x_4}^2\\
\{B, D\} &\longrightarrow\ b_2\\
\{C, D\} &\longrightarrow\ b_1.
\end{aligned}$$

Recall that the obtained coefficients are also needed, although we do not write them down.

Now, by using the 2-dimensional faces of F and the non-written above coefficients, we are able to produce four $\Bbbk$-linearly independent 1-syzygies of $\Bbbk[S]$ (see Proposition 16):

$$\begin{aligned}
\{A, B, C\} &\longrightarrow\ \mathbf{b}_1^{(1)} = [x_3, -x_2x_4, 0, x_1, 0, \ldots, 0] \in R^{s_1}\\
\{A, B, D\} &\longrightarrow\ \mathbf{b}_2^{(1)} = [x_2, x_1x_3, -x_4, 0, 0, \ldots, 0] \in R^{s_1}\\
\{A, C, D\} &\longrightarrow\ \mathbf{b}_3^{(1)} = [x_1, x_2^2, -x_3, 0, 0, \ldots, 0] \in R^{s_1}\\
\{B, C, D\} &\longrightarrow\ \mathbf{b}_4^{(1)} = [x_4, -x_3^2, 0, x_2, 0, \ldots, 0] \in R^{s_1}.
\end{aligned}$$

Again, we do not write down the obtained coefficients, although we insist that they are necessary to go further. Notice that the coordinates of $\mathbf{b}_i^{(1)}$, $i = 1, \ldots, 4$, has been completed with zeroes, because we do not know whether the rank of R^{s_1} is 4.

Finally, by using the 3-dimensional face of F and the non-written above coefficients, we get one 2-syzygy of $\Bbbk[S]$ (see by Corollary 19):

$$F = \{A, B, C, D\} \longrightarrow\ \mathbf{b}_1^{(2)} = [-x_2, x_3, -x_4, x_1, 0, \ldots, 0] \in R^{s_2}.$$

Therefore, we have obtained a chain complex of free R-modules

$$0 \to R \to R^4 \to R^4 \to R \to R/J, \tag{9}$$

where $J = (b_1, b_2, b_3, b_4) \subset R$, which is a subcomplex of the minimal free resolution of $\Bbbk[S]$.

In this case, it is not difficult to see that $I_S = J$ is a Gorenstein ideal of codimension 2 and thus (9) is its minimal free resolution.

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