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Cyclic behavior of linear fractional composition operators in the unit ball of $\mathbb{C}^N$ ☆

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Abstract

We characterize the cyclicity and hypercyclicity of composition operators induced by linear fractional self-maps of B_N on the Hardy space $H^2(B_N)$ based on the classification of linear fractional maps given by Bisi and Bracci.

Keywords: Linear fractional maps; Composition operators; Hypercyclicity; Cyclicity; Hardy space

1. Introduction

In the unit disc D of complex plane, the cyclic behavior of composition operators with linear fractional symbol has been studied by various authors (see [3,9], for example), where linear fractional self-maps of the unit disc play a basic role. In a recent paper, Cowen and MacCluer [8] introduced a class of holomorphic self-maps of the unit ball B_N, called linear fractional self-maps of B_N, which generalize both automorphisms of the ball and linear fractional maps of the unit disc, and can be represented as $(N+1) \times (N+1)$-matrices in a Kreĭn space. Linear fractional self-maps of B_N have been studied intensively (e.g. geometric characterization, continuous semigroups, the range; see [2,5,13]). In this paper, we study the cyclic behavior of composition operators on the Hardy space $H^2(B_N)$ induced by linear fractional self-maps of B_N, with the belief that these maps in higher dimensions will also play an important role in similar problems.

For linear fractional composition operators on $H^2(B_N)$, some results have been obtained as follows. Let φ be a linear fractional self-map of B_N. In [6] Chen et al. proved that the composition operator C_φ is hypercyclic if φ is an automorphism of B_N without interior fixed point. If the map φ has more than two fixed points in $\overline{B_N}$, Bisi and Bracci [2] pointed out that C_φ is non-cyclic, and it is hypercyclic if and only if its differential is injective at some point when φ is not an automorphism and has exactly two boundary fixed points. Bayart [1] observed that if φ has

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a unique boundary fixed point with the boundary dilatation coefficient 1, and if the restriction of φ to any non-trivial affine subset of B_N is not an automorphism, then C_φ is not hypercyclic.

Bourdon and Shapiro [3] completely characterized the cyclic and hypercyclic composition operators on $H^2(D)$ induced by linear fractional maps, in accordance with fixed-points location. However, in the unit ball, apart from the above-mentioned results, it seems that there does not exist any paper systematically studying the cyclic behavior of linear fractional composition operators on $H^2(B_N)$. In this paper, we try to characterize the cyclicity and hypercyclicity of linear fractional composition operators on $H^2(B_N)$ based on an elegant classification theorem (see Theorem 2.6 in Section 2) of linear fractional self-maps of B_N in [2]. For linear fractional map φ in the case $p_0 > 0$ and $p_1 \geqslant 1$ in the classification theorem, the cyclicity of C_φ has been studied by Bisi and Bracci [2], so we focus on those maps in the remaining cases. In Section 3, we get a necessary and sufficient condition for a composition operator induced by an automorphism with only one fixed point in B_N to be cyclic on $H^2(B_N)$. Our method is different from the corresponding result in one variable. We also find that the automorphism with only one fixed point in B_N has no fixed points on ∂B_N, i.e. such an automorphism is in the case $p_0 = 0$ of Theorem 2.6. If φ has only one fixed point on the boundary, the dynamics of C_φ are very difficult to understand. For our purposes, we divide this case into three subclasses according to Theorem 2.6. First, if φ is a parabolic linear fractional self-map of B_N, but not an automorphism, according to Corollary 5.9 of [1] C_φ is not hypercyclic if the restriction of φ to any non-trivial affine subset of B_N is not an automorphism, for the case where the restriction is an automorphism, we see that C_φ also fails to be hypercyclic on $H^2(B_2)$. Next, if φ has only one interior and one boundary fixed points, the non-hypercyclicity of C_φ will be shown. The above two results appear in Section 4. Finally, in Section 5 we study a class of special hyperbolic linear fractional self-maps of B_N, and prove that the induced composition operators are hypercyclic. In this situation, we give another proof of a result for the unit disc of [15], and use this method to deal with those special hyperbolic linear fractional maps.

2. Preliminary results

Definition 2.1. Let $A = (a_{jk})$ be an $N \times N$-matrix, $B = (b_j)$, $C = (c_j)$ be N-column vectors, and d be a complex number. A *linear fractional map* of $\mathbb{C}^N$ is a map of the form

$$\varphi(z) = \frac{Az + B}{\langle z, C\rangle + d},$$

where $\langle\cdot,\cdot\rangle$ indicates the usual Hermitian product in $\mathbb{C}^N$. The map φ is said to be a linear fractional map of $B_N = \{z \in \mathbb{C}^N\colon\ |z|^2 < 1\}$ whenever φ is defined on a neighborhood of B_N and $\varphi(B_N) \subset B_N$, in this case we write $\varphi \in \mathrm{LFM}(B_N)$.

In particular, an automorphism of B_N is a linear fractional map of B_N. The set of automorphisms of B_N will be denoted by $\mathrm{Aut}(B_N)$. Since $\mathrm{LFM}(B_N)$ is a semigroup, this allows us to classify linear fractional maps up to conjugation with $\mathrm{Aut}(B_N)$.

Recall that an *m-dimensional affine subset* of B_N is the intersection of B_N with an affine m-dimensional subspace of $\mathbb{C}^N$. A *slice* S (also called a *complex geodesic*) is a non-empty subset of B_N of the form $S = B_N \cap V$, where V is an one-dimensional affine subspace of $\mathbb{C}^N$. The prototype of a slice is $S_0 = B_N \cap \mathbb{C}e_1$, where $e_1 = (1, 0, \dots, 0)$. Note that for any slice S in B_N there exists an automorphism $\psi \in \mathrm{Aut}(B_N)$ such that $S = \psi(S_0)$. In [4], a holomorphic map $\varphi : B_N \mapsto B_N$ is said to be *rigid* if the image under φ of any complex geodesic is contained in a complex geodesic.

Proposition 2.2. *(See [2].) Let $\varphi \in \mathrm{LFM}(B_N)$ and let $D(\varphi)$ be its domain. Let G be an m-dimensional affine subspace of $\mathbb{C}^N$. Then $\varphi(G \cap D(\varphi))$ is contained in an m-dimensional affine subspace of $\mathbb{C}^N$. In particular φ is rigid.*

It is often very useful to transfer the problem to the Siegel half-plane $H_N = \{(w_1, \dots, w_N) = (w_1, w') \in \mathbb{C} \times \mathbb{C}^{N-1}\colon\ \operatorname{Re} w_1 > |w'|^2\}$, via the Cayley transform defined by

$$\sigma_C(z_1, z') = \left(\frac{1 + z_1}{1 - z_1}, \frac{z'}{1 - z_1}\right), \quad (z_1, z') \in \mathbb{C} \times \mathbb{C}^{N-1}.$$

It is well known that σ_C is a biholomorphic map of B_N onto H_N, which extends to a homeomorphism of $\overline{B_N}$ onto $H_N \cup \partial H_N \cup \{\infty\}$, the one-point compactification of $\overline{H_N}$. As a result, if $\varphi \in \mathrm{LFM}(B_N)$, the map

$\Phi = \sigma_C \circ \varphi \circ \sigma_C^{-1} : H_N \mapsto H_N$ is called a linear fractional self-map of H_N, the set of all of them will be denoted by $\mathrm{LFM}(H_N)$.

After transferring everything to H_N via the Cayley transform, we see that a slice $S \subset B_N$ such that $e_1 \in \overline{S}$ corresponds to a slice $S' \subset H_N$ given by $\{(w_1, w') \in H_N\colon\ w' = \text{const}\}$. The "prototype" slice $S_0 = B_N \cap \mathbb{C}e_1$ corresponds now to the slice $S'_0 = \{(w_1, w') \in H_N\colon\ w' = 0\}$ in H_N.

Now, we recall some results about fixed points and linear fractional maps. The following can be found in [2].

Theorem 2.3. *Let $\varphi \in \mathrm{LFM}(B_N)$ with no fixed points in B_N, then there exists a unique point $\tau \in \partial B_N$ such that $\varphi(\tau) = \tau$ and $\langle d\varphi_\tau(\tau), \tau\rangle = \lambda$ with $0 < \lambda \leqslant 1$.*

The unique point $\tau \in \partial B_N$ defined by Theorem 2.3 is called the Denjoy–Wolff point of φ, and λ is the boundary dilatation coefficient of φ. Some basic properties of Denjoy–Wolff points and boundary dilatation coefficients can be found in [4] and [10]. As customary, according to Theorem 2.3, the semigroup of linear fractional self-maps of B_N can be divided into three big families.

Definition 2.4. Let $\varphi \in \mathrm{LFM}(B_N)$. If φ has some fixed point in B_N we call it elliptic. If φ has no fixed points in B_N and λ is the boundary dilatation coefficient of φ at its Denjoy–Wolff point, we say that φ is hyperbolic if $\lambda < 1$, while it is parabolic if $\lambda = 1$.

Next, we introduce a classification theorem due to Bisi and Bracci [2].

Definition 2.5. Let

$$\mathcal{P}_0 = \operatorname{span}_{\mathbb{C}}\bigl\{x \in \partial B_N\colon\ \varphi(x) = x\bigr\}$$

and $p_0 = \dim_{\mathbb{C}} \mathcal{P}_0$. If $p_0 > 0$ and $\varphi(x_0) = x_0$, $x_0 \in \partial B_N$, let

$$\mathcal{P}_1 = \operatorname{span}_{\mathbb{C}}\bigl\{x - x_0\colon\ \varphi(x) = x,\ x \in \partial B_N\bigr\}$$

and $p_1 = \dim_{\mathbb{C}} \mathcal{P}_1$. Finally, let

$$\mathcal{P}_1^{\mathbb{R}} = \operatorname{span}_{\mathbb{R}}\bigl\{x - x_0\colon\ \varphi(x) = x,\ x \in \partial B_N\bigr\}$$

and $p_1^{\mathbb{R}} = \dim_{\mathbb{R}} \mathcal{P}_1^{\mathbb{R}}$.

Theorem 2.6. *Let φ be a linear fractional map of B_N. One and only one of the following cases is possible*:

(1) $p_0 = 0$ *if and only if φ has only one (isolated) fixed point in B_N and no fixed points on ∂B_N.*
(2) $p_0 > 0$ *if and only if φ has at least one fixed point on the boundary. In this case*:
  (i) $p_1 = 0$ *if and only if φ has only one fixed point on the boundary. In this case it is the unique fixed point of φ in $\overline{B_N}$ if and only if the boundary dilatation coefficient of φ at that point is less than or equal to* 1. *Otherwise φ has also an isolated fixed point inside B_N.*
  (ii) $p_1 = 1$ *if and only if one (and only one) of the two holds*:
    (a) $p_1^{\mathbb{R}} = 1$, *φ has only two fixed points on ∂B_N, and φ is conjugate to a map which has a hyperbolic automorphism (different from the identity) as first coordinate; i.e., φ is conjugate to a map of the form*

$$z \mapsto \left(\frac{az_1 + b}{bz_1 + a}, \frac{A_1 z'}{bz_1 + a}\right),$$

where $a = \cosh t$, $b = \sinh t$ with $t \in \mathbb{R} - \{0\}$ and A_1 is a $(N-1) \times (N-1)$ matrix with $\|A_1\| \leqslant 1$.
    (b) $p_1^{\mathbb{R}} = 2$, *φ is conjugate to a map of the form*

$$z \mapsto (z_1, A_1 z'),$$

where A_1 is a $(N-1) \times (N-1)$ matrix with $\|A_1\| \leqslant 1$.

(iii) $p_1 > 1$ *if and only if* φ *is conjugate to a map of the form*

$$z \mapsto \left(z_1, \dots, z_{p_1}, A_{p_1} z^{(p_1)}\right),$$

where A_{p_1} *is a* $(N - p_1) \times (N - p_1)$ *matrix with* $\|A_{p_1}\| \leqslant 1$ *and* $z^{(p_1)} = (z_{p_1+1}, \dots, z_N)$.

We say that a holomorphic function f on B_N belongs to the Hardy space $H^2(B_N)$ provided that

$$\|f\|_2^2 = \sup_{0<r<1} \int_{\partial B_N} \left|f(r\zeta)\right|^2 d\sigma(\zeta) < \infty,$$

where σ is the rotation-invariant positive Borel measure on ∂B_N with $\sigma(\partial B_N) = 1$. The space $H^2(B_N)$ is a Hilbert space. We refer to [14] for the properties of Hardy spaces.

Let φ be a holomorphic self-map of B_N, the composition operator C_φ on $H^2(B_N)$ is defined by

$$C_\varphi f = f \circ \varphi \quad \text{for } f \in H^2(B_N).$$

In general for $N > 1$, C_φ may fail to be bounded from $H^2(B_N)$ into itself. However, if $\varphi \in \mathrm{LFM}(B_N)$, then Cowen and MacCluer [8] showed that C_φ is bounded.

Recall that an operator T on a Hilbert space $\mathcal{H}$ is said to be cyclic if there is a vector $x \in \mathcal{H}$ such that $\{p(T)x\colon\ p \text{ polynomial}\}$ is dense in $\mathcal{H}$. Moreover T is called hypercyclic if the set $\{T^n x\colon\ n = 0, 1, 2, \dots\}$ is dense in $\mathcal{H}$.

Throughout this paper, we say that φ is conjugate to ψ (by T), if $\varphi = T \circ \psi \circ T^{-1}$, where T and ψ are linear fractional transformations.

3. Automorphisms in the case $p_0 = 0$

Let us assume that φ has only one (isolated) fixed point in B_N and no fixed points on ∂B_N, i.e. φ is in the case $p_0 = 0$ of Theorem 2.6. We see that C_φ is not hypercyclic on $H^2(B_N)$ by Proposition 1 of [6]. While in one-dimensional setting, Bourdon and Shapiro [3] have proved that

Proposition 3.1. *If* φ *is an elliptic automorphism of D, then* C_φ *is cyclic if and only if* φ *is conjugate* (*by automorphisms*) *to a rotation through an irrational multiple of* π.

We will generalize this result to the unit ball. First, we give a lemma about automorphisms of B_N with only one fixed point in B_N.

Lemma 3.2. *Suppose* $\varphi \in \mathrm{Aut}(B_N)$ *has only one fixed point in* B_N*, then* φ *fixes no points on* ∂B_N.

Proof. Suppose φ fixes $x \in B_N$ and has another fixed point $y \in \partial B_N$. Since φ is rigid, then φ fixes (as a set) the complex geodesic G passing through x and y. Therefore φ restricted to G is an automorphism of the unit disc with two fixed points. Thus it is the identity. Hence φ fixes any point $z \in G$, contradicting the hypothesis. □

Theorem 3.3. *Suppose that* φ *is an automorphism of* B_N*, with only one fixed point* $z_0 \in B_N$*. Let* $\{e^{i\theta_1}, \dots, e^{i\theta_N}\}$ *be the eigenvalues of the differential* $\varphi'(z_0)$*. Then* C_φ *is cyclic on* $H^2(B_N)$ *if and only if* $\theta_1, \dots, \theta_N, \pi$ *are rationally linearly independent real numbers.*

Proof. Without loss of generality, we assume that $\theta_j \in (0, 2\pi]$ for $j = 1, \dots, N$. There exists an automorphism ρ of B_N taking z_0 to 0. Since $\varphi(z_0) = z_0$, the map $\rho \circ \varphi \circ \rho^{-1} \in \mathrm{Aut}(B_N)$ fixes 0, and is therefore a unitary transformation. Thus there exists a unitary matrix V such that $V\rho\varphi\rho^{-1}V^{-1} \equiv U$ satisfies

$$U(z_1, \dots, z_N) = \left(e^{i\theta_1} z_1, \dots, e^{i\theta_N} z_N\right),$$

here, we have used the fact that the eigenvalues of U and the eigenvalues of $\varphi'(z_0)$ are the same, namely $\{e^{i\theta_1}, \dots, e^{i\theta_N}\}$. Thus $C_\varphi = C_{\phi^{-1} \circ U \circ \phi} = C_\phi C_U C_\phi^{-1}$ is similar to C_U, where $\phi = V\rho \in \mathrm{Aut}(B_N)$. Since cyclicity is similarity invariant, we only need to consider the cyclicity of C_U.

Suppose first that C_φ is cyclic. If some linear combination of $\theta_1, \dots, \theta_N$ and π with non-zero rational number coefficients is zero, up to multiplication the coefficients may be taken as integers, i.e., there exist integers k_i ($i = 1, \dots, N$) and n, not all zero, such that

$$\sum_{i=1}^{N} k_i \theta_i = 2n\pi.$$

We can rewrite as

$$\sum_{n_j \geqslant 0} n_j \theta_{j_0} + \sum_{m_l < 0} m_l \theta_{l_0} = 2n\pi,$$

where $n_j, m_l \in \{k_i,\ i = 1, \dots, N\}$ and $j_0, l_0 \in \{1, \dots, N\}$. So

$$\prod \left(e^{-i\theta_{j_0}}\right)^{n_j} = \exp\left(-i \sum_{n_j \geqslant 0} n_j \theta_{j_0}\right) = \exp\left(i\left(\sum_{m_l<0} m_l \theta_{l_0} - 2n\pi\right)\right) = \exp\left(i \sum_{m_l<0} m_l \theta_{l_0}\right) = \prod \left(e^{i\theta_{l_0}}\right)^{m_l}.$$

Now setting

$$s = \prod \left(e^{-i\theta_{j_0}}\right)^{n_j} = \prod \left(e^{i\theta_{l_0}}\right)^{m_l}$$

and

$$f(z_1, \dots, z_N) = \prod z_{j_0}^{n_j}.$$

By Lemma 8.1 of [7], we have $C_U^* = C_W$ with

$$W(z_1, \dots, z_N) = \left(e^{-i\theta_1} z_1, \dots, e^{-i\theta_N} z_N\right),$$

where C_U^* is the Hilbert space adjoint of C_U, which gives

$$C_U^* f = C_W f = f \circ W = f\left(e^{-i\theta_1} z_1, \dots, e^{-i\theta_N} z_N\right) = \prod \left(e^{-i\theta_{j_0}} z_{j_0}\right)^{n_j} = \prod \left(e^{-i\theta_{j_0}}\right)^{n_j} \prod z_{j_0}^{n_j} = sf.$$

On the other hand, the function $g(z_1, \dots, z_N) = \prod z_{l_0}^{-m_l}$ also satisfies

$$C_U^* g = C_W g = g \circ W = \prod \left(e^{-i\theta_{l_0}} z_{l_0}\right)^{-m_l} = \prod \left(e^{i\theta_{l_0}}\right)^{m_l} \prod z_{l_0}^{-m_l} = sg.$$

It is clear that the analytic functions f and g are in $H^\infty(B_N)$. Hence, s is an eigenvalue of C_U^* with multiplicity at least two, and C_U is not cyclic by Proposition 2.7 of [3].

Conversely, if $\theta_1, \dots, \theta_N$ and π are linearly independent over rational numbers, then C_U is cyclic. To see this, let $w = (1/2\sqrt{N}, \dots, 1/2\sqrt{N})$ be a point in B_N, and let K_w be the reproducing kernel at w for $H^2(B_N)$. Assume the function f in $H^2(B_N)$ is orthogonal to

$$\begin{aligned} \operatorname{Orb}(C_U, K_w) &= \left\{C_U^n K_w\colon\ n = 0, 1, 2, \dots\right\} \\ &= \left\{K_{w'}\colon\ w' = \left(e^{-in\theta_1}/2\sqrt{N}, \dots, e^{-in\theta_N}/2\sqrt{N}\right),\ n = 0, 1, 2, \dots\right\}. \end{aligned}$$

Kronecker's Theorem implies $\overline{\langle P\rangle} = T$ for the point $P = (\theta_1, \dots, \theta_N)$, where $\overline{\langle P\rangle}$ denotes the closure of the subgroup generated by P, and T is the torus $(0, 2\pi] \times \cdots \times (0, 2\pi]$ (N times). Hence f vanishes on the distinguished boundary of the polydisk $\{z \in \mathbb{C}^N\colon\ |z_k| < 1/2\sqrt{N},\ k = 1, \dots, N\}$. It follows that f must vanish on this polydisk, and hence must vanish identically on B_N. Thus C_U is cyclic on $H^2(B_N)$. $\square$

Remark 3.4. (1) According to Lemma 3.2, Theorem 3.3 gives a necessary and sufficient condition when a composition operator induced by an automorphism in the case $p_0 = 0$ of Theorem 2.6 is cyclic on $H^2(B_N)$. If $N = 1$, Theorem 3.3 is just Proposition 3.1.

(2) The result presented in the example at the end of [6] follows easily from Theorem 3.3.

4. Two classes of linear fractional maps in the case $p_0 > 0$ and $p_1 = 0$

Combining the classification of linear fractional maps in Theorem 2.6 and some results about the dynamics of the induced composition operators, we find that linear fractional maps in the case $p_0 > 0$ and $p_1 = 0$ have more complicate properties, a few results have been obtained about the cyclicity and hypercyclicity of composition operators stemming from these maps. By Theorem 2.6, they contain three families: parabolic linear fractional maps, hyperbolic linear fractional maps with only one fixed point on ∂B_N, and linear fractional maps fixing only one interior and one boundary points. Hyperbolic linear fractional maps with only one boundary fixed point are left to Section 5. In this section, we mainly consider the cyclic behavior of composition operators induced by the remaining two classes.

Parabolic linear fractional maps have been studied by Bayart [1]. If φ is a parabolic non-automorphic linear fractional self-map of B_N, he proved that C_φ is not hypercyclic if the restriction of φ to any non-trivial affine subset of B_N is not an automorphism. For the case where the restriction is an automorphism, in case $N = 2$, the following theorem shows that C_φ is still not hypercyclic on $H^2(B_2)$. The method of the proof follows Bayart.

Theorem 4.1. *Let φ be a parabolic linear fractional self-map of B_2. Suppose that φ is not an automorphism and the restriction of φ to some non-trivial affine subset of B_2 is an automorphism. Then C_φ is not hypercyclic on $H^2(B_2)$.*

Proof. For a parabolic non-automorphic linear fractional self-map φ of B_2, if φ does not fix (as a set) any non-trivial affine subset of B_2, then [1, Theorem 5.1] implies that C_φ is not hypercyclic.

Now, suppose φ fixes some non-trivial affine subset of B_2. Observe that the non-trivial affine subset S of B_2 is a slice, and, without loss of generality, we assume $S = \{z \in B_2\colon\ z_2 = 0\}$. Since the slice S is invariant for φ, the slice $S' = \{w \in H_2\colon\ w_2 = 0\}$ in H_2 is invariant for the conjugate map $\Phi = \sigma_C \circ \varphi \circ \sigma_C^{-1}$. Applying Proposition 4.2 of [5], there exist $a, b, d, \lambda \in \mathbb{C}$ such that

$$\Phi(w_1, w_2) = \big(w_1 + \langle w_2, b\rangle + a, \lambda w_2 + d\big), \quad (w_1, w_2) \in H_2,$$

with $|\lambda| \leqslant 1$. Since the slice S' is invariant for Φ, we have $d = 0$. On the other hand, the hypothesis implies that Φ restricted to S' is an automorphism, i.e., the function $\Phi(w_1, 0) = (w_1 + a, 0)$ is an automorphism on S', thus $\operatorname{Re} a = 0$. Since φ is a linear fractional map of B_2 which is an automorphism when restricted to S, by Lemma 3 of [12], the first coordinate function of φ depends only on z_1, so is the conjugate Φ. Hence, $b = 0$ and

$$\Phi(w_1, w_2) = (w_1 + a, \lambda w_2), \quad (w_1, w_2) \in H_2,$$

with $\operatorname{Re} a = 0$ and $|\lambda| \leqslant 1$. Since Φ is not an automorphism of H_2, we have $|\lambda| < 1$. As a consequence, we obtain

$$\varphi(z_1, z_2) = \sigma_C^{-1} \circ \Phi \circ \sigma_C = \left(\frac{(2-a)z_1 + a}{2 + a - az_1}, \frac{2\lambda z_2}{2 + a - az_1}\right), \quad (z_1, z_2) \in B_2.$$

If $\lambda = 0$, it is obvious that C_φ is not hypercyclic on $H^2(B_2)$, thus we only need to consider the case $0 \neq |\lambda| < 1$.

An easy computation shows that

$$\varphi_n(z) = \left(\frac{(2-na)z_1 + na}{2 + na - naz_1}, \frac{2\lambda^n z_2}{2 + na - naz_1}\right), \quad (z_1, z_2) \in B_2,$$

where $\varphi_n = \underbrace{\varphi \circ \varphi \circ \cdots \circ \varphi}_{n \text{ times}}$. Since $\operatorname{Re} a = 0$, there exist constants K_1, K_2 and K_3 such that

$$1 - \big|\varphi_{n,1}(0, z_2)\big| \sim \frac{K_1}{n^2}, \qquad \big|1 - \varphi_{n,1}(0, z_2)\big| \sim \frac{K_2}{n}$$

and

$$1 - \big|\varphi_n(0, z_2)\big| \sim \frac{K_3}{n^2}$$

for any $(0, z_2) \in B_2$ with $z_2 \neq 0$. In this setting, we write $z^\star = (0, z_2)$.

We estimate the derivatives of $\varphi_{n,1}$ and $\varphi_{n,2}$ at any given point $z^\star$. For any $k \geqslant 1$, there exist complex numbers c_1, c_2 so that

$$\partial_1^k \varphi_{n,1}(z^\star) = \frac{4k!(na)^{k-1}}{(na+2)^{k+1}} = \frac{c_1}{n^2} + O\left(\frac{1}{n^3}\right),$$
$$\partial_1^k \varphi_{n,2}(z^\star) = \frac{2\lambda^n k!(na)^k z_2}{(na+2)^{k+1}} = \frac{c_2\lambda^n}{n} + O\left(\frac{1}{n^2}\right).$$

On the other hand, [1, p. 661] gives

$$\left|\partial_2^m \partial_1^l f(z_1, z_2)\right| \leqslant C\|f\|_2 \frac{1}{(1-|z_1|^2)^{1+l+m/2}} \times \left(\frac{1}{1-(\frac{|z_2|^2}{1-|z_1|^2})^{1/2}}\right)^{3/2+l+m}$$

for any $l, m \geqslant 0$ and any $f \in H^2(B_2)$. Replace (z_1, z_2) by $\varphi_n(z^\star)$, we get

$$\frac{1}{(1-|z_1|^2)^{1+l+m/2}} = \frac{1}{(1-|\varphi_{n,1}(z^\star)|^2)^{1+l+m/2}} = O\left(n^{2+2l+m}\right)$$

and

$$\frac{|z_2|^2}{1-|z_1|^2} = 1 - \frac{1-|z|^2}{1-|z_1|^2} = 1 - \frac{1-|\varphi_n(z^\star)|^2}{1-|\varphi_{n,1}(z^\star)|^2} = O(1).$$

So

$$\left|\partial_2^m \partial_1^l f\left(\varphi_n(z^\star)\right)\right| \leqslant C_{l,m}\|f\|_2 n^{2+2l+m},$$

where C and $C_{l,m}$ are positive constants.

Set $g_n = f \circ \varphi_n$, for any $k \geqslant 1$, as in the proof of [1, Lemma 5.7], we have

$$\partial_1^k g_n(z^\star) = \sum_{1 \leqslant l+m \leqslant k} \alpha_{l,m,n} \partial_2^m \partial_1^l f\left(\varphi_n(z^\star)\right),$$

where $\alpha_{l,m,n}$ is a finite linear combination of terms like

$$\left(\partial_1^{\mu_1}\varphi_{n,1}(z^\star)\right)^{l_1} \ldots \left(\partial_1^{\mu_r}\varphi_{n,1}(z^\star)\right)^{l_r} \left(\partial_1^{\nu_1}\varphi_{n,2}(z^\star)\right)^{m_1} \ldots \left(\partial_1^{\nu_s}\varphi_{n,2}(z^\star)\right)^{m_s}$$

with $l_1 + \cdots + l_r = l$ and $m_1 + \cdots + m_s = m$. Since $0 \neq |\lambda| < 1$, the above argument gives

$$\alpha_{l,m,n} = O\left(\frac{\lambda^{nm}}{n^{2l+m}}\right),$$

and

$$\alpha_{l,m,n}\partial_2^m \partial_1^l f\left(\varphi_n(z^\star)\right) = O\left(\lambda^{nm} n^2\right).$$

Hence, for any $k \geqslant 1$, $\partial_1^k g_n(z^\star) = o(1)$.

Next, suppose that f is a hypercyclic vector for C_φ and set $g_n = C_\varphi^n f = f \circ \varphi_n$. If $h \in H^2(B_2)$ is a cluster point of $\{g_n\}$, since convergence in the norm of $H^2(B_2)$ implies pointwise convergence (of the derivatives) on B_2, then there is a constant $\varepsilon > 0$, for any $k \geqslant 1$,

$$\left|\partial_1^k h(z^\star)\right| < \varepsilon.$$

It is clear that there exist functions in $H^2(B_2)$ which do not satisfy this inequality. □

The following result is about the non-cyclicity of composition operators whose symbols are linear fractional self-maps of B_N with only one interior and one boundary fixed points.

Theorem 4.2. *Suppose $\varphi \in \mathrm{LFM}(B_N)$ has exactly one interior and one boundary fixed points. Then C_φ is not cyclic on $H^2(B_N)$.*

Proof. Without loss of generality, suppose $\alpha = e_1 \in \partial B_N$ and $\beta \in B_N$ are the fixed points of φ. Since φ is rigid, φ fixes (as a set) the complex geodesic S passing through α and β. We may assume $S = \{z \in B_N\colon z_2 = \cdots = z_N = 0\}$. Therefore, $\psi = \varphi|_S$ is a linear fractional self-map of the unit disc, with interior and boundary fixed points, and we have $\psi(z_1) = \varphi_1(z_1, 0')$ for $z_1 \in D$.

Now, for any $f \in A^2_{N-2}(D)$ (the weighted Bergman space with the weight $(1-|z_1|^2)^{N-2}$), define F on B_N by $F(z_1, z') = f(z_1)$, we have $F \in H^2(B_N)$ (see 1.4.4 in [14]). If C_φ is cyclic, suppose $G \in H^2(B_N)$ is a cyclic vector for C_φ, then there exists a sequence $\{p_n, p_n \text{ is polynomial}\}$ such that

$$\left\| p_n(C_\varphi)G - F \right\|_{H^2(B_N)} \to 0 \quad \text{as } n \to \infty.$$

We define the functions E_n on D by

$$E_n(z_1) = p_n(C_\varphi)G(z_1 e_1) - F(z_1 e_1) = p_n(C_\varphi)G(z_1, 0') - F(z_1, 0'),$$

then $E_n \in A^2_{N-2}(D)$ and satisfies

$$\left\| p_n(C_\varphi)G - F \right\|_{H^2(B_N)} \geqslant \|E_n\|_{A^2_{N-2}(D)}$$

(see, for example, Corollary 1.4 in [11]). Write $p_n(x) = \sum_{k \in I_n} a_k x^k$, where I_n is a finite set of non-negative integers. Then

$$E_n(z_1) = p_n(C_\varphi)G(z_1, 0') - F(z_1, 0') = \sum_{k \in I_n} a_k G\big(\varphi^k(z_1, 0')\big) - F(z_1, 0'),$$

here, we use φ^k to denote the kth iterate of φ. Since φ fixes the slice S, this implies $\varphi^k(z_1, 0') = (\psi^k(z_1), 0')$.

On the other hand, since $G \in H^2(B_N)$, the slice function g defined on D by $g(z_1) = G(z_1, 0')$ is in $A^2_{N-2}(D)$. Thus,

$$\begin{aligned} E_n(z_1) &= \sum_{k \in I_n} a_k G\big(\varphi^k(z_1, 0')\big) - F(z_1, 0') = \sum_{k \in I_n} a_k G\big(\psi^k(z_1), 0'\big) - f(z_1) \\ &= \sum_{k \in I_n} a_k g\big(\psi^k(z_1)\big) - f(z_1) = p_n(C_\psi)g(z_1) - f(z_1). \end{aligned}$$

Therefore,

$$\left\| p_n(C_\psi)g - f \right\|_{A^2_{N-2}(D)} = \|E_n\|_{A^2_{N-2}(D)} \leqslant \left\| p_n(C_\varphi)G - F \right\|_{H^2(B_N)} \to 0 \quad \text{as } n \to \infty,$$

and g is a cyclic vector for C_ψ on $A^2_{N-2}(D)$. However, C_ψ is not cyclic on $A^2_{N-2}(D)$ (see [9]). This completes the proof. □

5. A class of hyperbolic linear fractional self-maps of B_N

The following result has been proved by Shapiro [15, p. 114] from a geometric point of view. We give another proof by an analytic method, which provides a way to understand the hypercyclicity of composition operators induced by a class of hyperbolic linear fractional self-maps of B_N. In this section, we denote linear fractional self-maps of the unit disc D by $\mathrm{LFT}(D)$.

Proposition 5.1. *Suppose that $\varphi \in \mathrm{LFT}(D)$ has no fixed points in D. If φ is a hyperbolic non-automorphism, then C_φ is hypercyclic on $H^2(D)$.*

Proof. The hyperbolic non-automorphism $\varphi \in \mathrm{LFT}(D)$ has an attractive fixed point on ∂D, with the other fixed point outside $\overline{D}$. Without loss of generality, we may assume $\alpha = 1$ is the attractive fixed point and $\beta = -x$ $(x > 1)$ is the other fixed point. Upon conjugating φ by the transformation $\sigma(z) = \frac{1+z}{1-z}$, we come up with a linear fractional map Φ that maps the right half-plane Π into itself, fixes ∞ and $\frac{1-x}{1+x}$, with ∞ the attractive fixed point. Assume $\varphi'(1) = 1/\lambda$ with $\lambda > 1$. Hence,

$$\Phi(w) = \lambda w + (1-\lambda)\frac{1-x}{1+x}, \quad w \in \Pi.$$

Clearly, $\mathrm{Re}(1-\lambda)\frac{1-x}{1+x} > 0$ and Φ is not an automorphism of Π. However, Φ maps half-plane $\Pi' = \{w \in \mathbb{C}\colon \mathrm{Re}\, w > \frac{1-x}{1+x}\}$ onto itself, i.e., Φ is an automorphism of Π'.

An easy computation shows that $\sigma^{-1}(w) = \frac{w-1}{w+1}$ maps the line $\{w \in \mathbb{C}\colon \mathrm{Re}\, w = \frac{1-x}{1+x}\}$ to the circle $\{z \in \mathbb{C}\colon |z - \frac{1-x}{2}| = \frac{1+x}{2}\}$. We know that $D \subset \Delta = \{z \in \mathbb{C}\colon |z - \frac{1-x}{2}| < \frac{1+x}{2}\}$ and $\Pi \subset \Pi'$. With the conventions about ∞, each linear fractional transformation maps the Riemann sphere one-to-one and holomorphically onto itself. So σ maps Δ one-to-one and holomorphically onto Π'. Thus, the map $\psi = \sigma^{-1} \circ \Phi \circ \sigma$ is a automorphism of Δ, with two boundary fixed points α and β. Restricting to D, we have $\psi|_D = \sigma^{-1} \circ \Phi \circ \sigma|_D = \varphi$. Observe that α is the attractive fixed point of ψ, and ψ^{-1} is also an automorphism of Δ with β the attractive fixed point. Hence,

$$\varphi^{-n}(\zeta) = \underbrace{\varphi^{-1} \circ \cdots \circ \varphi^{-1}}_{n \text{ times}}(\zeta) = \psi^{-n}(\zeta) \to \beta$$

for each $\zeta \in \partial D \setminus \{\alpha\}$.

Next, the proof is similar to that of the Linear Fractional Hypercyclicity Theorem in [15, p. 114], and we omit it. □

Let $\Phi \in \mathrm{LFM}(H_N)$ be without fixed points in H_N, with Denjoy–Wolff point ∞ and boundary dilatation coefficient λ. Proposition 4.3 of [5] gives $\Phi \in \mathrm{Aut}(H_N)$ if and only if

$$\Phi(w_1, w') = \frac{1}{\lambda}\left(w_1 + \frac{2}{\sqrt{\lambda}}\langle Uw', d\rangle + c, \sqrt{\lambda}Uw' + d\right), \quad (w_1, w') \in H_N,$$

where $c \in \mathbb{C}$, $d \in \mathbb{C}^{N-1}$ with $\lambda\,\mathrm{Re}\, c = |d|^2$, and $U \in \mathbb{C}^{(N-1)\times(N-1)}$ is a unitary matrix. Note that if $\lambda = 1$ and $U \equiv I$ (the identity matrix), $\Phi \in \mathrm{Aut}(H_N)$ is called a Heisenberg translation of H_N. In which case, if $\mathrm{Re}\, c \geqslant |d|^2 > 0$, Φ is a generalized Heisenberg translation of H_N defined by Bayart [1]. He proved that a generalized Heisenberg translation of B_N (the conjugation of a generalized Heisenberg translation of H_N), not an automorphism, induces a composition operator which is not hypercyclic on $H^2(B_N)$. Similarly, we introduce the following definition

Definition 5.2. Let $\varphi \in \mathrm{LFM}(B_N)$. We say that φ is a generalized hyperbolic linear fractional self-map of B_N, if it is conjugated to a self-map of H_N of the form

$$\Phi_0(w_1, w') = \frac{1}{\lambda}\left(w_1 + \frac{2}{\sqrt{\lambda}}\langle Uw', d\rangle + c, \sqrt{\lambda}Uw' + d\right), \quad (w_1, w') \in H_N,$$

with $\lambda < 1$ and $\lambda\,\mathrm{Re}\, c \geqslant |d|^2$, $U \in \mathbb{C}^{(N-1)\times(N-1)}$ is a unitary matrix.

Note that the matrix $\sqrt{\lambda}U - \lambda I$ is invertible, and there exists a unique point $w_0 \in \mathbb{C}^{N-1}$ such that $(\sqrt{\lambda}U - \lambda I)w_0 = -d$. Consider the linear fractional map

$$\eta(w_1, w') = \bigl(w_1 - 2\langle w', w_0\rangle + |w_0|^2, w' - w_0\bigr), \quad (w_1, w') \in H_N.$$

Then $\eta \in \mathrm{Aut}(H_N)$ and η sends the slice $\{(w_1, w') \in H_N\colon w' = w_0\}$ to the slice $\{(w_1, w') \in H_N\colon w' = 0\}$. Thus $\Psi = \eta \circ \Phi_0 \circ \eta^{-1}$ is of the form

$$\Psi(w_1, w') = \left(\frac{1}{\lambda}w_1 + b, \frac{1}{\sqrt{\lambda}}Uw'\right), \quad (w_1, w') \in H_N, \tag{$*$}$$

with $b \in \mathbb{C}$ and $\mathrm{Re}\, b \geqslant 0$. Let $\psi = \sigma_C^{-1} \circ \Psi \circ \sigma_C$, we have $\varphi = \phi^{-1} \circ \psi \circ \phi$, where $\phi = \sigma_C^{-1} \circ \eta \circ \sigma_C \in \mathrm{Aut}(B_N)$. Hence, every generalized hyperbolic linear fractional self-map φ of B_N can be conjugate (by automorphisms) to another map $\psi \in \mathrm{LFM}(B_N)$, whose corresponding conjugate map of H_N has the simpler form $(*)$.

Next, we want to prove that the composition operator induced by a generalized hyperbolic linear fractional self-map of B_N is hypercyclic on $H^2(B_N)$. We first give a lemma needed later.

Lemma 5.3. *Suppose $\alpha \in \partial B_N$ and $\beta \in \mathbb{C}^N \setminus \overline{B_N}$. Then there exists an automorphism of B_N fixing α which takes β onto the part of the line through α and 0, which lies on the opposite side of the origin from α.*

Proof. $\mathrm{Aut}(B_N)$ acts transitively on $\partial B_N \times \mathbb{P}(\mathbb{C}^N)$, thus, up to conjugation with automorphisms, we can assume that $\alpha = e_1$ and $\beta \in \mathbb{C}e_1 \backslash \overline{B_N}$, namely $\beta = (\beta_1, 0') \in \mathbb{C} \times \mathbb{C}^{N-1}$ with $|\beta_1| > 1$. Transferring to the Siegel half-plane H_N via the Cayley transform σ_C, the point e_1 corresponds to ∞, and β to

$$\sigma_C(\beta) = \left(\frac{1+\beta_1}{1-\beta_1}, 0'\right).$$

Define a Heisenberg translation

$$\eta(w_1, w') = (w_1 + c, w'), \quad (w_1, w') \in H_N,$$

where $c = -\frac{2i\,\mathrm{Im}\,\beta_1}{|1-\beta_1|^2}$, which fixes ∞ and takes $\sigma_C(\beta)$ to the point

$$\gamma = \left(\frac{1-|\beta_1|^2}{|1-\beta_1|^2}, 0'\right).$$

Next, since $|\beta_1| > 1$, for $t > \frac{|\beta_1|^2-1}{|1-\beta_1|^2} > 0$, the non-isotropic dilation

$$\delta_t(w_1, w') = (w_1/t, w'/\sqrt{t})$$

sends γ to

$$\delta_t(\gamma) = \left(\frac{1-|\beta_1|^2}{t|1-\beta_1|^2}, 0'\right),$$

and $-1 < \frac{1-|\beta_1|^2}{t|1-\beta_1|^2} < 0$.

Let $\phi = \sigma_C^{-1} \circ \delta_t \circ \eta \circ \sigma_C$, then $\phi \in \mathrm{Aut}(B_N)$, with the fixed point e_1, takes β to

$$\phi(\beta) = \left(\frac{1-|\beta_1|^2 - t|1-\beta_1|^2}{1-|\beta_1|^2 + t|1-\beta_1|^2}, 0'\right)$$

with $\frac{1-|\beta_1|^2-t|1-\beta_1|^2}{1-|\beta_1|^2+t|1-\beta_1|^2} < -1$. Hence ϕ is the desired automorphism of B_N. □

Theorem 5.4. *Let φ be a generalized hyperbolic linear fractional self-map of B_N. Then C_φ is hypercyclic on $H^2(B_N)$.*

Proof. As shown before, up to conjugation with an automorphism of B_N, we may assume that φ is conjugated to a map $\Phi \in \mathrm{LFM}(H_N)$ of the form

$$\Phi(w_1, w') = (\lambda w_1 + b, \sqrt{\lambda} U w'), \quad (w_1, w') \in H_N,$$

where $b \in \mathbb{C}$ with $\mathrm{Re}\, b \geqslant 0$, $U \in \mathbb{C}^{(N-1)\times(N-1)}$ is a unitary matrix, and $1/\lambda$ is the boundary dilatation coefficient of φ, $\lambda > 1$. In fact, if $\mathrm{Re}\, b = 0$, then φ is an automorphism of B_N, and by [6] C_φ is hypercyclic. We only need to consider the case $\mathrm{Re}\, b > 0$, in this setting, the map Φ has attractive fixed point ∞ and an exterior fixed point. Thus φ fixes the point $\beta = (\beta_1, 0')$ outside $\overline{B_N}$, with Denjoy–Wolff point $\alpha = e_1 \in \partial B_N$.

By Lemma 5.3 we can assume $\beta = (-r, 0')$ with $r > 1$. Then Φ fixes the point $\tau = (\frac{1-r}{1+r}, 0') \notin \overline{H_N}$, and

$$\Phi(w_1, w') = \left(\lambda w_1 + (1-\lambda)\frac{1-r}{1+r}, \sqrt{\lambda} U w'\right), \quad (w_1, w') \in H_N.$$

From this form, we see that Φ is an automorphism of the half-plane

$$\Omega = \left\{(w_1, w') \in \mathbb{C} \times \mathbb{C}^{N-1}\colon\ \mathrm{Re}\, w_1 > |w'|^2 + \frac{1-r}{1+r}\right\}.$$

Recall that the Cayley transform $\sigma_C(z_1, z') = (\frac{1+z_1}{1-z_1}, \frac{z'}{1-z_1})$ is a biholomorphic map from B_N onto H_N. On the other hand, σ_C is an one-to-one and holomorphic map from $\mathbb{C} \times \mathbb{C}^{N-1} \setminus \{(1, z'), z' \in \mathbb{C}^{N-1}\}$ onto $\mathbb{C} \times \mathbb{C}^{N-1} \setminus \{(-1, w'), w' \in \mathbb{C}^{N-1}\}$. It is clear that $\sigma_C^{-1}(w_1, w') = (\frac{w_1-1}{w_1+1}, \frac{2w'}{w_1+1})$ maps

$$\partial\Omega = \left\{(w_1, w') \in \mathbb{C} \times \mathbb{C}^{N-1}\colon\ \mathrm{Re}\, w_1 = |w'|^2 + \frac{1-r}{1+r}\right\}$$

to the set

$$\left\{(z_1, z') \in \mathbb{C} \times \mathbb{C}^{N-1}\colon\ \frac{|z_1 - \frac{1-r}{2}|^2}{(\frac{1+r}{2})^2} + \frac{|z'|^2}{\frac{1+r}{2}} = 1\right\}.$$

Therefore, σ_C is a biholomorphic transform from the complex ellipsoid

$$\Delta = \left\{(z_1, z') \in \mathbb{C} \times \mathbb{C}^{N-1}\colon\ \frac{|z_1 - \frac{1-r}{2}|^2}{(\frac{1+r}{2})^2} + \frac{|z'|^2}{\frac{1+r}{2}} < 1\right\}$$

onto Ω, and $B_N \subset \Delta$, $H_N \subset \Omega$. Thus $\psi = \sigma_C^{-1} \circ \Phi \circ \sigma_C$ is defined on Δ, and is an automorphism of Δ with two boundary fixed points $\alpha = e_1$ and $\beta = (-r, 0')$.

A straightforward induction argument shows that for any $n \geqslant 0$,

$$\Phi^{-n}(w_1, w') = \left(\lambda^{-n} w_1 + \left(1 - \lambda^{-n}\right)\frac{1-r}{1+r}, \lambda^{-\frac{n}{2}}\left(U^*\right)^n w'\right), \quad (w_1, w') \in \Omega,$$

where U^* denotes the adjoint of the unitary matrix U. Hence,

$$\psi^{-n}(z_1, z') = \sigma_C^{-1} \circ \Phi^{-n} \circ \sigma_C = \left(\frac{(1+\lambda^n r)z_1 + r(1-\lambda^n)}{(1-\lambda^n)z_1 + (r+\lambda^n)}, \frac{(1+r)\lambda^{\frac{n}{2}}(U^*)^n z'}{(1-\lambda^n)z_1 + (r+\lambda^n)}\right)$$

for $(z_1, z') \in \Delta$. Since $\lambda > 1$, for any $z \in \Delta$, the first component of ψ^{-n} is easily seen to go to $-r$ as $n \to \infty$. Moreover

$$\left|\frac{(1+r)\lambda^{\frac{n}{2}}(U^*)^n z'}{(1-\lambda^n)z_1 + (r+\lambda^n)}\right| = \frac{(1+r)\lambda^{\frac{n}{2}}|z'|}{|(1-\lambda^n)z_1 + (r+\lambda^n)|} \to 0$$

as $n \to \infty$. Thus $\psi^{-n}(z) \to \beta = (-r, 0')$.

Note that $\psi|_{B_N} = \sigma_C^{-1} \circ \Phi \circ \sigma_C|_{B_N} = \varphi$. Thus $\varphi^{-n}(z) = \psi^{-n}(z) \to \beta$ for any $z \in B_N$. Next, we want to find the dense sets X and Y, and a map S which satisfy the hypotheses of the Hypercyclicity Criterion (see [15, p. 109]). We will consider the same space X as in [6], namely

$$X = \left\{f \in A(B_N) \text{ the ball algebra: } f(e_1) = 0\right\}.$$

We see that X is dense in $H^2(B_N)$, and $C_\varphi^n \to 0$ on X. In a similar way, let Y be the set of functions that are continuous on the closed ellipsoid $\overline{\Delta}$, analytic on the interior, and vanish at β. To see that Y is dense in $H^2(B_N)$, suppose $f \in H^2(B_N)$ is orthogonal to Y. Since Y contains the subset $\{(z_1 + r)z_1^k,\ z^\alpha,\ \alpha = (\alpha_1, \ldots, \alpha_N) \neq (k, 0'),\ k = 0, 1, 2, \ldots\}$, f is also orthogonal to this subset.

Let $f = \sum f_k$, where

$$f_k = C_k^1 z_1^k + \sum_{|\alpha| = k,\, \alpha \neq (k, 0')} C(\alpha) z^\alpha.$$

Then for any non-negative integer k and $\alpha \neq (k, 0')$,

$$0 = \left\langle f, z^\alpha\right\rangle = C(\alpha)\left\|z^\alpha\right\|_2^2$$

and

$$0 = \left\langle f, (z_1 + r)z_1^k\right\rangle = C_{k+1}^1\left\|z_1^{k+1}\right\|_2^2 + rC_k^1\left\|z_1^k\right\|_2^2.$$

Hence, $C(\alpha) = 0$ and

$$C_k^1\left\|z_1^k\right\|_2^2 = (-r)^k C_0^1,$$

where $C_0^1 = f(0)$. So $f = \sum f_k = \sum C_k^1 z_1^k$ and

$$\|f\|_2^2 = \sum\|f_k\|_2^2 = \sum\left|C_k^1\right|^2\left\|z_1^k\right\|_2^2 = \sum\frac{r^{2k}|C_0^1|^2}{\|z_1^k\|_2^2} = \left|C_0^1\right|^2\sum\frac{r^{2k}(N-1+k)!}{(N-1)!k!}.$$

Since $r > 1$,

$$\sum \frac{r^{2k}(N-1+k)!}{k!} \geqslant \sum r^{2k}(k+1)^{N-1} = \infty.$$

Therefore $f(0) = C_0^1 = 0$, and $C_k^1 = 0$ for any $k \in \mathbb{N}$. Thus the only function in $H^2(B_N)$ orthogonal to Y is the zero function, for Y is a linear subspace of $H^2(B_N)$. It follows that Y is dense in $H^2(B_N)$.

Finally define the map $S : Y \mapsto Y$ by

$$Sf(z) = f\big(\varphi^{-1}(z)\big) = f\big(\psi^{-1}(z)\big), \quad z \in B_N.$$

Since $\psi^{-1}(\Delta) \subset \Delta$, we see that S is defined on Y and $C_\varphi S$ is identity on Y. In addition, we have proved that $\varphi^{-n} \to \beta$ uniformly on compact subset of the unit ball. Therefore, $S^n \to 0$ on Y, and so by the Hypercyclicity Criterion C_φ is hypercyclic on $H^2(B_N)$. □

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