The intersection of essential approximate point spectra of operator matrices

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Abstract

When $A \in B(\mathcal{H})$ and $B \in B(\mathcal{K})$ are given, we denote by $M_C$ the operator acting on the infinite-dimensional separable Hilbert space $\mathcal{H} \oplus \mathcal{K}$ of the form $M_C = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$. In this paper, it is shown that there exists some operator $C \in B(\mathcal{K}, \mathcal{H})$ such that $M_C$ is upper semi-Fredholm and $\text{ind}(M_C) \leq 0$ if and only if there exists some left invertible operator $C \in B(\mathcal{K}, \mathcal{H})$ such that $M_C$ is upper semi-Fredholm and $\text{ind}(M_C) \leq 0$. A necessary and sufficient condition for $M_C$ to be upper semi-Fredholm and $\text{ind}(M_C) \leq 0$ for some $C \in \text{Inv}(\mathcal{K}, \mathcal{H})$ is given, where $\text{Inv}(\mathcal{K}, \mathcal{H})$ denotes the set of all the invertible operators of $B(\mathcal{K}, \mathcal{H})$. In addition, we give a necessary and sufficient condition for $M_C$ to be upper semi-Fredholm and $\text{ind}(M_C) \leq 0$ for all $C \in \text{Inv}(\mathcal{K}, \mathcal{H})$.

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1. Introduction

The study of upper triangular operator matrices arises naturally from the following fact: if $T$ is a Hilbert space operator and $M$ is an invariant subspace for $T$, then $T$ has the following $2 \times 2$ upper triangular operator matrix representation:

$$T = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} : \mathcal{M} \oplus \mathcal{M}^\perp \to \mathcal{M} \oplus \mathcal{M}^\perp,$$

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and one way to study operator is to see them as entries of simpler operators. Recently, many authors have paid much attention to $2 \times 2$ upper triangular operator matrices (see [2–5,7,9,10]). For a given pair $(A, B)$ of operators, Du and Pan (see [5]) give a necessary and sufficient condition for which $M_C$ is invertible for some $C \in B(K, \mathcal{H})$, Han et al. (see [9]) extended the result for operators $A, B, C$ on Banach space. For the essential spectrum $\sigma_e(T)$, the Weyl spectrum $\sigma_w(T)$ and the Browder spectrum $\sigma_b(T)$ of $T$, analogous results have been obtained in many literature (see [2,3,5,7]).

Throughout this paper, let $\mathcal{H}$ and $\mathcal{K}$ be complex separable Hilbert spaces, let $B(\mathcal{H}, \mathcal{K})$, $B_l(\mathcal{H}, \mathcal{K})$ and Inv$(\mathcal{H}, \mathcal{K})$, respectively, denote the set of bounded linear operators, left invertible bounded linear operators and invertible bounded linear operators from $\mathcal{H}$ to $\mathcal{K}$, respectively, and abbreviate $B(\mathcal{H}, \mathcal{H})$ to $B(\mathcal{H})$. If $A \in B(\mathcal{H})$, $B \in B(\mathcal{K})$ and $C \in B(\mathcal{K}, \mathcal{H})$, we define an operator $M_C$ acting on $\mathcal{H} \oplus \mathcal{K}$ by the form

$$M_C := \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}.$$ 

For an operator $T$, we use $N(T)$ and $R(T)$ to denote the null space and the range of $T$, respectively. Let $n(T)$ be the nullity of $T$ which is equal to $\dim N(T)$, and let $d(T)$ be the deficiency of $T$ which is equal to $\dim N(T^*)$. An operator $T \in B(\mathcal{H}, \mathcal{K})$ (or $B(\mathcal{H})$) is said to be upper semi-Fredholm if $R(T)$ is closed and $N(T)$ has finite dimension and lower semi-Fredholm if $R(T)$ is closed and $N(T^*)$ has finite dimension. An operator $T$ is called Fredholm if it is both upper semi-Fredholm and lower semi-Fredholm. Let $\Phi_+(H)$ ($\Phi_-(H)$) denotes the set of all upper (lower) semi-Fredholm operators. For an operator $T$, the left (right) essential spectrum $\sigma_{le}(T)$ ($\sigma_{re}(T)$) is defined by

$$\sigma_{le}(T)(\sigma_{re}(T)) = \{\lambda \in \mathbb{C}: T - \lambda \text{ is not upper (lower) semi-Fredholm}\}.$$ 

If $T$ is a semi-Fredholm operator, we define the index of $T$ by $\text{ind}(T) = n(T) - d(T)$. An operator $T \in B(\mathcal{H}, \mathcal{K})$ is called Weyl if it is a Fredholm operator of index zero.

Let $\Phi_+(H)$ ($\Phi_+(H, K)$) (introduced in [11]) be the class of all $T \in \Phi_+(H)$ ($T \in \Phi_+(H, K)$) with $\text{ind}(T) \leq 0$ for any $T \in B(\mathcal{H})$ ($T \in B(\mathcal{H}, \mathcal{K})$), let $\Phi^+(H)$ ($\Phi^+(H, K)$) be the class of all $T \in \Phi^-(H)$ ($T \in \Phi^-(H, K)$) with $\text{ind}(T) \geq 0$ for any $T \in B(\mathcal{H})$ ($T \in B(\mathcal{H}, \mathcal{K})$), let

$$\sigma_{ea}(T) = \{\lambda \in \mathbb{C}: T - \lambda \text{ is not in } \Phi_+(H)\},$$

$$\sigma_{SF^+}(T) = \{\lambda \in \mathbb{C}: T - \lambda \text{ is not in } \Phi^+(H)\},$$

$$\sigma_w(T) = \sigma_{ea}(T) \cup \sigma_{SF^+}(T).$$

Cao and Meng (in [2]) give a necessary and sufficient condition for which $M_C \in \Phi_+(H)$ for some $C \in B(\mathcal{K}, \mathcal{H})$ and characterize the set of $\bigcap_{C \in B_l(\mathcal{K}, \mathcal{H})} \sigma_{ea}(M_C)$. In this paper, our main goal is to characterize the intersection of $\bigcap_{C \in B_l(\mathcal{K}, \mathcal{H})} \sigma_{ea}(M_C)$ and $\bigcap_{C \in \text{Inv}(\mathcal{K}, \mathcal{H})} \sigma_{ea}(M_C)$. This paper is organized as follows. In Section 2, we give a necessary and sufficient condition for which $M_C \in \Phi_+(H)$ for some $C \in B_l(\mathcal{K}, \mathcal{H})$ and get

$$\bigcap_{C \in B_l(\mathcal{K}, \mathcal{H})} \sigma_{ea}(M_C) = \bigcap_{C \in B_l(\mathcal{K}, \mathcal{H})} \sigma_{ea}(M_C).$$

In Section 3, we give a necessary and sufficient condition for which $M_C \in \Phi_+(H)$ for some $C \in \text{Inv}(\mathcal{K}, \mathcal{H})$ and get

$$\bigcap_{C \in \text{Inv}(\mathcal{K}, \mathcal{H})} \sigma_{ea}(M_C) = \bigcap_{C \in B_l(\mathcal{K}, \mathcal{H})} \sigma_{ea}(M_C) \cup \{\lambda \in \mathbb{C}: B - \lambda \text{ is compact}\}.$$
In Section 4, we give a necessary and sufficient condition for which $M_C \in \Phi_+^-(H)$ for all $C \in \Inv(K, H)$. In addition, the idea in this paper is different from [2].

2. $\bigcap_{C \in B_r(K, H)} \sigma_{ea}(M_C)$

In order to prove our main results, we begin with some lemmas.

**Lemma 2.1.** Let $A \in B(H)$, $B \in B(K)$ and $C \in B(K, H)$. If $C$ as an operator from $N(B) \oplus N(B)^\perp$ into $R(A)^\perp \oplus R(A)$ has the following operator matrix:

$$C = \begin{pmatrix} C_1 & C_2 \\ C_3 & C_4 \end{pmatrix},$$

then

(a) $M_C \in \Phi_+^-(H \oplus K)$ if and only if

(i) $A \in \Phi_+(H)$;

(ii) $M_1 \in \Phi_+^-(N(A) \oplus N(B) \oplus N(B)^\perp, R(A)^\perp \oplus R(B)^\perp \oplus R(B))$ where

$$M_1 := \begin{pmatrix} 0 & C_1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & B_1 \end{pmatrix} : N(A) \oplus N(B) \oplus N(B)^\perp \to R(A)^\perp \oplus R(B)^\perp \oplus R(B),$$

where $B$ as an operator from $N(B) \oplus N(B)^\perp$ into $R(B)^\perp \oplus R(B)$ has the operator matrix $B = \begin{pmatrix} 0 & 0 \\ 0 & B_1 \end{pmatrix}$.

(b) If $R(B)$ is closed, then $M_C \in \Phi_+^-(H \oplus K)$ if and only if

(i) $A \in \Phi_+(H)$;

(ii) $C_1$ is an operator with $R(C_1)$ is closed, $n(C_1) < \infty$, and $n(C_1) + n(A) \leq d(C_1) + d(B)$.

**Proof.** (a) **Sufficiency.** Since $A \in \Phi_+(H)$, then $R(A)$ is closed. The space $H \oplus K$ can be decomposed as the following direct sums:

$$H \oplus K = N(A) \oplus N(A)^\perp \oplus N(B) \oplus N(B)^\perp = R(A)^\perp \oplus R(A) \oplus \overline{R(B)} \oplus R(B)^\perp.$$

Thus $M_C$ as an operator from $N(A) \oplus N(A)^\perp \oplus N(B) \oplus N(B)^\perp$ into $R(A)^\perp \oplus R(A) \oplus \overline{R(B)} \oplus R(B)^\perp$ has the following operator matrix:

$$M_C = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix} = \begin{pmatrix} 0 & 0 & C_1 & C_2 \\ 0 & A_1 & C_3 & C_4 \\ 0 & 0 & 0 & B_1 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

where $A_1$ is an operator from $N(A)^\perp$ onto $R(A)$ and $B_1$ is an operator from $N(B)^\perp$ into $\overline{R(B)}$.

By the assumption that $A \in \Phi_+(H)$, $A_1$ is an invertible operator. In this case, we have

$$\begin{pmatrix} 0 & 0 & C_1 & C_2 \\ 0 & A_1 & C_3 & C_4 \\ 0 & 0 & 0 & B_1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} I & 0 & 0 & 0 \\ 0 & I & -A_1^{-1}C_3 & -A_1^{-1}C_4 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{pmatrix} = \begin{pmatrix} 0 & 0 & C_1 & C_2 \\ 0 & A_1 & 0 & 0 \\ 0 & 0 & 0 & B_1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$
where
\[
\begin{pmatrix}
I & 0 & 0 & 0 \\
0 & I & -A_1^{-1}C_3 & -A_1^{-1}C_4 \\
0 & 0 & I & 0 \\
0 & 0 & 0 & I
\end{pmatrix}
\]
is an invertible operator from \(N(A) \oplus N(A)^\perp \oplus N(B) \oplus N(B)^\perp\) onto \(N(A) \oplus N(A)^\perp \oplus N(B) \oplus N(B)^\perp\). Thus \(M_C \in \Phi_+^-(H \oplus K)\) if and only if
\[
\begin{pmatrix}
0 & 0 & C_1 & C_2 \\
0 & A_1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & B_1
\end{pmatrix}
\in \Phi_+^-(N(A) \oplus N(B) \oplus N(B)^\perp, R(A)^\perp \oplus R(A) \oplus R(B)^\perp \oplus \overline{R(B)}).
\]
It follows that if \(A \in \Phi_+(H)\) then \(M_C \in \Phi_+^-(H \oplus K)\) if and only if
\[
M_1 = \begin{pmatrix}
0 & C_1 & C_2 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & B_1
\end{pmatrix}
\in \Phi_+^-(N(A) \oplus N(B) \oplus N(B)^\perp, R(A)^\perp \oplus R(A) \oplus R(B)^\perp \oplus \overline{R(B)}).
\]

**Necessity.** Clearly, \(A \in \Phi_+(H)\). From the discussion above, it is not difficult to get (ii).

(b) If \(R(B)\) is closed, then \(B_1\) as an operator from \(N(B)^\perp\) into \(R(A)^\perp\) is invertible. Thus
\[
\begin{pmatrix}
I & 0 & -C_2B_1^{-1} \\
0 & I & 0 \\
0 & 0 & I
\end{pmatrix}
\begin{pmatrix}
0 & C_1 & C_2 \\
0 & 0 & 0 \\
0 & 0 & B_1
\end{pmatrix} = \begin{pmatrix}
0 & C_1 & 0 \\
0 & 0 & 0 \\
0 & 0 & B_1
\end{pmatrix}.
\]
Since \(n(A) < \infty\) and \(B_1\) is invertible, we conclude that \(M_1 \in \Phi_+^-(N(A) \oplus N(B) \oplus N(B)^\perp, R(A)^\perp \oplus R(B)^\perp \oplus \overline{R(B)})\) if and only if \(R(C_1)\) is closed, \(n(C_1) < \infty\) and \(n(C_1) + n(A) \leq d(C_1) + d(B)\). \(\square\)

**Corollary 2.2.** Let \((A, B)\) be a given pair of operators. If \(A \in \Phi_+(H), R(B)\) is closed and \(d(A) + d(B) < n(A) + n(B)\), then for all \(C \in B(K, H)\), \(M_C \notin \Phi_+^-(H \oplus K)\).

**Proof.** Suppose that \(C\) has the operator matrix form (1) for all \(C \in B(K, H)\).

(i) \(n(B) = \infty\). Since \(d(A) < \infty\), then \(n(C_1) = \infty\) for all \(C\). By Lemma 2.1, \(M_C \notin \Phi_+^-(H \oplus K)\).

(ii) \(n(B) < \infty, n(A) < \infty, d(A) < \infty\) and \(d(B) < \infty\). Since \(C_1\) is an operator from \(N(B)\) into \(R(A)^\perp\), then
\[
n(B) = n(C_1) + \dim N(C_1)^\perp \quad \text{and} \quad d(A) = d(C_1) + \dim R(C_1).
\]
Thus \(n(C_1) + n(A) > d(B) + d(C_1)\) since \(n(B) + n(A) > d(A) + d(B)\) and \(\dim N(C_1)^\perp = \dim R(C_1)\). From Lemma 2.1, \(M_C \notin \Phi_+^-(H \oplus K)\) for all \(C\). \(\square\)

**Corollary 2.3.** If \(R(B)\) is closed, \(A \in \Phi_+(H)\) and \(n(B) + n(A) \leq d(B) + d(A)\), then \(M_C \notin \Phi_+^-(H \oplus K)\) for any \(C \in B(K, H)\) if and only if \(d(A) < \infty\) and \(n(B) = d(B) = \infty\).

**Proof.** Suppose that \(C\) has the operator matrix form (1).
Sufficiency is clear, since \( n(B) = d(B) = \infty \) and \( d(A) < \infty \), then \( n(C_1) = \infty \). By Lemma 2.1, \( M_C \notin \Phi_+^-(H \oplus K) \) for all \( C \).

Necessity. Suppose that \( d(A) < \infty \) and \( n(B) = d(B) = \infty \) are not satisfied. There are four cases to consider.

**Case 1.** \( n(B) = d(A) = \infty \).

Assume that \( n(A) \leq d(B) \). Let \( S \) be an unitary operator from \( N(B) \) onto \( R(A)^\perp \). Since \( A \in \Phi_+(H) \), \( \dim R(A) = \infty \), let \( S_1 \) be a left invertible operator from \( N(B)^\perp \) into \( R(A) \). Define an operator \( C_0 \) by

\[
C_0 = \begin{pmatrix} S & 0 \\ 0 & S_1 \end{pmatrix} : N(B) + N(B)^\perp \to R(A)^\perp + R(A)
\]

then \( MC_0 \in \Phi_+^-(H \oplus K) \) by Lemma 2.1.

If \( n(A) > d(B) \) and \( \{e_i\}_{i=1}^\infty \) and \( \{f_i\}_{i=1}^\infty \) are orthogonal bases of \( N(B) \) and \( R(A)^\perp \), respectively, denote \( n(A) - d(B) = m \), and define \( C_1 \) as an operator from \( N(B) \) into \( R(A)^\perp \) by

\[
C_1(e_i) = f_{m+i}, \quad i = 1, 2, \ldots
\]

Clearly, \( n(C_1) = 0 \) and \( n(C_1^*) = m \), then \( n(C_1) + n(A) = d(C_1) + d(B) \). Define an operator \( C_0 \) by \( C_0 = \left( \begin{array}{cc} C_1 & 0 \\ 0 & S_1 \end{array} \right) \). From Lemma 2.1, \( MC_0 \in \Phi_+^-(H \oplus K) \).

**Case 2.** \( n(B) < \infty \) and \( d(A) = \infty \).

It is easy to show that \( M_C \in \Phi_+^-(H \oplus K) \), for all \( C \in \mathcal{B}(\mathcal{K}, \mathcal{H}) \).

**Case 3.** \( n(B) < \infty \), \( d(A) < \infty \) and \( d(B) = \infty \).

It is clear that \( M_C \in \Phi_+^-(H \oplus K) \), for all \( C \in \mathcal{B}(\mathcal{K}, \mathcal{H}) \).

**Case 4.** \( n(B) < \infty \), \( d(A) < \infty \) and \( d(B) < \infty \).

As the similar way with the proof of Corollary 2.2(ii), we can prove that \( n(C_1) < \infty \) and \( n(C_1) + n(A) \leq d(C_1) + d(B) \) for all \( C \). \( \square \)

The following theorem is our main result in this section.

**Theorem 2.4.** For a given pair \((A, B)\) of operators, we have

\[
\bigcap_{C \in \mathcal{B}(\mathcal{K}, \mathcal{H})} \sigma_{\text{sc}}(MC) = \sigma_{\text{le}}(A) \cup \Phi_+\text{lw}(A, B) \cup \Upsilon_\text{lw}(A, B) \cup \Psi_1(A, B),
\]

where

\[
\Psi_1(A, B) = \{ \lambda \in \mathbb{C}: R(B - \lambda) \text{ is not closed and } d(A - \lambda) < \infty \},
\]

\[
\Phi_+\text{lw}(A, B) = \{ \lambda \in \mathbb{C}: R(B - \lambda) \text{ is closed and } n(B - \lambda) + n(A - \lambda) > d(B - \lambda) + d(A - \lambda) \},
\]

\[
\Upsilon_\text{lw}(A, B) = \{ \lambda \in \mathbb{C}: R(B - \lambda) \text{ is closed, } n(B - \lambda) = d(B - \lambda) = \infty \text{ and } d(A - \lambda) < \infty \}.
\]
Proof. For convenience, we divide the proof into two steps.

Step 1. If \( \lambda \in \Psi_l(A, B) \setminus \sigma_{le}(A) \), then for all \( C \in B(K, H) \), \( MC - \lambda \notin \Phi_\mp(H \oplus K) \).

Suppose that \( MC - \lambda \) has the operator matrix (3) and \( C \) has the operator matrix (1). By Lemma 2.1, for all \( C \in B(K, H) \), \( MC - \lambda \) is in \( \Phi_\mp+ (H \oplus K) \) if and only if

\[
\begin{pmatrix}
0 & C_1 & C_2 \\
0 & 0 & 0 \\
0 & 0 & B_1 - \lambda
\end{pmatrix}
\notin \Phi_\mp(N(A - \lambda) \oplus N(B - \lambda) \oplus N(B - \lambda)^\perp, R(A - \lambda)^\perp \oplus R(B - \lambda)^\perp \oplus \overline{R(B - \lambda)}),
\]

for all \( C_1 \in B(N(B - \lambda), R(A - \lambda)^\perp) \), \( C_2 \in B(N(B - \lambda)^\perp, R(A - \lambda)^\perp) \).

Conversely, assume that there exist \( C_0 \in B(N(B - \lambda), R(A - \lambda)^\perp) \) and \( C_2^0 \in B(N(B - \lambda)^\perp, R(A - \lambda)^\perp) \) such that

\[
\begin{pmatrix}
0 & C_1 & C_2 \\
0 & 0 & 0 \\
0 & 0 & B_1 - \lambda
\end{pmatrix}
\in \Phi_\mp(N(A) \oplus N(B) \oplus N(B)^\perp, R(A)^\perp \oplus R(B)^\perp \oplus \overline{R(B)}).
\]

Then it is upper semi-Fredholm. By the assumption that \( \lambda \in \Psi_l(A, B) \setminus \sigma_{le}(A) \), we have \( d(A - \lambda) < \infty \). It follows that \( C_0 \) and \( C_2^0 \) are compact operators. Using 3.11 in Chapter XI of [1], we conclude that

\[
\begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & B_1 - \lambda
\end{pmatrix}
\]

is upper semi-Fredholm. Thus \( R(B_1 - \lambda) \) is closed. But \( \lambda \in \Psi_l(A, B) \setminus \sigma_{le}(A) \) implies that \( R(B - \lambda) \) is not closed. This is a contradiction.

Step 2. If \( \lambda \in \{ \lambda \in \mathbb{C} : R(B - \lambda) \) is not closed, \( d(A - \lambda) = \infty \} \setminus \sigma_{le}(A) \), then there exists \( C_0 \in B_1(K, H) \), such that \( MC_0 - \lambda \notin \Phi_\mp(H \oplus K) \).

Let \( H_1 \) be a closed subspace of \( R(A - \lambda)^\perp \) with \( \dim H_1 = \dim N(B - \lambda) \) and \( \dim R(A - \lambda)^\perp \oplus H_1 = \dim N(B - \lambda)^\perp \). Let \( C_1 \) and \( C_2 \) be unitary operators from \( N(B - \lambda) \) onto \( H_1 \) and from \( N(B - \lambda)^\perp \) onto \( R(A - \lambda)^\perp \oplus H_1 \), respectively. Define

\[
C_0 = \begin{pmatrix}
C_1 \\
0
\end{pmatrix} : N(B - \lambda) \oplus N(B - \lambda)^\perp \to R(A - \lambda)^\perp \oplus R(A - \lambda).
\]

Clearly,

\[
\begin{pmatrix}
C_1^* & 0 \\
C_2^* & 0
\end{pmatrix} \begin{pmatrix}
C_1 & C_2 \\
0 & 0
\end{pmatrix} = \begin{pmatrix}
I & 0 \\
0 & I
\end{pmatrix},
\]

where

\[
\begin{pmatrix}
C_1^* & 0 \\
C_2^* & 0
\end{pmatrix} : R(A - \lambda)^\perp \oplus R(A - \lambda) \to N(B - \lambda) \oplus N(B - \lambda)^\perp.
\]

Thus \( C_0 \) is left invertible. Since

\[
\begin{pmatrix}
I & 0 & 0 \\
0 & I & 0 \\
-(B_1 - \lambda)C_2^* & 0 & I
\end{pmatrix} \begin{pmatrix}
0 & C_1 & C_2 \\
0 & 0 & 0 \\
0 & 0 & B_1 - \lambda
\end{pmatrix} = \begin{pmatrix}
0 & C_1 & C_2 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
\]

for all \( C \in B(K, H) \), it follows that \( MC_0 - \lambda \) is in \( \Phi_\mp(H \oplus K) \) if and only if

\[
\begin{pmatrix}
0 & C_1 & C_2 \\
0 & 0 & 0 \\
0 & 0 & B_1 - \lambda
\end{pmatrix}
\notin \Phi_\mp(N(A - \lambda) \oplus N(B - \lambda) \oplus N(B - \lambda)^\perp, R(A - \lambda)^\perp \oplus R(B - \lambda)^\perp \oplus \overline{R(B - \lambda)}),
\]

for all \( C_1 \in B(N(B - \lambda), R(A - \lambda)^\perp) \), \( C_2 \in B(N(B - \lambda)^\perp, R(A - \lambda)^\perp) \).

Conversely, assume that there exist \( C_0 \in B(N(B - \lambda), R(A - \lambda)^\perp) \) and \( C_2^0 \in B(N(B - \lambda)^\perp, R(A - \lambda)^\perp) \) such that

\[
\begin{pmatrix}
0 & C_1 & C_2 \\
0 & 0 & 0 \\
0 & 0 & B_1 - \lambda
\end{pmatrix}
\in \Phi_\mp(N(A) \oplus N(B) \oplus N(B)^\perp, R(A)^\perp \oplus R(B)^\perp \oplus \overline{R(B)}).
\]

Then it is upper semi-Fredholm. By the assumption that \( \lambda \in \Psi_l(A, B) \setminus \sigma_{le}(A) \), we have \( d(A - \lambda) < \infty \). It follows that \( C_0 \) and \( C_2^0 \) are compact operators. Using 3.11 in Chapter XI of [1], we conclude that

\[
\begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & B_1 - \lambda
\end{pmatrix}
\]

is upper semi-Fredholm. Thus \( R(B_1 - \lambda) \) is closed. But \( \lambda \in \Psi_l(A, B) \setminus \sigma_{le}(A) \) implies that \( R(B - \lambda) \) is not closed. This is a contradiction.
and
\[
\begin{pmatrix}
0 & C_1 & C_2 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix} \in \Phi_+\left((N(A - \lambda) \oplus N(B - \lambda)) \oplus N(B - \lambda)^\perp, R(A - \lambda)^\perp \oplus R(B - \lambda)^\perp \oplus R(B - \lambda)\right),
\]
\[M_{C_0} - \lambda \in \Phi_+(H \oplus K), \text{ by Lemma 2.1(a)}.\]
Finally, by Step 1, we can conclude that
\[
\bigcap_{C \in B_1(K, H)} \sigma_{\text{ca}}(M_C) \supseteq \Psi_{l}(A, B) \setminus \sigma_{\text{le}}(A).
\]
By Corollaries 2.2 and 2.3, it is easy to see that
\[
\bigcap_{C \in B_1(K, H)} \sigma_{\text{ca}}(M_C) \supseteq \left(\Psi_{l}(A, B) \setminus \sigma_{\text{le}}(A)\right) \cup \sigma_{\text{le}}(A) \cup \Phi_{\text{lw}}(A, B) \cup \Upsilon_{\text{lw}}(A, B)
\]
\[= \sigma_{\text{le}}(A) \cup \Phi_{\text{lw}}(A, B) \cup \Upsilon_{\text{lw}}(A, B) \cup \Psi_{l}(A, B).\]
By Corollary 2.3 and Step 2, we get that
\[
\bigcap_{C \in B_1(K, H)} \sigma_{\text{ca}}(M_C) \subseteq \sigma_{\text{le}}(A) \cup \Phi_{\text{lw}}(A, B) \cup \Upsilon_{\text{lw}}(A, B) \cup \Psi_{l}(A, B).
\]
Combining the two inclusions above, we obtain
\[
\bigcap_{C \in B_1(K, H)} \sigma_{\text{ca}}(M_C) = \sigma_{\text{le}}(A) \cup \Phi_{\text{lw}}(A, B) \cup \Upsilon_{\text{lw}}(A, B) \cup \Psi_{l}(A, B). \quad \square
\]
The following corollaries are immediate from Theorem 2.4.

**Corollary 2.5.** (See [2].) For given \(A \in B(H), B \in B(K)\), we have
\[
\bigcap_{C \in B_1(K, H)} \sigma_{\text{ca}}(M_C) = \sigma_{\text{le}}(A) \cup \Phi_{\text{lw}}(A, B) \cup \Upsilon_{\text{lw}}(A, B) \cup \Psi_{l}(A, B).
\]

**Corollary 2.6.** (See [2].) For a given pair \((A, B)\) of operators, we have
\[
\bigcap_{C \in B_1(K, H)} \sigma_{\text{SF}^+}(M_C) = \bigcap_{C \in B(K, H)} \sigma_{\text{SF}^+}(M_C)
\]
\[= \sigma_{\text{re}}(B) \cup \Phi_{\text{rw}}(A, B) \cup \Upsilon_{\text{rw}}(A, B) \cup \Psi_{r}(A, B),\]
where
\[
\Phi_{\text{rw}}(A, B) = \{ \lambda \in \mathbb{C}: R(A - \lambda) \text{ is closed}, \ n(B - \lambda) + n(A - \lambda) < d(B - \lambda) + d(A - \lambda) \},
\]
\[
\Upsilon_{\text{rw}}(A, B) = \{ \lambda \in \mathbb{C}: R(A - \lambda) \text{ is closed, } n(A - \lambda) = d(A - \lambda) = \infty, n(B - \lambda) < \infty \},
\]
\[
\Psi_{r}(A, B) = \{ \lambda \in \mathbb{C}: R(A - \lambda) \text{ is closed, } n(B - \lambda) < \infty \}.
\]
It is a natural question that whether the equation 
\[ \bigcap_{C \in \text{Inv}(\mathcal{K}, \mathcal{H})} \sigma_{ea}(M_C) = \bigcap_{C \in B(\mathcal{K}, \mathcal{H})} \sigma_{ea}(M_C) \]
holds?

3. \( \bigcap_{C \in \text{Inv}(\mathcal{K}, \mathcal{H})} \sigma_{ea}(M_C) \)

In this section, our main result is:

**Theorem 3.1.** For given pair of operators \((A, B)\), we have
\[ \bigcap_{C \in \text{Inv}(\mathcal{K}, \mathcal{H})} \sigma_{ea}(M_C) = \bigcap_{C \in B(\mathcal{K}, \mathcal{H})} \sigma_{ea}(M_C) \cup \{ \lambda \in \mathbb{C} : B - \lambda \text{ is compact} \}. \]

We need the following lemmas.

**Lemma 3.2.** Let \( A \in B(\mathcal{H}) \), \( B \in B(\mathcal{K}) \) and \( C \in B(\mathcal{K}, \mathcal{H}) \). If \( C \) has the operator matrix (1), then \( MC \) is invertible if and only if \( A \) is left invertible, \( B \) is right invertible and \( C_1 \) is invertible.

**Proof.** *Sufficiency.* Since \( A \) is left invertible, \( A_1 \) is invertible. Then
\[
\begin{pmatrix}
0 & C_1 & C_2 \\
A_1 & C_3 & C_4 \\
0 & 0 & B_1
\end{pmatrix}
\begin{pmatrix}
I & -A_1^{-1}C_3 & -A_1^{-1}C_4 \\
0 & I & 0 \\
0 & 0 & I
\end{pmatrix}
= \begin{pmatrix}
0 & C_1 & C_2 \\
A_1 & 0 & 0 \\
0 & 0 & B_1
\end{pmatrix}.
\]
Since \( B \) is right invertible, \( B_1 \) is invertible. Then
\[
\begin{pmatrix}
I & 0 & -C_2B_1^{-1} \\
0 & I & 0 \\
0 & 0 & I
\end{pmatrix}
\begin{pmatrix}
0 & C_1 & C_2 \\
A_1 & 0 & 0 \\
0 & 0 & B_1
\end{pmatrix}
= \begin{pmatrix}
0 & C_1 & 0 \\
A_1 & 0 & 0 \\
0 & 0 & B_1
\end{pmatrix}.
\]
Therefore, if \( C_1 \) is invertible, then \( MC \) is invertible.

*Necessity.* If \( MC \) is invertible, then \( A \) is left invertible, \( B \) is right invertible. By the proof of sufficiency, we have that \( C_1 \) is invertible. \( \square \)

**Lemma 3.3.** If \( A \in \Phi_+(H), n(B) + n(A) \leq d(A) + d(B) \) and \( R(B) \) is closed, then \( MC \notin \Phi_-(H) \) for any \( C \in \text{Inv}(\mathcal{K}, \mathcal{H}) \) if and only if one of the following conditions holds:

(i) \( \dim N(B)^{\perp} < \infty \),
(ii) \( n(B) = d(B) = \infty \) and \( d(A) < \infty \).

**Proof.** *Necessity.* Assume that (ii) is not satisfied. To show that (i) holds, we will prove that \( \dim N(B)^{\perp} = \infty \) then there exist some \( C \in \text{Inv}(\mathcal{K}, \mathcal{H}) \) such that \( MC \in \Phi_-(H \oplus K) \). By Corollary 2.3, we only need to show that if \( n(B) = d(A) = \infty \) and \( \dim N(B)^{\perp} = \infty \) then there exist some \( C \in \text{Inv}(\mathcal{K}, \mathcal{H}) \) such that \( MC \in \Phi_-(H \oplus K) \).

**Case 1.** \( n(A) \leq d(B) \). Let \( S \) be an unitary operator from \( N(B) \) onto \( R(A)^{\perp} \) and \( S_1 \) an invertible operator from \( N(B)^{\perp} \) onto \( R(A) \), since \( \dim R(A) = \infty \). Set
\[
C_0 = \begin{pmatrix} S & 0 \\ 0 & S_1 \end{pmatrix} : N(B) \oplus N(B)^{\perp} \rightarrow R(A)^{\perp} \oplus R(A)
\]
then \( MC_0 \in \Phi_-(H \oplus K) \) by Lemma 2.1.
Lemma 3.6. Let $w_1$ be a left invertible operator from $N(B)$ into $R(A) \perp$ with $n(w_1^\perp) = 2n$, $w_3$ be a right invertible operator from $N(B) \perp$ into $R(A)$ with $n(w_3) = 2n$, and $w_2$ be an invertible operator from $N(B)^\perp$ into $R(A) \perp$ such that $P_{N(w_1^\perp)}w_2|_{N(w_3)}$ is an invertible operator from $N(w_3)$ onto $N(w_1^\perp)$, respectively, where $P_{N(w_1^\perp)}$ is the orthogonal projection onto $N(w_1^\perp)$. Clearly, $n(w_1) + n(A) \leq d(w_1) + d(B)$. Set
\[
C_0 = \begin{pmatrix}
w_1 & w_2 \\
0 & w_3
\end{pmatrix}: N(B) \oplus N(B)^\perp \to R(A) \perp \oplus R(A).
\]
It is easy to see that $C_0$ is invertible, by Lemma 3.2. From Lemma 2.1, $M_{C_0} \in \Phi_+^-(H \oplus K)$.

Sufficiency. By Corollary 2.3, we only need to show that if $\dim N(B)^\perp < \infty$, then $M_C \notin \Phi_+^-(H)$ for all $C \in \text{Inv}(K, \mathcal{H})$. Since $A \in \Phi_+(H)$, $\dim R(A) = \infty$. By the contrary, assume that $M_C \in \Phi_+^-(H \oplus K)$, where $C \in \text{Inv}(K, \mathcal{H})$. We have that $C_1$ is an operator with $R(C_1)$ is closed, $n(C_1) < \infty$, by Lemma 2.1. Suppose that $C_1^+C_1 = I_{N(B)} + K_0$ (see [8], Atkinson’s theorem), where $K_0$ is a compact operator from $N(B)$ into $N(B)$. Thus
\[
\begin{pmatrix}
I & 0 \\
-C_3C_1^+ & I
\end{pmatrix}
\begin{pmatrix}
C_1 & C_2 \\
C_3 & C_4
\end{pmatrix}
= \begin{pmatrix}
C_1 & C_2 \\
-C_3K_0 & C_4 - C_3C_1^+C_2
\end{pmatrix}
\]
is invertible. Using 3.11 in Chapter XI of [1], we get that
\[
\begin{pmatrix}
C_1 & 0 \\
0 & 0
\end{pmatrix}
: N(B) \oplus N(B)^\perp \to R(A)^\perp \oplus R(A)
\]
is Fredholm. But this is a contradiction with the fact that $\dim R(A) = \infty$. \hfill \Box

Lemma 3.4. If $A \in \Phi_+(H)$ and $B$ is compact, then for all $C \in \text{Inv}(K, \mathcal{H})$, $M_C \notin \Phi_+^-(H \oplus K)$.

Proof. Suppose, contrary to the assertion, that $M_{C_0} \in \Phi_+^-(H \oplus K)$, for some $C_0 \in \text{Inv}(K, \mathcal{H})$.
\[
\begin{pmatrix}
I & 0 \\
-BC_0^{-1} & I
\end{pmatrix}
\begin{pmatrix}
A & C_0 \\
0 & B
\end{pmatrix}
\begin{pmatrix}
I & 0 \\
-C_0^{-1}A & I
\end{pmatrix}
= \begin{pmatrix}
0 & C_0 \\
-BC_0^{-1}A & 0
\end{pmatrix},
\]
then $-BC_0^{-1}A \in \Phi_+(H, K)$. This is a contradiction with compactness of $-BC_0^{-1}A$. Hence, for all $C \in \text{Inv}(K, \mathcal{H})$, $M_C \notin \Phi_+^-(H \oplus K)$. \hfill \Box

Lemma 3.5. [6] Let $V$ be a linear subspace of $\mathcal{H}$. These are equivalent:

(1) Any bounded operator $A$ on $\mathcal{H}$ with $R(A) \subseteq V$ is compact;
(2) $V$ contains no closed infinite-dimensional subspace.

Lemma 3.6. If $A \in \Phi_+(H)$, $R(B)$ is not closed and $d(A) = \infty$, then $B$ is not compact if and only if there exists $C \in \text{Inv}(K, \mathcal{H})$ such that $M_C \in \Phi_+^-(H \oplus K)$.

Proof. Sufficiency. If $B$ is compact, by Lemma 3.4, $M_C \notin \Phi_+^-(H \oplus K)$, for any $C \in \text{Inv}(K, \mathcal{H})$.

Necessity. If $B$ is not compact, by Lemma 3.5, $R(B)$ contains a closed infinite-dimensional subspace. No loss of generality, suppose that $K_1$ is closed subspace of $R(B)$ with $\dim K_1 = \infty$ and $\dim K_1^\perp = \infty$. Let $H_1 = \{x \in N(B)^\perp: Bx \in K_1\}$. Thus $H_1$ is a closed subspace of
$N(B)\perp$ and $\dim H_1 = \infty$. Denote $H_1\perp = N(B)\perp \ominus H_1$. No loss of generality, we may assume that $\dim H_1\perp = \infty$. (Otherwise, suppose that $\{e_n\}_{n=1}^{\infty}$ is an orthonormal basis of $H_1$. Denote $H_0 = \text{span}\{e_n: n = 2, 4, 6, \ldots\}$ and $K_0 = \{Bx: x \in H_0\}$, then $H_1$ and $K_1$ can be instead by $H_0$ and $K_0$, respectively.) Since $d(A) = \infty$, let $R(A)^* = H_2 \ominus H_2\perp$ with $\dim H_2 = \dim N(B)$ and $\dim H_2\perp = \infty$. Define an operator $C: \mathcal{K} \to \mathcal{H}$ by

$$C = \begin{pmatrix} V_1 & 0 & 0 \\ 0 & V_2 & 0 \\ 0 & 0 & V_3 \end{pmatrix}: N(B) \ominus H_1\perp \ominus H_1 \to H_2 \ominus H_2\perp \ominus R(A),$$

where $V_1$, $V_2$ and $V_3$ are unitary operators. Obviously, $C$ is invertible. Suppose that $B_1 = B|_{N(B)\perp}$, then

$$B_1 = \begin{pmatrix} B_{11} & B_{12} \\ B_{13} & 0 \end{pmatrix}: H_1\perp \ominus H_1 \to K_1 \ominus K_1\perp,$n

where $B_{12}$ is an invertible operator from $H_1$ onto $K_1$. Hence $M_1$ (as Lemma 2.1) has the following operator matrix form:

$$M_1 = \begin{pmatrix} 0 & V_1 & 0 & 0 \\ 0 & 0 & V_2 & 0 \\ 0 & 0 & B_{11} & B_{12} \\ 0 & 0 & B_{13} & 0 \end{pmatrix}: N(A) \ominus N(B) \ominus H_1\perp \ominus H_1 \to H_2 \ominus H_2\perp \ominus K_1 \ominus K_1\perp.$n

Let

$$W = \begin{pmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & -B_{11}V_2^* & I & 0 \\ 0 & -B_{13}V_2^* & 0 & I \end{pmatrix}: H_2 \ominus H_2\perp \ominus K_1 \ominus K_1\perp \to H_2 \ominus H_2\perp \ominus K_1 \ominus K_1\perp.$n

Then

$$WM_1 = \begin{pmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & -B_{11}V_2^* & I & 0 \\ 0 & -B_{13}V_2^* & 0 & I \end{pmatrix} \begin{pmatrix} 0 & V_1 & 0 & 0 \\ 0 & 0 & V_2 & 0 \\ 0 & 0 & B_{11} & B_{12} \\ 0 & 0 & B_{13} & 0 \end{pmatrix} = \begin{pmatrix} 0 & V_1 & 0 & 0 \\ 0 & 0 & V_2 & 0 \\ 0 & 0 & B_{11} & B_{12} \\ 0 & 0 & 0 & 0 \end{pmatrix}.$n

It is easy to show that

$$\begin{pmatrix} 0 & V_1 & 0 & 0 \\ 0 & 0 & V_2 & 0 \\ 0 & 0 & 0 & B_{12} \\ 0 & 0 & 0 & 0 \end{pmatrix} \in \Phi_+^-(N(A) \ominus N(B) \ominus H_1\perp \ominus H_1, H_2 \ominus H_2\perp \ominus K_1 \ominus K_1\perp).$$

Therefore, $M_1 \in \Phi_+^-(N(A) \ominus N(B) \ominus H_1\perp \ominus H_1, H_2 \ominus H_2\perp \ominus K_1 \ominus K_1\perp).$ By Lemma 2.1, $M_C \in \Phi_+^-(H \ominus K)$. $\square$

**Proof of Theorem 3.1.** By Lemma 3.4, it is clear that

$$\bigcap_{C \in \text{Inv}(\mathcal{K}, \mathcal{H})} \sigma_{ea}(M_C) \supseteq \bigcap_{C \in B(\mathcal{K}, \mathcal{H})} \sigma_{ea}(M_C) \cup \{\lambda \in \mathbb{C}: B - \lambda \text{ is compact}\}.$$n

For the converse, let $\lambda \notin \bigcap_{C \in B(\mathcal{K}, \mathcal{H})} \sigma_{ea}(M_C) \cup \{\lambda \in \mathbb{C}: B - \lambda \text{ is compact}\}$. $\square$

**Case 1.** $R(B - \lambda)$ is not closed. Then $d(A - \lambda) = \infty$, $A - \lambda \in \Phi_+(H)$ and $B - \lambda$ is not compact. By Lemma 3.6, there exists $C \in \text{Inv}(\mathcal{K}, \mathcal{H})$ such that $M_C - \lambda \in \Phi_+^-(H \ominus K)$. 
Case 2. \( R(B - \lambda) \) is closed. Since \( B - \lambda \) is not compact, then \( \dim N(B - \lambda) = \infty \). By Lemma 3.3, there exists \( C \in \text{Inv}(K, H) \) such that \( M_C - \lambda \in \Phi_+ (H \oplus K) \), since \( A - \lambda \in \Phi_+ (H) \) and \( d(A - \lambda) + d(B - \lambda) \geq n(A - \lambda) + n(B - \lambda) \). \( \square \)

In the similar way, we have

**Corollary 3.7.** For a given pair of operators \((A, B)\), we have

\[
\bigcap_{C \in \text{Inv}(K, H)} \sigma_{SF}^+(M_C) = \bigcap_{C \in B(K, H)} \sigma_{SF}^+(M_C) \cup \{\lambda \in \mathbb{C} : A - \lambda \text{ is compact}\}.
\]

**Theorem 3.8.** For a given pair of operators \((A, B)\), we have

\[
\bigcap_{C \in \text{Inv}(K, H)} \sigma_{w}^+(M_C) = \bigcap_{C \in B(K, H)} \sigma_{w}^+(M_C) \cup \{\lambda \in \mathbb{C} : A - \lambda \text{ or } B - \lambda \text{ is compact}\}.
\]

4. \( \bigcup_{C \in \text{Inv}(K, H)} \sigma_{ea}(M_C) \)

**Theorem 4.1.** For a given pair of operators \((A, B)\), \( M_C \in \Phi_+^-(H \oplus K) \) for all \( C \in \text{Inv}(K, H) \) if and only if the following conditions hold:

(i) \( A \in \Phi_+ (H) \);
(ii) \( B \in \Phi_+ (H) \);
(iii) \( \text{ind}(A) + \text{ind}(B) \leq 0 \).

**Proof.** Sufficiency is clear, since \( M_C = \begin{pmatrix} I & 0 \\ 0 & B \end{pmatrix} \begin{pmatrix} I \\ C \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix} \).

Necessity. It is clear that \( A \in \Phi_+ (H) \) and \( n(B) < \infty \). We firstly show that \( R(B) \) is closed. Assume to the contrary that \( R(B) \) is not closed. By Theorem 2.4 and Lemma 3.4, \( d(A) = \infty \) and \( B \) is not compact. Thus \( R(B) \) contains a closed infinite-dimensional subspace \( K_1 \) with \( \dim K_1 = \infty \). Let \( H_1 = \{ x \in N(B) : Bx \in K_1 \} \). Using the same technique as Lemma 3.6, we may assume that \( H_1 \) is a closed subspace of \( N(B) \), \( \dim H_1 = \infty \) and \( \dim H_1 = \infty \). Since \( d(A) = \infty \), set \( R(A) = H_2 \oplus H_2 \) with \( \dim H_2 = \dim N(B) \) and \( \dim H_2 = \infty \). Thus

\[
M_C = \begin{pmatrix} 0 & 0 & C_{11} & C_{12} \\ 0 & A_1 & C_{21} & C_{22} \\ 0 & 0 & B_1 & B_2 \\ 0 & 0 & 0 & B_4 \end{pmatrix} : N(A) \oplus N(A) \oplus H_1 \oplus H_1 \rightarrow R(A) \oplus R(A) \oplus K_1 \oplus K_1,
\]

where \( B_1 \) is an invertible operator from \( H_1 \) into \( K_1 \). Since \( A_1 \) is invertible, \( M_C \in \Phi_+^-(H \oplus K) \) if and only if

\[
\begin{pmatrix} 0 & C_{11} & C_{12} \\ 0 & B_1 & B_2 \\ 0 & 0 & B_4 \end{pmatrix} \in \Phi_+ (N(A) \oplus H_1 \oplus H_1, R(A) \oplus K_1 \oplus K_1).\]
Since
\[
\begin{pmatrix}
I & -C_{11}B_1^{-1} & 0 \\
0 & I & 0 \\
0 & 0 & I
\end{pmatrix}
\begin{pmatrix}
0 & C_{11} & C_{12} \\
0 & B_1 & B_2 \\
0 & 0 & B_4
\end{pmatrix}
\begin{pmatrix}
I & 0 & 0 \\
0 & I & -B_1^{-1}B_2 \\
0 & 0 & I
\end{pmatrix}
= \begin{pmatrix}
0 & 0 & -C_{11}B_1^{-1}B_2 + C_{12} \\
0 & B_1 & 0 \\
0 & 0 & B_4
\end{pmatrix},
\] (4)

\[MC \in \Phi_+^-(H \oplus K)\] if and only if
\[
\begin{pmatrix}
0 & 0 & -C_{11}B_1^{-1}B_2 + C_{12} \\
0 & B_1 & 0 \\
0 & 0 & B_4
\end{pmatrix} \in \Phi_+^-(N(A) \oplus H_1 \oplus H_1^\perp, R(A)^\perp \oplus K_1 \oplus K_1^\perp).
\]

Since \(d(A) = \infty\), define an operator \(C_0 : K \rightarrow \mathcal{H}\) by
\[
C_0 = \begin{pmatrix}
V_2 & V_2B_1^{-1}B_2 \\
0 & V_1
\end{pmatrix} : H_1 \oplus H_1^\perp \rightarrow R(A)^\perp \oplus R(A),
\]
where \(V_1\) and \(V_2\) are unitary operators. It is easy to show that \(C_0\) is invertible. By Eq. (4), \(MC_0 \in \Phi_+^-(H \oplus K)\) if and only if
\[
M_0 := \begin{pmatrix}
0 & 0 & 0 \\
0 & B_1 & 0 \\
0 & 0 & B_4
\end{pmatrix} \in \Phi_+^-(N(A) \oplus H_1 \oplus H_1^\perp, R(A)^\perp \oplus K_1 \oplus K_1^\perp).
\]

Thus \(B_4 \in \Phi_+(H_1^\perp, K_1^\perp)\). It follows from \(\begin{pmatrix} B_1 & 0 \\
0 & B_4 \end{pmatrix} \in \Phi_+(K)\) that \(R(B)\) is closed. This is a contradiction. Thus \(R(B)\) is closed.

Since \(n(M_0) = n(A) + n(B_4) < \infty\) and \(d(M_0) = d(A) + d(B_4)\), we get that
\[
n(M_0) - d(M_0) = n(A) - d(A) + n(B_4) - d(B_4) = \text{ind}(A) + \text{ind}(B),
\]
the last equation follows from that \(B_1\) is invertible. Thus \(\text{ind}(A) + \text{ind}(B) \leq 0\). \(\Box\)

**Corollary 4.2.** For a given pair of operators \((A, B)\),
\[
\bigcup_{C \in \text{Inv}(K, \mathcal{H})} \sigma_{ea}(MC) = \sigma_{ea} \begin{pmatrix} A & 0 \\
0 & B \end{pmatrix}.
\]

In a similar way, we may obtain the next corollaries.

**Corollary 4.3.** For given pair of operators \((A, B)\),
\[
\bigcup_{C \in \text{Inv}(K, \mathcal{H})} \sigma_{SF}^+(MC) = \sigma_{SF}^+ \begin{pmatrix} A & 0 \\
0 & B \end{pmatrix}.
\]

**Corollary 4.4.** For given pair of operators \((A, B)\),
\[
\bigcup_{C \in \text{Inv}(K, \mathcal{H})} \sigma_w(MC) = \sigma_w \begin{pmatrix} A & 0 \\
0 & B \end{pmatrix}.
\]
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References