Removable Singularities of Solutions
to a Class of Quasi-linear
Non-uniformly Elliptic Equations

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1. Introduction

Let \( \Omega \) be a bounded open connected subset (domain) of \( \mathbb{R}^n, n \geq 2 \), and \( K \) be a compact subset of \( \Omega \). Suppose \( u \in C^2(\Omega \sim K) \) satisfies the minimal surface equation in \( \Omega \sim K \), i.e.,

\[
\sum_{i=1}^{n} D_i((1 + |Du|^2)^{-1/2} D_i u) = 0 \quad \text{in} \quad \Omega \sim K.
\]

Here \( Du = (D_1 u, \ldots, D_n u) \) denotes the gradient of \( u \). In 1951, Bers first proved the startling result that if \( K \) consists only of a single point, then \( u \) can be extended to be a \( C^2 \) solution of (1.1) in all of \( \Omega \) and hence the singular set is removable [1]. An essential feature of Bers' theorem is that no a priori assumption is made on the growth behaviour of \( u \) near the singular set. Since then various authors have extended this result by either enlarging the suitable class of equations or the size of \( K \) [3-5, 7, 11, 13, 14]. In 1965, Nitsche and De Giorgi and Stampacchia proved independently that any compact subset \( K \) in \( \Omega \) of vanishing \((n - 1)\)-dimensional Hausdorff measure is removable [5, 7]. In 1966, Serrin generalized their theorem to include equations of bounded prescribed mean curvature [10, 11]. In 1980, Vazquez and Veron generalized it further to include the capillarity equation [14]. It should be remarked that in [10] all we need is \(-\text{sign}(z)\) \(B(x, z, p) \leq f(x) \in L^{n+n'}(\Omega)\) for some \( \varepsilon > 0 \) [10, p. 288]. Hence [10, 14] cover almost the same class of interesting cases.

We would like to emphasize that in all of the above-mentioned references (except [13]) the singular set \( K \) is required to lie strictly inside \( \Omega \). (For \( n = 2 \), Nitsche observed that the singular set can approach the boundary because one can always find a Jordan curve bypassing the set of vanishing linear measure [7, p. 209].) The difficulty lies in the fact that when \( K \) is
allowed to approach the boundary, the usual test function technique is no longer applicable. The paper [13] seems to be the only one dealing with the latter situation.

In this paper, we shall show that for a wide class of equations the singular set \( K \) can indeed be allowed to approach the boundary, thus giving a partial answer to a question raised in [11]. Special examples are given in Section 6. In particular, our results extend those in [5, 7, 13, 14].

2. PRELIMINARIES

We consider quasi-linear equations which can be written in the divergence form

\[
\text{div } A(x, Du) = B(x, u).
\]

(2.1)

Here \( A = (A_1, \ldots, A_n) \) is a given vector-valued function of \( (x, p) \in \Omega \times \mathbb{R}^n \) and \( B \) is a given scalar function \( (x, z) \in \Omega \times \mathbb{R} \). From now on, \( \Omega \) will always denote a bounded open connected subset (domain) and \( K \) is a compact subset of \( \mathbb{R}^n \), \( n \geq 2 \). The standard summation convention that repeated indices indicate summation from 1 to \( n \) is followed.

**Definition 2.1.** A function \( u \in W^{1,1}_{\text{loc}}(\Omega) \) is called a (weak) solution of (2.1) in \( \Omega \) if

\[
A(x, Du(x)) \in L^\infty(\Omega), \quad B(x, u(x)) \in L^1(\Omega)
\]

(2.2)

and

\[
\int_{\Omega} A_i(x, Du) D_i \varphi + B(x, u) \varphi \, dx = 0 \quad \text{for all } \varphi \in C_0^1(\Omega).
\]

(2.3)

**Remark 2.2.** By an approximation argument, in view of (2.2), the test function \( \varphi \) in (2.3) can be taken to belong to \( W^{1,1}_{0}(\Omega) \cap L^\infty(\Omega) \).

**Definition 2.3.** A function \( u \in W^{1,1}_{\text{loc}}(\Omega) \) is said to satisfy the differential inequality

\[
\text{div } A(x, Du) \leq B(x, u) \quad \text{in } \Omega
\]

(2.4)

if (2.2) holds and

\[
\int_{\Omega} A_i(x, Du) D_i \varphi + B(x, u) \varphi \, dx \geq 0 \quad \text{for all non-negative } \varphi \in C_0^1(\Omega).
\]

(2.5)
In this case, we call \( u \) a super-solution of (2.1) in \( \Omega \). A sub-solution is defined similarly. Note that Remark 2.2 is still valid.

We now list the various conditions imposed on \( A \) and \( B \):

\[
|A(x, p)| \leq a \quad \text{for all } (x, p) \in \Omega \times \mathbb{R}^n, \tag{A1}
\]

\[
p_i A_i(x, p) \geq |p| - a_i \quad \text{for all } (x, p) \in \Omega \times \mathbb{R}^n, \tag{A2}
\]

where \( a, a_i \) are non-negative constants.

\[
(p - q) \cdot (A(x, p) - A(x, q)) > 0 \quad \text{for all } x \in \Omega \text{ and } p \neq q \text{ in } \mathbb{R}^n. \tag{A3}
\]

\[
(p - q) \cdot (A(x, p) - A(x, q)) \geq 0 \quad \text{for all } (x, p, q) \in \Omega \times \mathbb{R}^n \times \mathbb{R}^n. \tag{A3}'
\]

\( B(x, z) \) is non-decreasing in \( z \) for each \( x \in \Omega \). \( \tag{B1} \)

\( B(x, z) \) is strictly increasing in \( z \) for each \( x \in \Omega \). \( \tag{B1}' \)

\[
B(x, z) \geq -c \quad \text{(resp. } \leq c \text{)} \quad \text{for all } z > 0 \text{ (resp. } z < 0 \text{), } x \in \Omega, \tag{B2}
\]

where \( c > 0 \) is a constant.

Remark 2.4. Note that (A3) is stronger than (A3)' while (B1)' is stronger than (B1).

Definition 2.5. Equation (2.1) is said to possess the local existence property if for any given \( x^0 \in \Omega, M > 0 \), there exists an \( r_0 = r_0(x^0, M) > 0 \) such that for any \( 0 < r \leq r_0 \) a solution \( v \in C^2(B_r(x^0)) \cap C^0(\overline{B}_r(x^0)) \) exists solving the following Dirichlet problem:

\[
\text{div } A(x, Dv) = B(x, v) \quad \text{in } B_r(x^0)
\]

\[
v = \phi \quad \text{on } \partial B_r(x^0).
\]

Here \( B_r(x^0) = \{ x \in \mathbb{R}^n : |x - x^0| < r \} \), \( \phi \in C^0(\partial B_r(x^0)) \) with \( \| \phi \|_{L^\infty(\partial B_r(x^0))} \leq M. \)

3. An Extended Comparison Principle

Proposition 3.1. Suppose that \( K \) is a compact subset of \( \mathbb{R}^n \), with \( H^{n-1}(K) = 0 \) and let \( v, w \in W^{1,1}_{\text{loc}}(\Omega \sim K) \) satisfy in the weak sense the following inequalities:

\[
\text{div } A(x, Dv) \leq B(x, v), \quad \text{div } A(x, Dw) \geq B(x, w) \quad \text{in } \Omega \sim K. \tag{3.1}
\]

Furthermore we assume conditions (A1), (A3), (B1) (or (A3)' and (B1)') and

\[
\limsup[w(x) - v(x)] \leq M \quad \text{for any approach to a point in } \partial \Omega \sim K, \tag{3.2}
\]
where $M \geq 0$ is a constant. Then

$$w(x) - v(x) \leq M \quad \text{a.e. } \Omega \sim K.$$  \hfill (3.3)

**Proof.** Let $M < M_1 < M_2$. Let $\varepsilon > 0$ be given. Since $H^{n-1}(K) = 0$, we can find points $x^1, \ldots, x^m \in K$ and positive real numbers $r_1, \ldots, r_m$ such that

$$K \subseteq D(\varepsilon) = \bigcup_{i=1}^{m} B_{r_i}(x^i) \subseteq D'({\varepsilon}) = \bigcup_{i=1}^{m} B_2(x^i)$$  \hfill (3.4)

and $\sum_{i=1}^{m} r_i^{n-1} < \varepsilon$. There exists a Lipschitz function $\eta_\varepsilon$ such that

$$\begin{align*}
\eta_\varepsilon(x) &\equiv 1 \quad \text{on } \mathbb{R}^n \sim D'(\varepsilon), \\
\eta_\varepsilon(x) &\equiv 0 \quad \text{on } D(\varepsilon), \\
0 &\leq \eta_\varepsilon(x) \leq 1 \quad \text{on } \mathbb{R}^n
\end{align*}$$

and

$$\int_{\mathbb{R}^n} |D\eta_\varepsilon| \to 0 \quad \text{as } \varepsilon \to 0.$$  \hfill (3.5)

(For the detailed construction of $\eta_\varepsilon$'s, we refer to [9, 13].) Define

$$u(x) = \begin{cases} 
v(x) + M_2 & \text{if } w(x) \geq v(x) + M_2 \\
w(x) & \text{if } v(x) + M_1 < w(x) < v(x) + M_2 \\
v(x) + M_1 & \text{if } w(x) \leq v(x) + M_1.
\end{cases}$$  \hfill (3.6)

Then

$$\int_{\Omega \sim D'(\varepsilon)} (D_1 u - D_1 v)(A_1(x, Du) - A_1(x, Dv)) \, dx$$

$$\leq \int_{\Omega \sim D(\varepsilon)} (D_1 u - D_1 v) \eta_\varepsilon(A_1(x, Dw) - A_1(x, Dv)) \, dx$$

$$= \int_{\Omega \sim D(\varepsilon)} D_1((u - v - M_1) \eta_\varepsilon)(A_1(x, Dw) - A_1(x, Dv)) \, dx$$

$$- \int_{\Omega \sim D(\varepsilon)} (u - v - M_1) D_1 \eta_\varepsilon(A_1(x, Dw) - A_1(x, Dv)) \, dx$$

$$\leq - \int_{\Omega \sim D(\varepsilon)} (u - v - M_1) \eta_\varepsilon(B(x, w) - B(x, v)) \, dx$$

$$+ \int_{\Omega \sim D(\varepsilon)} (u - v - M_1) D_1 \eta_\varepsilon(A_1(x, Dw) - A_1(x, Dv)) \, dx.$$
Letting \( \varepsilon \to 0 \) (if we assume (A3), (B1)), we conclude that
\[
Du - Dv \equiv 0 \quad \text{a.e.} \quad \Omega \sim K,
\]
that is, \( u(x) - v(x) \equiv \text{constant} \) a.e. \( \Omega \sim K \). But near \( \partial \Omega \sim K \), \( u(x) = v(x) + M_1 \). Hence
\[
u(x) - v(x) \equiv M_1 \quad \text{a.e.} \quad \Omega \sim K,
\]
which is equivalent to
\[
w(x) \leq v(x) + M_1 \quad \text{a.e.} \quad \Omega \sim K.
\]
We can then let \( M_1 \downarrow M \) to complete the proof. If we assume (A3)', (B1)', then (3.7) would give a contradiction if \( w(x) > v(x) + M_1 \) holds in a subset of positive measure.

4. LOCAL BONDEDNESS OF SOLUTIONS

**Lemma 4.1.** Assume:

(i) \( K \) is a compact subset of \( \mathbb{R}^n \), with \( H^{n-1}(K) = 0 \),

(ii) \( u \) is a \( C^2(\Omega \sim K) \) function satisfying
\[
\text{div } A(x, Du) = B(x, u) \quad \text{in} \quad \Omega \sim K,
\]

(iii) (A1), (A2), (B2).

Then \( u \in L^\infty_{\text{loc}}(\Omega) \).

**Proof.** We shall show that \( u \) is locally bounded from above and from below on \( \Omega \). Let \( x^0 \in K \cap \Omega \) and \( r_0 > 0 \) be such that \( B_{2r_0}(x^0) \subseteq \Omega \). Denote the graph of \( u \) by
\[
G = \{(x, u(x)) : x \in \Omega \sim K\}.
\]
We adopt the notation that if \( F \) is defined on \( \Omega \subseteq \mathbb{R}^n \), then \( F^* \) denotes the function defined on \( \Omega \times \mathbb{R} \) by \( F^*(x, t) = F(x) \) for \( x \in \Omega, \ t \in \mathbb{R} \). We rewrite the equation as

\[
\text{div}_{\mathbb{R}^{n+1}} A^*(x, Du) = B^*(x, u), \tag{4.3}
\]

where we define \( A^*_{n+1}(x, p) = -(a_1 + 1) \). This arbitrary choice of \( A^*_{n+1} \) will be useful in later computations. Let \( x^1 = (y^1, u(y^1)) \in G \cap (\mathcal{B}_1(x^0) \times \mathbb{R}) \) be such that \( u(y^1) > \rho_0 \). (Otherwise \( u \) would be locally bounded from above.)

For \( 0 < r < r_0 \), we define \( S_r(x^1) = \{ x \in \mathbb{R}^{n+1} : |x - x^1| < r \} \). We claim that \( \partial S_r(x^1) \cap G \) is a \( C^2 \) \((n-1)\)-dimensional submanifold of \( \mathbb{R}^{n+1} \) for almost all \( r, \ 0 < r < r_0 \). In fact, we consider the smooth map \( f : G \cap S_{\rho_0}(x^1) \to Y = (0, \rho_0) \), defined by \( f(x) = |x - x^1| \) for \( x \in G \cap S_{\rho_0}(x^1) \). By Sard's theorem and the pre-image theorem \( G \cap \partial S_r(x^1) = f^{-1}(\{r\}) \) is a \( C^2 \) \((n-1)\)-dimensional submanifold for almost all \( r \in (0, \rho_0) \). (We refer to [6, pp. 21 and 39] for the statements of the above theorems.) Let \( U = \{(x, t) \in (\Omega \cup K) \times \mathbb{R} : t < u(x)\} \). Using the fact that for open subsets \( P, Q \),

\[
(\partial P \cap Q) \cup (P \cap \partial Q) \subseteq (\partial P \cap Q) \subseteq (\partial P \cap Q) \cup (P \cap \partial Q) \cup (\partial P \cap \partial Q)
\]

we get

\[
\partial(S_r(x^1) \cap U) \subseteq ((\partial S_r(x^1)) \cap U) \cup (K \times \mathbb{R})
\]

\[
\cup (S(x^1) \cap G) \cup (\partial S_r(x^1) \cap G). \tag{4.4}
\]

Since \( K \times \mathbb{R} \) and \( G \cap \partial S_r(x^1) \) have zero \( n \)-dimensional Hausdorff measure, we have

\[
H^n(\partial(S_r(x^1) \cap U)) = H^n(\partial S_r(x^1)) \cap U) + H^n(S_r(x^1) \cap G) \tag{4.5}
\]

for a.e. \( r \in (0, r_0) \).

We note that \( u(x) > 0 \) when \( (x, u(x)) \in S_r(x^1) \). Hence by (B2), it follows that \( B(x, u(x)) > -c \). By the divergence theorem and (4.5), we obtain

\[
\int_{S_r(x^1) \cap U} \nabla \cdot A^*(x, Du) + B^*(x, u) \, dL^{n+1} = \int_{\partial(S_r(x^1)) \cap U} \eta^* (x, Du) \cdot n(x) \, dH^n
\]

\[
= \int_{(\partial S_r(x^1)) \cap U} \eta^* A^*(x, Du) \cdot n(x) \, dH^n
\]

\[
+ \int_{S_r(x^1) \cap G} \eta^* A^*(x, Du) \cdot (1 + |Du|^2)^{-1/2} (-Du, 1) \, dH^n. \tag{4.6}
\]
Here $L^{n+1}$ denotes the usual $(n+1)$-dimensional Lebesgue measure and $n(x)$ is the unit outward normal. After rearranging, we obtain

$$
\int_{S(x') \cap G} \eta^* A^*(x, Du) \cdot (1 + |Du|^2)^{-1/2} (Du, -1) dH^n
+ \int_{S(x') \cap U} Dn^* \cdot A^*(x, Du) dL^{n+1}
= \int_{S(x') \cap U} \eta^* \cdot -B^*(x, u) dL^{n+1}
+ \int_{(\partial S(x')) \cap U} \eta^* A^*(x, Du) \cdot n(x) dH^n. \tag{4.7}
$$

We observe that

$$(1 + |Du|^2)^{-1/2} A^*(x, Du) \cdot (Du, -1) = (1 + |Du|^2)^{-1/2} \{A(x, Du) \cdot Du + a_i + 1\}
\geq (1 + |Du|^2)^{-1/2} \{|Du| - a_i + a_1 + 1\} \geq 1. \tag{4.8}
$$

Letting $\varepsilon \to 0$, (4.7) gives

$$H^n(S(x') \cap G) \leq c L^{n+1}(S(x') \cap U) + (a + a_1 + 1) H^n((\partial S(x')) \cap U). \tag{4.9}
$$

Set $h(r) = L^{n+1}(S(x') \cap U)$. Using the isoperimetric inequality ([1], 3.2.39, 3.2.43) and (4.9), we get

$$h(r)^{n+1} = \{L^{n+1}(S(x') \cap U)\}^{n+1} \leq c(n) H^n((\partial S(x')) \cap U)
\leq c(n) \{H^n((\partial S(x')) \cap U) + H^n(S(x') \cap G)\}
\leq c(n) h'(r) + c(n) \{ch(r) + (a + a_1 + 1) h'(r)\}, \tag{4.10}
$$

where $c(n)$ is a constant depending on $n$. This implies

$$h(r)^{n+1} - c_1(n, c) h(r) \leq c_2(n, a, a_1) h'(r). \tag{4.11}
$$

For sufficiently small $r$, namely, when $h(r) \leq \omega_{n+1} r^{a+1} \leq (2c_1)^{-n-1}$, we have

$$h(r)^{n+1} \leq c_3(n, a, a_1, c) h'(r) \quad \text{for} \quad 0 < r \leq (2c_1 \omega_{n+1}^{1/(n+1)})^{-1}. \tag{4.12}
$$

Solving (4.12), we get

$$h(r) = L^{n+1}(S(x') \cap U) \geq c_4(n, a, a_1, c) r^{n+1}. \tag{4.13}
$$
From this volume bound, one concludes that \( u \) is bounded from above on \( B_r(x^0) \sim K \): from the continuity of \( u \) on \( \overline{B}_{2r}(x^0) \sim K \) we know that
\[
H^n \{ x \in B_{2r}(x^0) \sim K : u(x) \geq \tau \} \leq 2^{-1} c_4 r^n
\]
for sufficiently large \( \tau > 0 \). In case \( u(y^1) \geq \tau + r \) for some \( y^1 \in B_r(x^0) \sim K \), we can apply (4.13) with \( x^1 = (y^1, u(y^1)) \) and obtain a contradiction. Thus
\[
\sup \{ u(x) : x \in B_r(x^0) \sim K \} \leq \tau + r.
\]
To prove that \( u \) is locally bounded from below in \( \Omega \), we proceed as above with \( u(y^1) < -r_0 \) and \( U \) replaced by
\[
V = \{ (x, t) \in (\Omega \sim K) \times \mathbb{R} : t > u(x) \}.
\]

5. Main Theorems

**Theorem 5.1.** Assume:

(i) \( K \) is a compact subset of \( \mathbb{R}^n \), with \( H^{n-1}(K) = 0 \),
(ii) \( u \in C^2(\Omega \sim K) \) is a solution of \( \text{div} \ A(x, Du) = B(x, u) \) in \( \Omega \sim K \),
(iii) (A1), (A2), (A3), (B1), (B2), (or with (A3), (B1) replaced by (A3)', (B1)')
(iv) the local existence property; cf. Definition 2.4.

Then \( u \) can be defined on \( K \) so that the resulting function is a \( C^2 \) solution of (5.1) in all of \( \Omega \).

**Proof.** Take an arbitrary point \( x'' \in K \cap \Omega \). Choose \( r > 0 \) small enough so that \( B_{2r}(x^0) \subseteq \Omega \). By Lemma 4.1, \( u \) is bounded in \( B_r(x^0) \sim K \). By [8, pp. 604–606] and (iv) there exists \( v \in C^2(B_r(x^0)) \cap C^0(\overline{B}_r(x^0) \sim K) \) solving the following Dirichlet problem with incomplete boundary data:
\[
\begin{align*}
\text{div} \ A(x, Dv) &= B(x, v) \quad \text{in} \; B_r(x^0), \\
v &= u \quad \text{on} \; \partial B_r(x^0) \sim K.
\end{align*}
\]

By Proposition 3.1, \( v(x) = u(x) \) in \( B_r(x^0) \sim K \). Hence we can simply define \( u(x) = v(x) \) on \( B_r(x^0) \cap K \) to make it a \( C^2 \) solution in \( B_r(x^0) \). Since \( x^0 \) is arbitrary, the assertion is proved.
Theorem 5.2. Assume:

(i) \( K \) is a compact subset of \( \mathbb{R}^n \), with \( H^{n-1}(K) = 0 \),
(ii) \( u \in C^2(\Omega \sim K) \) is a solution of \( \text{div} \ A(x, Du) = B(x, \mu) \) in \( \Omega \sim K \),
(iii) (A1), (A2), (B2),
(iv) \( B(x, u) \) is continuous in \( (x, u) \in \Omega \times \mathbb{R} \).

Then \( u \) can be extended to be a weak solution of (5.1) in all of \( \Omega \).

Proof. By Lemma 4.1, \( u \) is bounded in \( B_i(x^0) \) (notations as in the proof of Theorem 5.1). Hence \( B(x, u(x)) \) is bounded in \( B_i(x^0) \). By Theorem 10' of [10], \( u \) can be extended to be a weak solution of (5.1) in \( B_i(x^0) \). Since \( x^0 \) is arbitrary, the assertion is proved.

Remark 5.3. If we assume the singular set \( K \) to lie strictly inside \( \Omega \), then many cases can be handled by the comparison principle alone without resort to the local existence theory. In particular the following gives a relatively direct proof of the Nitsche–De Giorgi–Stampacchia theorem.

Theorem 5.4. Assume:

(i) \( K \) is a compact subset of \( \Omega \), with \( H^{n-1}(K) = 0 \),
(ii) \( u \in C^0(\Omega \sim K) \) is a (weak) solution of \( \text{div} \ A(Du) = B(u) \) in \( \Omega \sim K \),
(iii) (A1), (A3), (B1) (or (A3)' and (B1)'),
(iv) \( B(u) \) is continuous.

Then \( u \) can be continuously extended to be a (weak) solution of (5.4) in all of \( \Omega \).

Proof. Since \( K \) lies strictly inside \( \Omega \), we can assume without loss of generality that \( u \) is continuous near \( \partial \Omega \). Hence \( |u| \) is uniformly bounded on \( \partial \Omega \). By (iv) and Proposition 3.1, we conclude that \( u \in L^\infty(\Omega) \). By Theorem 10' of [10], \( u \) can be extended to a (weak) solution in all of \( \Omega \). Since \( A, B \) are independent of \( x \), both \( u(x) \) and \( u(x + h) \) are solutions of (5.4). By Proposition 3.1 again, it follows that for sufficiently small \( h \in \mathbb{R}^n \),

\[
|u(x + h) - u(x)| \leq \omega(h),
\]

where \( \omega(h) \) is the modulus of continuity of \( u \) on \( \partial \Omega \). Hence \( u \) can be continuously extended to all of \( \Omega \).
Theorem 5.5. Assume:

(i) all the hypotheses of Theorem 5.4,
(ii) both $A$, $B$ are of class $C^2$ with $D_p A_j(p) \xi_i \xi_j > 0$ for $0 \neq \xi \in \mathbb{R}^n$,
(iii) $u \in C^{0,1}(\Omega \sim K)$.

Then $u$ can be extended to be a $C^2$ solution in all of $\Omega$.

Proof. From (iii) and (5.5) we know that $u$ can be extended to be a Lipschitz solution in all of $\Omega$. Since the extension satisfies a uniform Lipschitz condition in $\Omega$, in view of (ii) Eq. (5.4) becomes uniformly elliptic. We can then apply standard elliptic theory to conclude the proof.

6. Some Examples

Example 6.1. $A_j(p) = (1 + |p|^2)^{-1/2} p_j$. Here (A1), (A2), and (A3) are satisfied. When $B(x, u) \equiv 0$, (2.1) is the minimal surface equation. When $B(x, u) = ku$, with $k > 0$, (2.1) is the capillarity equation. When $B(x, u) \equiv \text{constant}$, this is the equation of constant mean curvature. It is easy to check that all these $B$'s satisfy (B1), (B2).

Example 6.2. More generally we consider the Euler-Lagrange equations of parametric elliptic functionals; namely,

$$
\sum_{i=1}^{n+1} D_i F_{q_i}(x, Du(x), -1) = B(x, u),
$$

where $F$ is a $C^3$ function on $\Omega \times (\mathbb{R}^{n+1} \sim \{0\})$ such that for all $(x, q) \in \Omega \times (\mathbb{R}^{n+1} \sim \{0\})$ the following conditions are satisfied:

$$
F(x, \mu q) = \mu F(x, q) \quad \text{for all } \mu > 0.
$$

where $\xi' = \xi - (\xi \cdot |q|^{-1} q)|q|^{-1}$, $\xi \in \mathbb{R}^{n+1}$,

$$
|F_q(x, \zeta)| + \sum_{l=1}^{n+1} \left| \sum_{i=1}^{n} F_{q_i x_l}(x, \zeta) \right| + \sum_{l=1}^{n} \sum_{i,j=1}^{n} \left| F_{q_l q_j x_l}(x, \zeta) \right| \leq M
$$

for all $x \in \Omega$, $\zeta \in S^n = \{q \in \mathbb{R}^{n+1}: |q| = 1\}$ and for some $M > 0$,

$$
F(x, q) \geq |q| \quad \text{for all } (x, q) \in \Omega \times (\mathbb{R}^{n+1} \sim \{0\})
$$

$B(x, u) \in C^1(\Omega \times \mathbb{R})$ with $D_u B(x, u) \geq 0$ for all $(x, u) \in \Omega \times \mathbb{R}$.
and
\[
\lim_{u \to +\infty} \inf B(x, u) > -\infty, \quad \lim_{u \to -\infty} \sup B(x, u) < \infty.
\]

By [12, p. 853], the local existence theorem holds. (A1), (A2), (A3) can be checked to hold as well (see [13] for some of the details). When \( B(x, u) = B(x) \) in (6.6), we recover L. Simon’s result in [13].

Remark 6.3. Some slightly more general forms of the equations
\[
\text{div } A(x, Du) = B(x, u)
\]
can also be considered [12, p. 843]. To prove the local existence property we only need to establish upper and lower barriers [12, p. 849]. For sufficiently small balls, say
\[
B_R = \left\{ (x_1, \ldots, x_n) \in \mathbb{R}^n : |(x_1, \ldots, x_n) - (0, \ldots, 0, R)| < R \right\},
\]
we can use barrier functions of the form
\[
K(d(x) + |x'|) + \varphi(0) + \varepsilon,
\]
where \( d(x) = \text{dist}(x, \partial B_R) \), \( \varphi \) is as in (2.6) and \( K \) is some large positive constant. We skip the details.

Example 6.4. \( A_i(p) = |p|^{-1} p_i, \ B(u) = u|u|^{q-1} \), where \( q \geq 1 \). It is easy to check that (A1), (A3)', (B1)' are satisfied and that Theorem 5.4(iv) holds.

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References