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Stable matchings and linear inequalities

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Abstract

The theory of linear inequalities and linear programming was recently applied to study the stable marriage problem which until then has been studied by mostly combinatorial methods. Here we extend the approach to the general stable matching problem in which the structure of matchable pairs need not be bipartite. New issues arise in the analysis and we combine linear algebra and graph theory to explore them.

1. Introduction

The *stable matching problem* describes a situation where agents are to be matched while having preferences over potential mates. The data for the model consists of a set of agents, a set of pairs of agents that are matchable and a list of strict preference orders of the agents over their matchable mates. The goal is to find a *stable matching*, i.e., a matching such that no two agents prefer each other over their respective outcome in the matching, where singlehood is considered to be worse than being paired with a matchable mate. This model includes the original *stable marriage problem* and *stable roommates problem* introduced by Gale and Shapley [7] as special cases. In the stable roommates problem all possible pairs are admissible, whereas in the stable marriage problem the agents are labeled either as *men* or *women* and only man–woman pairs are allowable.

It was observed by Gale and Shapley [7] that stable matching problems that arise from certain two-sided markets, e.g., matching students to colleges, can be modeled as stable marriage problems. Further, they described an algorithm that computes a stable matching for any given stable marriage problem. Roth [16] discovered that essentially the same algorithm has in fact been in use since 1952, ten years before the

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seminal paper of Gale and Shapley was published, by the National Resident Matching Program to assign medical students to their hospital residencies. Continuing work, e.g., [17], demonstrates the applicability and importance of stable matching problems.

Gale and Shapley [7] also obtained an example of a stable roommates problem which has no stable matching. Irving [10] gave the first polynomial algorithm that finds a stable matching for a stable roommates problem or determines that no such matching exists, and Gusfield and Irving [8] extended this algorithm to a general stable matching problem. Alternative algorithms are presented in Abeledo [1]. For more on stable matchings and stable marriages, see the books by Knuth [14], Gusfield and Irving [8], Roth and Sotomayor [19], and the references therein.

Despite their apparent simplicity, stable matching problems have a wealth of structural properties. For the past three decades the general approach for exploring these properties has relied mostly on combinatorial arguments. A new approach for studying the stable marriage problem was recently introduced by Vande Vate [21], who characterized stable marriages via the extreme points of a certain polytope when the number of men and women coincide, singlehood is prohibited and preferences are complete. The result was modified and extended to cover the general case via a simplified proof in Rothblum [20]. Vande Vate's approach was inspired by earlier work of Irving et al. [11], who showed how to reduce the stable marriage problem to a minimum cut problem such that stable marriages correspond to the s–t minimum cuts. Roth et al. [18] used the theory of linear inequalities and of linear programming to obtain new results and to derive new proofs of known results for the stable marriage problem. It is our purpose here to combine this linear algebraic approach with some elements of graph theory to study the stable matching problem.

In Section 2 we define a polytope, for each particular stable matching problem, that we call the fractional stable matching polytope. We show that this polytope contains the incidence vectors of all stable matchings of the problem. Further, we prove that the extreme points of the fractional stable matching polytope are always half-integral and we specify other structural properties of this polytope. There is an interesting relation between the results we obtain for fractional stable matching polytopes and classic results concerning the polytopes associated with matching problems. In Sections 3 and 4 we apply linear programming theory to obtain new proofs of known results on stable matchings and to extend these results to all points of the fractional stable matching polytope. For example, we show that the median (properly defined) on each triplet of points in the polytope is also in the polytope. Finally, in Section 5 we obtain a characterization of the extreme points of the fractional stable matching polytope.

2. Preliminaries and background

In this section we review some known results on polytopes associated with matchings on graphs. We then formally define stable matching problems using graph-theoretic terminology.

We begin by summarizing some basic definitions of graph theory. A *graph* G is an ordered pair (V, E), where V is a finite set called the *set of vertices* and E is a subset of $\{\{v, w\}: v, w \in V\}$ called the *set of edges*. Let $G = (V, E)$ be a graph. A graph $G' = (V', E')$ is a *subgraph* of G, denoted by $G' \subseteq G$, if $V' \subseteq V$ and $E' \subseteq E$. Vertices u and v are *adjacent* in G if $\{u, v\} \in E$. For $v \in V$, we denote by $N(v)$ the set of *neighbors* of v, i.e. $N(v) = \{u: \{v, u\} \in E\}$. We say that vertex v and edge e are *incident* if $v \in e$. Two distinct edges having nonempty intersection are called *adjacent*. Let $E' \subseteq E$. We denote by $V_{E'}$ the vertices that are contained in the edges of E'. Then, $G_{E'} \equiv (V_{E'}, E')$ is called the *subgraph of G spanned by E'*.

Let $G = (V, E)$ be a graph and let $v_1, v_2, \ldots, v_k$ be distinct vertices in V such that $E' = \{\{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_{k-1}, v_k\}\} \subseteq E$. Then $G_{E'}$ is a *path* with *end vertices* v_1 and v_k. Further, if $e = \{v_1, v_k\} \in E$, then $G_{E' \cup \{e\}}$ is a *cycle*. Thus, a path (cycle) is determined by an ordered list of its vertices where any two consecutive vertices are adjacent in the path (cycle). A cycle is called *odd* or *even* according to the number of edges it contains. Of course, the number of edges of a cycle is equal to the number of its vertices. A graph is *connected* if for every pair of distinct vertices u and v there is a path with end vertices u and v. A *connected component* of a graph G is a maximal connected subgraph of G. A graph is *bipartite* if its vertex set V can be partitioned into two sets such that there are no edges whose two vertices are in the same set. It is well known that a graph is bipartite if and only if its does not contain an odd cycle.

A *matching* in a graph $G = (V, E)$ is a set of edges $\mu \subseteq E$ such that no two edges have a common vertex. A matching μ defines a one-to-one mapping $\mu(\cdot)$ from the set V onto itself where $\mu(v) = u$ if $\{u, v\} \in \mu$ an $\mu(v) = v$ if no edge in μ contains v. We call $\mu(v)$ the *outcome of v under* the matching μ. Given a matching μ we say that a vertex v is *single* or *unmatched* in μ if $\mu(v) = v$. Otherwise, we say that v is *matched* and $\mu(v)$ is then called the *mate* of v in μ. In this case, we say that v is *matched* to $\mu(v)$ in μ. Each matching can also be represented by an *incidence vector* $x = (x_{u,v})_{\{u,v\} \in E} \in \{0, 1\}^E$, whose coordinates are indexed by the edges of the graph and $x_{u,v} = 1$ if $\{u, v\} \in \mu$, and $x_{u,v} = 0$ otherwise. The *matching polytope* of a graph G, denoted $M(G)$, is defined by

$$M(G) \equiv \text{conv}\{x \in R^E: x \text{ is the incidence vector of a matching in } G\},$$

where "conv" refers to the convex hull. The incidence vectors of the matchings in G are known to constitute the extreme points of $M(G)$.

We define the *support* of a nonnegative vector $x \in R^E$ as the set of edges $E_+(x) \equiv \{\{u, v\} \in E: x_{u,v} > 0\}$. Also, for $x \in R^E$ and $\delta \in R$, we define the set of edges $E_\delta(x) \equiv \{\{u, v\} \in E: x_{u,v} = \delta\}$. Henceforth, we shall only consider vectors x in R^E such that $0 \leqslant x \leqslant 1$. We call such vectors *half-integral* if they belong to $\{0, 1/2, 1\}^E$.

The following classic theorems deal with the description of matching polytopes using systems of linear inequalities.

Theorem 2.1 (Birkhoff [4]). *Let $G = (V, E)$ be a bipartite graph, then the matching polytope $M(G)$ is the set of solutions of the following linear inequality system:*

$$\sum_{u \in N(v)} x_{u,v} \leqslant 1 \quad \textit{for each } v \in V \tag{1}$$

$$x_{u,v} \geqslant 0 \quad \textit{for each } \{u, v\} \in E \tag{2}$$

In the case of nonbipartite graphs constraints (1) and (2) are insufficient for describing the matching polytope and one has to introduce additional constraints.

Theorem 2.2 (Edmonds [5]). *Let $G = (V, E)$ be a graph, then the matching polytope $M(G)$ is the set of solutions of the system of linear inequalities consisting of* (1), (2), *and*

$$\sum_{\{u,v\} \in E:\ u,v \in S} x_{u,v} \leqslant \tfrac{1}{2}(|S| - 1) \quad \textit{for each } S \subseteq V \textit{ with } |S| \textit{ odd and } |S| \geqslant 3. \tag{3}$$

The constraints of (3) are called the "odd set" constraints. Their removal for a nonbipartite graph will result in a larger polytope than the one defined by (1) and (2) and this larger polytope will have some nonintegral extreme points. We remark that Theorem 2.2 gives an NP-*description* of the matching polytope (see [15]), i.e., given a graph G, a vector $a \in R^E$ and a scalar $b \in R$ we can determine in polynomial time whether the inequality $ax \geqslant b$ belongs to the class of inequalities characterizing the matching polytope $M(G)$ via Theorem 2.2.

The polytope defined by (1) and (2), for an arbitrary graph G, is called the *fractional matching polytope* of the graph G and is denoted $FM(G)$. A vector $x \in FM(G)$ is called a *fractional matching*. The following result characterizes the extreme points of $FM(G)$.

Theorem 2.3 (Balinski [3]). *Let $G = (V, E)$ be a graph and let x belong to $FM(G)$. Then x is an extreme point of $FM(G)$ if and only if x is half-integral and the set of edges $E_{1/2}(x)$ spans vertex disjoint odd cycles.*

It is useful for our analysis of stable matching problems to view them from a graph theoretic perspective. Any stable matching problem can be represented by a pair $(G; P)$, where $G = (V, E)$ is a graph and P is a mapping on V such that, for each vertex $v \in V$, $P(v)$ is a strict linear order on $N(v) \cup \{v\}$ which has v as the last element in the order. In this case we call G the *acceptability graph*, P the *preference profile* and $P(v)$ the *preference* of vertex v. In particular, we refer to a stable matching problem $(G; P)$ as a *stable marriage problem* when its acceptability graph is bipartite and a corresponding bipartition of V into two vertex sets, M for "men" and W for "women", is specified.

Let $(G; P)$ be a stable matching problem. For $v \in V$, we represent $P(v)$ by listing the vertices in $N(v)$ in decreasing preference order. It is not necessary to include v itself in the list since we know it is always last. We call this list *v's preference list* and we call the collection of the preference lists for all $v \in V$ the *preference table* of the problem $(G; P)$.

We also denote $P(v)$, the preference of v, by $<_v$; in particular, for u, w in $N(v) \cup \{v\}$, we write $u <_v w$ if $P(v)$ orders w before u, i.e. if v prefers w to u. Note that, as $<_v$ ranks v last, we have that $u >_v v$ for each $u \in N(v)$. We express by $u \leqslant_v t$ that either $u <_v t$ or $u = t$. Finally, for a vertex v and a nonempty set of vertices $S \subseteq N(v)$, let $\max_v S$ and $\min_v S$ denote, respectively, the most preferred and the least preferred element in S with respect to the preference $<_v$. Further, we define $\max_v \emptyset = \min_v \emptyset = \{v\}$.

A pair $\{u, v\} \in E$ is a *blocking pair* for a matching μ if

$$\mu(u) <_u v \quad \text{and} \quad \mu(v) <_v u,$$

i.e., $\{u, v\}$ is a blocking pair for μ if both vertices prefer being matched to each other over their outcome under μ. A matching μ is *stable* if it has no blocking pair. Equivalently, μ is a stable matching for $(G; P)$ if the following *stability condition* holds for each $\{u, v\} \in E$:

$$\mu(u) \geqslant_u v \quad \text{or} \quad \mu(v) \geqslant_v u. \tag{4}$$

Gale and Shapley [7] proved that a stable matching problem with a bipartite acceptability graph always has a stable matching. The following result by Abeledo and Isaak [2] shows that if the acceptability graph is not bipartite, then there is a preference profile for which no stable matchings exists. The proof is included here for the sake of completeness.

Theorem 2.4. *Let G be a graph. Then $(G; P)$ has a stable matching under all profiles P if and only if G is bipartite.*

Proof. The Gale–Shapley algorithm guarantees existence of a stable matching when the graph G is bipartite; see Gale and Shapley [7]. To prove necessity, assume G is nonbipartite. Then G has an odd cycle, say $v_1 v_2, \ldots, v_{2p+1}$. We will show there is a profile for which there is no stable matching in G. Consider the profile defined by the following rule: for a vertex $v \notin \{v_1, \ldots, v_{2p+1}\}$, let

$$P(v) = \text{any ordering of } N(v)$$

and for $i = 1, \ldots, 2p + 1$, let

$$P(v_i) = v_{i+1}, v_{i-1}, \text{ followed by any ordering of } N(v_i) \backslash \{v_{i-1}, v_{i+1}\},$$

where $v_0 \equiv v_{2p+1}$ and $v_{2p+2} \equiv v_1$. Consider a matching μ. Then there is at least one vertex among $\{v_1, \ldots, v_{2p+1}\}$ which will not be matched to another vertex on the cycle. Without loss of generality, we can assume that v_2 is such a vertex. Then v_2 prefers v_1 to $\mu(v_2)$ (whether or not v_2 is matched under μ). Also v_2 is v_1's first choice, implying that $\mu(v_1) <_{v_1} v_2$. Thus, the stability condition (4) for the pair $\{v_1, v_2\}$ is not satisfied and the matching is not stable. $\square$

3. Stable matchings and linear inequalities

Vande Vate [21] and Rothblum [20] initiated the study of stable matching problems from the perspective of polyhedral combinatorics by characterizing the stable matchings of stable marriage problems as the extreme points of a certain polytope. Here, we extend this approach to general stable matching problems by defining, for each such problem, a corresponding polytope that we call the fractional stable matching polytope. We prove some properties of fractional stable matching polytopes, for example we show that they are nonempty and that their extreme points are half-integral. Fractional stable matching polytopes will remain the main object of concern of the subsequent sections of this paper.

The *stable matching polytope* $SM(G;P)$ of a stable matching problem $(G;P)$ is defined as the convex hull of the incidence vectors of the stable matchings. A description of the stable matching polytope for stable marriage problems in terms of explicit linear inequalities was recently obtained by Vande Vate [21] and Rothblum [20]. It resembles Theorem 2.1 which applies to matchings in a bipartite graph.

Theorem 3.1 (Rothblum). *Let $(G;P)$ be a stable matching problem where the graph $G=(V,E)$ is bipartite, then the stable matching polytope $SM(G;P)$ is described by the following linear inequality system*:

$$\sum_{u \in N(v)} x_{u,v} \leqslant 1 \qquad \text{for each } v \in V, \tag{5}$$

$$x_{u,v} \geqslant 0 \qquad \text{for each } \{u,v\} \in E, \tag{6}$$

$$\sum_{i >_u v} x_{u,i} + \sum_{j >_v u} x_{v,j} + x_{u,v} \geqslant 1 \qquad \text{for each } \{u,v\} \in E. \tag{7}$$

where $i >_u v$ denotes $\{i \in N(u)\colon i >_u v\}$ and $j >_v u$ denotes $\{j \in N(v)\colon j >_v u\}$.

Constraints (7) are called the *stability constraints*. The result of Gale and Shapley [7], showing that every stable marriage problem has a stable matching, proves that when G is bipartite the polytope described by (5)–(7) is nonempty. The first step in establishing Theorem 3 in Rothblum [20], was the observation that a matching is stable if and only if its incidence vector is an integer solution of (5)–(7). We next extend this observation to the case where the acceptability graph is not necessarily bipartite.

Lemma 3.2. *The incidence vectors of stable matchings are precisely the integer solutions of inequalities* (5)–(7).

Proof. Obviously, an integer vector satisfies constraints (5) and (6) if and only if it is the incidence vector of a matching in the corresponding graph. We further observe that stability asserts that if $\{u,v\} \in E$ and the first two terms on the left-hand side of (7) vanish, i.e. v is not matched to a vertex it prefers to u and u is not matched to a vertex it prefers to v, then necessarily u and v are matched to each other, i.e. $x_{u,v} = 1$. So stability is equivalent to (7). $\square$

Lemma 3.2 implies that all vectors in the stable matching polytope must satisfy inequalities (5)–(7). But in the nonbipartite case these inequalities, in general, do not describe the stable matching polytope. The following example proposed by Isaak [12] shows that the polytope defined by (5)–(7) can have fractional extreme points, and can therefore be larger than the stable matching polytope.

Example 1. Let $V = \{1, 2, 3, 4\}$, $G = K_4$, i.e., G is the complete graph with four vertices, and let the profile P be defined by the following preference table, where to each vertex there corresponds a row that lists its neighbours in decreasing preference order:

$$P(1) = 2, 4, 3,$$

$$P(2) = 3, 4, 1,$$

$$P(3) = 1, 4, 2,$$

$$P(4) = 1, 2, 3.$$

Since the graph is complete and has an even number of nodes, it follows that in any stable matching μ all vertices must be matched. Otherwise, if there is a vertex v so that $\mu(v) = v$, then there is at least one other vertex u with $\mu(u) = u$ and the stability condition (4) for the pair $\{u, v\}$ is not satisfied. The only three matchings under which all vertices are matched are: $\mu_1 = \{\{1, 4\}, \{2, 3\}\}$, $\mu_2 = \{\{1, 3\}, \{2, 4\}\}$ and $\mu_3 = \{\{1, 2\}\}, \{3, 4\}\}$. It is easily verified that of these only μ_1 is stable. We also observe that the system of inequalities (5)–(7) for this example is

$$\sum_{u \in N(v)} x_{u,v} \leqslant 1 \quad \text{for each } v \in V,$$

$$x_{2,3} + x_{2,4} + x_{1,2} \geqslant 1,$$

$$x_{1,2} + x_{1,4} + x_{1,3} \geqslant 1,$$

$$x_{1,2} + x_{1,4} \geqslant 1,$$

$$x_{1,3} + x_{3,4} + x_{2,3} \geqslant 1,$$

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$$x_{1,3} + x_{1,4} + x_{2,4} + x_{3,4} \geqslant 1,$$

$$x_{u,v} \geqslant 0 \quad \text{for each } \{u, v\} \in E.$$

As μ_1 is the only stable matching, its incidence vector is the only integral solution of the above system of linear inequalities (see Lemma 3.2). Further, it is easily seen that the above system of inequalities reduces to

$$x_{1,2} + x_{1,4} = 1,$$

$$x_{1,2} + x_{2,3} = 1,$$

$$x_{2,3} + x_{3,4} = 1,$$

$$x_{2,3} + x_{1,4} \geqslant 1,$$

$$x_{1,3} = x_{2,4} = 0,$$

$$x_{1,2}, x_{1,4}, x_{2,3}, x_{3,4} \geqslant 0.$$

And we can parametrize the solutions of this system as

$$x_{1,2} = x_{3,4} = \alpha,$$

$$x_{1,4} = x_{2,3} = 1 - \alpha,$$

$$x_{1,3} = x_{2,4} = 0,$$

where

$$1/2 \leqslant \alpha \leqslant 1.$$

Hence, the polytope defined (5)–(7) has precisely two extreme points y and z which are given by

$$y_{1,4} = y_{2,3} = 1, \qquad y_{1,2} = y_{1,3} = y_{2,4} = y_{3,4} = 0,$$

and

$$z_{1,4} = z_{1,2} = z_{2,3} = z_{3,4} = 1/2, \qquad z_{1,3} = z_{2,4} = 0.$$

We note that the points in the stable matching polytope must obviously satisfy Edmonds' odd set constraints. But, the fractional extreme point z in Example 1 does not violate the odd set constraints. Thus, adding the odd set constraints (3) to constraints (5)–(7) will not necessarily give a description of the stable matching polytope.

The next example gives a stable matching problem that does not have a stable matching, though the corresponding polytope defined by constraints (3), (5)–(7) is nonempty.

Example 2. Consider the stable matching problem $(G; P)$ where G has six vertices, $V = \{1, \ldots, 6\}$, and P is given by the following preference table:

$$P(1) = 2, 3, 5, 6,$$

$$P(2) = 3, 4, 6, 1,$$

$P(3) = 4, 5, 1, 2,$

$P(4) = 5, 6, 2, 3,$

$P(5) = 6, 1, 3, 4,$

$P(6) = 1, 2, 4, 5.$

It can be verified that the vector x with coordinates $x_{u,v} = 1/2$, for $\{u, v\} \in \{\{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 5\}, \{5, 6\}, \{6, 1\}\}$ and $x_{u,v} = 0$ for all other arcs $\{u, v\}$ satisfies constraints (3), (5)–(7); and by exhaustive search we can check that there is no stable matching for the problem.

The above two examples raise the question of whether it is possible, by introducing additional sets of linear inequalities, to give an NP-*description* of the stable matching polytope $SM(G; P)$ for an arbitrary stable matching problem $(G; P)$, (see the discussion following Theorem 2.2). Recently Feder [6] proved that the optimal stable matching problem, i.e., the problem of finding a stable matching that maximizes a linear function, is NP-*hard*. Feder's result makes the search for any NP-*description* of the stable matching polytope hopeless unless NP = co-NP. This statement follows from a theorem by Karp and Papadimitriou [13] (see also [15, p. 332]).

Let $(G; P)$ be a given stable matching problem. We then call solutions of (5)–(7) *fractional stable matchings*. The set of all fractional stable matchings will be called the *fractional stable matching polytope* and will be denoted $FSM(G; P)$. We next derive some properties of this polytope.

Theorem 3.3. *Let $(G; P)$ be a stable matching problem. Then the fractional stable matching polytope $FSM(G; P)$ is a nonempty polytope containing $SM(G; P)$. Further, if x is an extreme point of $FSM(G; P)$ then x is half-integral.*

Proof. We first observe that Lemma 3.2 shows that all incidence vectors of stable matchings of $(G; P)$ are in $FSM(G; P)$, immediately implying that $SM(G; P) \subseteq FSM(G; P)$.

To see that $FSM(G; P) \neq \emptyset$, let $G = (V, E)$. We will construct a bipartite graph $\hat{G} = (\hat{V}, \hat{E})$ and will define a stable matching problem $(\hat{G}; \hat{P})$ on this graph. For each vertex $v \in V$, create two vertices m_v, w_v in $\hat{V}$ and for each edge $\{u, v\} \in E$ create the edges $\{m_v, w_u\}$ and $\{m_u, w_v\}$ in $\hat{E}$. The profile $\hat{P}$ is next defined by having $w_s <_{m_u} w_t$ and $m_s <_{w_u} m_t$ whenever $s <_u t$. As $\hat{G}$ is bipartite, the stable matching polytope $SM(\hat{G}; \hat{P})$ is nonempty and is the solution set of constraints (5)–(7) applied to $(\hat{G}; \hat{P})$. Let $\hat{x} \in R^{\hat{E}}$ be a point in $SM(\hat{G}; \hat{P})$. We obtain $x \in R^E$ be defining, for each $\{u, v\} \in E$,

$$x_{u,v} = \tfrac{1}{2}(\hat{x}_{m_u, w_v} + \hat{x}_{m_v, w_u}).$$

We will show that $x \in FSM(G; P)$.

Let $\{u, v\} \in E$. Using the definition of $\hat{P}$ and the fact that $\hat{x}$ satisfies constraints (7) for $(\hat{G}; \hat{P})$ for the pairs $\{m_u, w_v\}$ and $\{m_v, w_u\}$ in $\hat{E}$, we have that

$$\sum_{i >_u v} \hat{x}_{m_u, w_i} + \sum_{j >_v u} \hat{x}_{m_j, w_v} + \hat{x}_{m_u, w_v} \geqslant 1$$

and

$$\sum_{i >_u v} \hat{x}_{m_i, w_u} + \sum_{j >_v u} \hat{x}_{m_v, w_j} + \hat{x}_{m_v, w_u} \geqslant 1.$$

Combining these two inequalities shows that

$$\sum_{i >_u v} \tfrac{1}{2}(\hat{x}_{m_u, w_i} + \hat{x}_{m_i, w_u}) + \sum_{j >_v u} \tfrac{1}{2}(\hat{x}_{m_j, w_v} + \hat{x}_{m_v, w_j}) + \tfrac{1}{2}(\hat{x}_{m_u, w_v} + \hat{x}_{m_v, w_u}) \geqslant 1.$$

By the definition of x, the above is equivalent to

$$\sum_{i >_u v} x_{u,i} + \sum_{j >_v u} x_{v,j} + x_{u,v} \geqslant 1,$$

proving that x satisfies constraints (7) for $(G; P)$. The corresponding conclusions for constraints (5) and (6) applied to $(G; P)$ follow trivially. So, x belongs to $FSM(G; P)$, establishing the nonemptiness of this polytope.

To see that the extreme points of $FSM(G; P)$ are half-integral, suppose that x is an extreme point of $FSM(G; P)$. We obtain $\hat{x} \in \hat{E}$ by defining, for each $\{u, v\} \in E$,

$$\hat{x}_{m_u, w_v} = \hat{x}_{m_v, w_u} = x_{u, v}$$

Then

$$\sum_{w_j >_{m_u} w_v} \hat{x}_{m_u, w_j} + \sum_{w_i >_{w_v} m_u} \hat{x}_{m_i, m_v} + \hat{x}_{m_u, w_v} = \sum_{j >_u v} x_{u,j} + \sum_{i >_v u} x_{i,v} + x_{u,v} \geqslant 1$$

and

$$\sum_{w_j >_{m_v} w_u} \hat{x}_{m_v, w_j} + \sum_{w_i >_{w_u} m_v} \hat{x}_{m_i, w_u} + \hat{x}_{m_v, w_u} = \sum_{j >_v u} x_{v,j} + \sum_{i >_u v} x_{i,u} + x_{u,v} \geqslant 1.$$

Hence, $\hat{x}$ satisfies (7). Clearly $\hat{x}$ also satisfies (5) and (6), hence $\hat{x} \in SM(\hat{G}; \hat{P})$ and therefore $\hat{x}$ can be expressed as a convex combination of extreme points of $SM(\hat{G}; \hat{P})$. Let $\hat{y}^k$, for $k \in K$, be extreme points of $SM(\hat{G};\hat{P})$ such that

$$\hat{x} = \sum_{k \in K} \lambda_k \hat{y}^k,$$

where $\sum_{k \in K} \lambda_k = 1$ and, for each $k \in K$, $\lambda_k > 0$. For $k \in K$, we define $y^k \in K$, we define $y^k \in R^E$ by letting, for each $\{u, v\} \in E$,

$$y^k_{u,v} = \tfrac{1}{2}(\hat{y}^k_{u_m, v_w} + \hat{y}^k_{v_m, u_w}).$$

Then the earlier arguments show that $y^k \in FSM(G; P)$ and, as $\hat{y}^k$ is integral by Theorem 3.1, it follows that $y^k \in \{0, 1/2, 1\}^E$. Further, for each $\{u, v\} \in E$

$$x_{u,v} = \tfrac{1}{2}(x_{m_u, w_v} + \hat{x}_{m_v, w_u}) = \tfrac{1}{2}\left(\sum_{k \in K} \lambda_k \hat{y}^k_{m_u, w_v} + \sum_{k \in K} \lambda_k \hat{y}^k_{m_v, w_u}\right) = \sum_{k \in K} \lambda_k y^k_{u,v}.$$

Thus,

$$x = \sum_{k \in K} \lambda_k y^k.$$

Since the right-hand side of the above equation is a convex combination of the y^k's, the extremality of x for $FSM(G; P)$ implies that the y^k's are identical and, therefore, x coincides with them. In particular, $x \in \{0, 1/2, 1\}^E$ as asserted. □

The last conclusion of Theorem 3.3 will be refined in Theorem 6.12, where we characterize the extreme points of $FSM(G; P)$.

Hartmann independently proved in [9] the nonemptiness of $FSM(G; P)$ and the existence of half-integral solutions in $FSM(G; P)$. We also note that the half-integrality of the extreme points of $FSM(G; P)$ resembles Theorem 2.3 of Balinski. But, in contrast with Theorem 2.3, $FSM(G; P)$ can have an extreme point x where the set of edges $E_{1/2}(x)$ contains even cycles, as is shown by the extreme point z of Example 1. Of course, not all half-integral points in $FSM(G; P)$ are extreme points. In particular, if $(G; P)$ has more than one stable matching then the midpoint between the incidence vectors of any two stable matchings is a half-integral stable matching which is not an extreme point of $FSM(G; P)$. Of course, the half-integrality of the extreme points of $FSM(G; P)$ implies that Phase I of the simplex method can be used to compute half-integral points of $FSM(G; P)$.

We next obtain a necessary condition for half-integral points to be in $FSM(G; P)$. We will need an additional definition. Let $C = v_1, v_2, \dots, v_k$ be a cycle in G, we say that C has *cyclic preferences* in $(G; P)$ if

$$v_{i-1} <_{v_i} v_{i+1} \quad \text{for each } i = 1, \dots, k$$

or

$$v_{i-1} >_{v_i} v_{i+1} \quad \text{for each } i = 1, \dots, k,$$

where $v_{k+1} \equiv v_1$. The following theorem identifies a common property of all half-integral points in $FSM(G; P)$.

Theorem 3.4. *Let $(G; P)$ be a stable matching problem and let x be a half-integral point in $FSM(G; P)$. Then the set of edges $E_{1/2}(x)$ forms vertex disjoint cycles, each having cyclic preferences.*

Proof. The matching constraints (5) imply that a vertex is incident to at most two edges in $E_{1/2}(x)$. Also, if a vertex is incident to a edge in $E_{1/2}(x)$ then it is not incident to an edge in $E_1(x)$. Hence, the edges in $E_{1/2}(x)$ can be partitioned into vertex disjoint paths and cycles.

Let $\{v_1, v_2\}$ be an arbitrary edge in $E_{1/2}(x)$. The constraint (7) applied to $\{v_1, v_2\}$ implies that $\sum_{i >_{v_1} v_2} x_{v_1,i} + \sum_{j >_{v_2} v_1} x_{v_2,j} \geq 1/2$. By possibly interchanging the roles of

v_1 and v_2 we conclude, without loss of generality, that $\sum_{j >_{v_1} v_1} x_{v_2,j} \geqslant 1/2$. It follows that there exists a vertex $v_3 \in V$ such that $\{v_2, v_3\} \in E_{1/2}(x)$ and $v_3 >_{v_2} v_1$. Next, as $x_{v_1,v_2} = x_{v_2,v_3} = 1/2$ constraint (5) implies that $x_{v_2,i} = 0$ for $i \neq v_1, v_3$. In particular, as $v_3 >_{v_2} v_1$, $\sum_{j >_{v_2} v_3} x_{v_2,j} = 0$. So the stability constraint (7) applied to $\{v_2, v_3\}$ implies that $\sum_{j >_{v_3} v_2} x_{v_3,j} \geqslant 1/2$. Thus, there exists a vertex $v_4 \in V$ such that $\{v_3, v_4\} \in E_{1/2}(x)$ and $v_4 >_{v_3} v_2$. Repeating this argument, inductively, we obtain a sequence of edges $\{v_i, v_{i+1}\} \in E_{1/2}(x)$, for $i \geqslant 2$, such that the set of vertices $\{k \in V\colon x_{v,k} > 0\} = \{v_{i-1}, v_{i+1}\}$ and $v_{i-1} <_{v_i} v_{i+1}$. Since the number of edges is finite, we conclude that a vertex must recur. Let $v_s = v_t$ be the first recurring vertex where $s < t$. Then $t - s \geqslant 2$. If $s \geqslant 2$, then

$$\{v_{s-1}, v_{s+1}\} = \{k \in V\colon x_{v,k} > 0\} = \{k \in V\colon x_{v,k} > 0\} = \{v_{t-1}, v_{t+1}\},$$

implying that v_{t-1} is a recurring vertex, a contradiction which proves that $s = 1$. So the sequence forms a cycle which has cyclic preferences, whose edge are in $E_{1/2}(x)$ and contain the edge $\{v_1, v_2\}$. Hence, every edge in $E_{1/2}(x)$ belongs to such a cycle. □

4. Stable matchings and linear programming

Roth et al. [18] use duality theory of linear programming to derive new proofs of some results on the stable marriage problem. We extend their approach here to the nonbipartite case, obtaining additional insight on the structure of fractional stable matchings. The proofs of the following Lemmas are similar to results in [18] and are included for completeness.

Henceforth, let $(G; P)$ be a given stable matching problem with acceptability graph $G = (V, E)$. Consider the linear program:

$$\text{(LP)} \qquad \text{maximize} \quad \sum_{\{u,v\} \in E} x_{u,v}$$
$$\text{subject to} \quad x \in FSM(G; P).$$

The dual problem has variables $(\alpha, \gamma) \in R^V \times R^E$, and is given by

$$\text{(DLP)} \qquad \text{minimize} \sum_{v \in V} \alpha_v - \sum_{\{u,v\} \in E} \gamma_{u,v}$$
$$\text{subject to}\ \alpha_u + \alpha_v - \sum_{j <_u v} \gamma_{j,u} - \sum_{i <_v u} \gamma_{i,v} - \gamma_{u,v} \geqslant 1 \quad \text{for each } \{u, v\} \in E,$$
$$\alpha_v \geqslant 0 \qquad \text{for each } v \in V$$
$$\gamma_{u,v} \geqslant 0 \qquad \text{for each } \{u, v\} \in E.$$

There is an unusual property of the above pair of primal and dual linear programs: each fractional stable matching is an optimal solution of (LP) and is also included in an optimal solution of (DLP).

Lemma 4.1. *Each $x \in FSM(G; P)$ is an optimal solution of (LP) and (α, x) is an optimal solution of (DLP), where*

$$\alpha_v = \sum_{j \in N(v)} x_{j,v} \quad \text{for each } v \in V. \tag{8}$$

Proof. Let $x \in FSM(G; P)$ and let α be defined by (8). Let $\{u, v\} \in E$, to see that (α, x) is feasible for (DLP), note that

$$\begin{aligned}
&\alpha_u + \alpha_v - \sum_{i <_u v} x_{i,u} - \sum_{j <_v u} x_{j,v} - x_{u,v} \\
&\quad = \sum_{i \in N(u)} x_{i,u} + \sum_{j \in N(v)} x_{j,v} - \sum_{i <_u v} x_{i,u} - \sum_{j <_v u} x_{j,v} - x_{u,v} \\
&\quad = \sum_{i >_u v} x_{i,u} + \sum_{j >_v u} x_{j,v} + x_{u,v} \geqslant 1,
\end{aligned}$$

where the last inequality holds since x satisfies (7). To see that solutions x and (α, x) are optimal for (LP) and (DLP), respectively; observe that

$$\sum_{v \in V} \alpha_v - \sum_{\{u,v\} \in E} x_{u,v} = 2 \sum_{\{u,v\} \in E} x_{u,v} - \sum_{\{u,v\} \in E} x_{u,v} = \sum_{\{u,v\} \in E} x_{u,v};$$

so x has the same objective in LP as (α, x) has in DLP. Thus, the weak duality theorem of linear programming implies that x is optimal for (LP) and (α, x) is optimal for (DLP). □

The following result was proved independently by Hartman [9].

Theorem 4.2. *There is a partition of V into V^0 and V^1, such that for each fractional stable matching x*

$$\sum_{j \in N(v)} x_{j,v} = 0 \quad \text{if } v \in V^0,$$

$$\sum_{j \in N(v)} x_{j,v} = 1 \quad \text{if } v \in V^1.$$

Proof. For $\delta \in \{0, 1\}$ let $V^\delta = \{v \in V\colon \sum_{j \in N(v)} x_{j,v} = \delta, \forall x \in FSM(G; P)\}$. Suppose v is a vertex such that $v \notin V^0$. Then $\sum_{j \in N(v)} x'_{j,v} > 0$ for some fractional stable matching x'. By Lemma 4.1 there is an optimal dual solution (α', x') with $\alpha'_v > 0$. Note that α'_v is the dual variable that corresponds to the primal constraint (5) for v. Hence, by the complementary slackness theorem, for every optimal solution x for (LP)

$$\sum_{j \in N(v)} x_{j,v} = 1. \tag{9}$$

As Lemma 4.1 shows that every $x \in FSM(G; P)$ is optimal for (LP), we conclude that each such x satisfies (9), i.e., $v \in V^1$. □

As a consequence of Theorem 4.2, we obtain sufficient conditions for nonexistence of a stable matching.

Corollary 4.3 *For each $x \in FSM(G; P)$*

$$\sum_{\{u,v\} \in E} x_{u,v} = 1/2 \sum_{v \in V^1} \sum_{j \in N(v)} x_{j,v} = |V^1|/2.$$

In particular, if $|V^1|$ is odd, then $(G; P)$ does not have a stable matching.

Proof. By Theorem 4.2, each $x \in FSM(G; P)$ has

$$\sum_{\{u,v\} \in E} x_{u,v} = 1/2 \sum_{v \in V^1} \sum_{j \in N(v)} x_{j,v} = |V^1|/2.$$

If $|V^1|$ is odd, it is clear that no $x \in FSM(G; P)$ is integral. □

The following corollary specializes Theorem 4.2 to stable matchings, showing that the set of matched vertices for every stable matching is the same. The result was previously proved by Gusfield and Irving [8]. Their proof relied on Irving's combinatorial algorithm for solving the stable matching problem.

Corollary 4.4. *Let μ be a stable matching. Then $\mu(v) = v$ if $v \in V^0$ and $\mu(v) \neq v$ if $v \in V^1$.*

Theorem 4.5. *Let $\{u, v\} \in E$. If there exists some $x' \in FSM(G;P)$ with $x'_{u,v} > 0$, then for every $x \in FSM(G;P)$:*

$$\sum_{i >_u v} x_{i,u} + \sum_{i >_v u} x_{i,u} + x_{u,v} = 1\,. \tag{10}$$

Proof. By Lemma 4.1 (α', x') is an optimal solution of (DLP), where α' is defined from x' via (8). Recall that $\gamma_{u,v} = x'_{u,v}$ is the dual variable that corresponds to the primal constraint (7) for the pair $\{u, v\}$. Since $x'_{u,v}$ is positive, by the Complementary slackness theorem, every optimal solution of (LP) satisfies (7) for the pair $\{u, v\}$ as an equality. As Lemma 4.1 shows that every $x \in FSM(G;P)$ is optimal for (LP), we conclude that each such x satisfies Eq. (10) for the pair $\{u, v\}$. □

The following two corollaries specialize Theorem 4.5 to stable matchings. The first one was originally proved by Gusfield and Irving [8]. The second extends the decomposition principle of Knuth [14] from stable marriage problems to stable matching problems.

Corollary 4.6. *If vertices u and v are matched in a stable matching μ, then there is no stable matching that both u and v prefer to μ.*

Proof. If vertices u and v are matched in a stable matching μ with incidence vector x, then $x_{u,v} = 1 > 0$. Lemma 3.2 and Theorem 4.5 combine to show that Eq. (10) holds for the incidence vector of any stable matching. Now suppose both u and v prefer their outcomes in stable matching μ' to being matched to each other. Let x' be the incidence vector corresponding to μ', then

$$\sum_{i >_u v} x_{i,u} + \sum_{j >_v u} x'_{j,u} + x'_{u,v} = 2,$$

contradicting (10). □

Corollary 4.7. *Let μ and μ' be two stable matchings and define $V(\mu) \equiv \{v \in V\colon \mu(v) >_v \mu'(v)\}$ and $V(\mu') \equiv \{v \in V\colon \mu'(v) >_v \mu(v)\}$. Then μ and μ' map $V(\mu)$ onto $V(\mu')$ and $V(\mu')$ onto $V(\mu)$.*

Proof. It follows from Corollary 4.4 that $V(\mu')$ and $V(\mu)$ are contained in V^1. Let x be the incidence vector of μ and let x' be the incidence vector of μ'. Suppose $u = \mu(v)$ and $u' = \mu'(v)$. Since $x_{u,v} = 1 > 0$, by Theorem 4.5

$$\sum_{i >_u v} x'_{i,u} + \sum_{j >_v u} x'_{j,u} + x'_{u,v} = 1 .$$

Now $v \in V(\mu)$ if and only if

$$\sum_{j >_u v} x'_{j,v} + x'_{u,v} = 0$$

and $u \in V(\mu')$ if and only if

$$\sum_{i >_u v} x'_{i,u} = 1 .$$

Thus, $v \in V(\mu)$ if and only if $\mu(v) = u \in V(\mu')$. The other statement follows similarly. □

5. The median property

It was observed by John Conway (see [14, 8]) that the set of stable matchings for a stable marriage problem forms a distributive lattice under a natural partial order. This lattice structure was extended to all points in the stable matching polytope of a stable marriage problem by Roth et al. [18]. The lattice structure of the stable marriage problem does not carry over to the general stable matching problem, but, in this case, Gusfield and Irving [8] prove that there is a (weaker) semilattice structure. For this purpose, Gusfield and Irving [8] show that given a triplet of stable matchings a stable matching can be constructed via assigning each vertex its median choice over its outcomes in the three matchings. The outcome is then a matching which is called the *median* of the three original matchings. Here we show how this construction can be extended to the set of fractional stable matchings.

Let S be a set of three real numbers. Recall that the *median of* S, denoted med S, is defined as the second largest (or smallest) number among the three elements of S. The following two simple lemmas will be useful for our development.

Lemma 5.1. *Suppose* $r \in R$. *Let* (a_1, a_2), (b_1, b_2) *and* (c_1, c_2) *be three real solutions of the equation* $y_1 + y_2 = r$. *Then* $\operatorname{med}\{a_1, b_1, c_1\} + \operatorname{med}\{a_2, b_2, c_2\} = r$.

Proof. We may assume that $a_1 \leqslant b_1 \leqslant c_1$ and, hence, $\operatorname{med}\{a_1, b_1, c_1\} = b_1$. Since $(b_1 - a_1) + (b_2 - a_2) = 0$, it follows that $a_1 \leqslant b_1$ if and only if $b_2 \leqslant a_2$. Similarly, $(b_1 - c_1) + (b_2 - c_2) = 0$ implies that $b_1 \leqslant c_1$ if and only if $c_2 \leqslant b_2$. Thus, $c_2 \leqslant b_2 \leqslant a_2$ and $\operatorname{med}\{a_2, b_2, c_2\} = b_2$. So, $\operatorname{med}\{a_1, b_1, c_1\} + \operatorname{med}\{a_2, b_2, c_2\} = b_1 + b_2 = r$. □

Lemma 5.2. *Suppose* $r \in R$. *Let* (a_1, a_2), (b_1, b_2) *and* (c_1, c_2) *be three real solutions of the inequality* $y_1 + y_2 \geqslant r$. *Then* $\operatorname{med}\{a_1, b_1, c_1\} + \operatorname{med}\{a_2, b_2, c_2\} \geqslant r$.

Proof. We may assume that $a_1 \leqslant b_1 \leqslant c_1$ and, hence, $\operatorname{med}\{a_1, b_1, c_1\} = b_1$. Then $b_1 + b_2 \geqslant r$ and $b_1 + a_2 \geqslant a_1 + a_2 \geqslant r$ implying that $b_1 + \min\{a_2, b_2\} \geqslant r$. Since $\operatorname{med}\{a_2, b_2, c_2\} \geqslant \min\{a_2, b_2\}$, we conclude that $b_1 + \operatorname{med}\{a_2, b_2, c_2\} \geqslant r$. □

The following lemma will allow us to extend the concept of median to triplets of fractional stable matchings.

Lemma 5.3. *Let* $x^1, x^2, x^3 \in FSM(G;\, P)$ *be three fractional stable matchings and let* $\{u, v\} \in E$. *Then*

$$\operatorname{med}\left\{\sum_{i \geqslant_u v} x^k_{u,i}\colon k = 1,2,3\right\} - \operatorname{med}\left\{\sum_{i >_u v} x^k_{u,i}\colon k = 1,2,3\right\},$$

$$= \operatorname{med}\left\{\sum_{j \geqslant_v u} x^k_{v,j}\colon k = 1,2,3\right\} - \operatorname{med}\left\{\sum_{j >_v u} x^k_{v,j}\colon k = 1,2,3\right\}. \tag{11}$$

Proof. We consider two cases. First, suppose that $x_{u,v} = 0$ for every $x \in FSM(G;P)$. Then, $x^1_{u,v} = x^2_{u,v} = x^3_{u,v} = 0$ implying that, for $k = 1,2,3$:

$$\sum_{i \geqslant_u v} x^k_{i,u} = \sum_{i >_u v} x^k_{i,u}.$$

Thus,

$$\operatorname{med}\left\{\sum_{i \geqslant_u v} x_{u,i}^k : k = 1,2,3\right\} = \operatorname{med}\left\{\sum_{i >_u v} x_{u,i}^k : k = 1,2,3\right\}$$

as the two above medians are of the same three numbers, respectively. A similar argument, exchanging the roles of u and v, shows that

$$\operatorname{med}\left\{\sum_{j \geqslant_v u} x_{v,j}^k : k = 1,2,3\right\} = \operatorname{med}\left\{\sum_{j >_v u} x_{v,j}^k : k = 1,2,3\right\},$$

and (11) follows immediately.

Next, suppose that there exists $x' \in FSM(G;P)$ such that $x'_{u,v} > 0$. Then Theorem 4.5 shows that for every $x \in FSM(G;P)$

$$\sum_{i >_u v} x_{u,i} + \sum_{j >_v u} x_{v,j} + x_{u,v} = 1 .$$

In particular, this applies to x^1, x^2, x^3. So for $k = 1,2,3$

$$\sum_{i \geqslant_u v} x_{u,i}^k + \sum_{j >_v u} x_{v,j}^k = 1 . \tag{12}$$

Hence, by Lemma 5.1

$$\operatorname{med}\left\{\sum_{i \geqslant_u v} x_{u,i}^k : k = 1,2,3\right\} + \operatorname{med}\left\{\sum_{j >_v u} x_{v,j}^k : k = 1,2,3\right\} = 1, \tag{13}$$

and a symmetric argument shows that

$$\operatorname{med}\left\{\sum_{j \geqslant_v u} x_{j,v}^k : k = 1,2,3\right\} + \operatorname{med}\left\{\sum_{i >_u v} x_{i,u}^k : k = 1,2,3\right\} = 1. \tag{14}$$

Subtracting (14) from (13) and rearranging the terms establishes (11). □

Let $x^1, x^2, x^3 \in FSM(G;P) \subset R^E$ be three fractional stable matchings. We define the median of x^1, x^2 and x^3, denoted $\operatorname{med}\{x^1, x^2, x^3\}$, component-wise by letting for each $\{u, v\} \in E$

$$[\operatorname{med}\{x^1,x^2,x^3\}]_{u,v} = \operatorname{med}\left\{\sum_{i \geqslant_u v} x_{u,i}^k : k = 1,2,3\right\} - \operatorname{med}\left\{\sum_{i >_u v} x_{u,i}^k : k = 1,2,3\right\}. \tag{15}$$

In order for this definition to apply we have to argue that the right-hand side of (15) is symmetric in u and v. Indeed, Lemma 5.3 shows that this is the case.

A useful property of the median of triplets of fractional stable matchings is given in the next lemma.

Lemma 5.4. *Let $x^1, x^2, x^3 \in FSM(G; P)$ be three fractional stable matchings. Then*

$$\sum_{i \geqslant_u v} [\mathrm{med}\{x^1, x^2, x^3\}]_{u,i} = \mathrm{med}\left\{ \sum_{i \geqslant_u v} x^k_{u,i} \colon k = 1,2,3 \right\} \tag{16}$$

and

$$\sum_{i >_u v} [\mathrm{med}\{x^1, x^2, x^3\}]_{u,i} = \mathrm{med}\left\{ \sum_{i >_u v} x^k_{u,i} \colon k = 1,2,3 \right\}. \tag{17}$$

Proof. The definition of $\mathrm{med}\{x^1, x^2, x^3\}$ in (16) implies that

$$\begin{aligned}
&\sum_{i \geqslant_u v} [\mathrm{med}\{x^1, x^2, x^3\}]_{u,i} \\
&\quad = \sum_{i \geqslant_u v} \left[\mathrm{med}\left\{ \sum_{j \geqslant_u i} x^k_{u,j} \colon k = 1,2,3 \right\} - \mathrm{med}\left\{ \sum_{j >_u i} x^k_{u,j} \colon k = 1,2,3 \right\} \right] \\
&\quad = \sum_{i \geqslant_u v} \mathrm{med}\left\{ \sum_{j \geqslant_u i} x^k_{u,j} \colon k = 1,2,3 \right\} - \sum_{i \geqslant_u v} \mathrm{med}\left\{ \sum_{j >_u i} x^k_{u,j} \colon k = 1,2,3 \right\}.
\end{aligned} \tag{18}$$

Observing that

$$\left\{ \left(\sum_{j >_u i} x^1_{u,j}, \sum_{j >_u i} x^2_{u,j}, \sum_{j >_u i} x^3_{u,j} \right) \colon i \geqslant_u v \right\} = \left\{ \left(\sum_{j \geqslant_u i} x^1_{u,j}, \sum_{j \geqslant_u i} x^2_{u,j}, \sum_{j \geqslant_u i} x^3_{u,j} \right) \colon i >_u v \right\},$$

we conclude that

$$\sum_{i \geqslant_u v} \mathrm{med}\left\{ \sum_{j >_u i} x^k_{u,j} \colon k = 1,2,3 \right\} = \sum_{i >_u v} \mathrm{med}\left\{ \sum_{j \geqslant_u i} x^k_{u,j} \colon k = 1,2,3 \right\}. \tag{19}$$

Combining (18) and (19) we see that

$$\begin{aligned}
&\sum_{i \geqslant_u v} [\mathrm{med}\{x^1, x^2, x^3\}]_{u,i} \\
&\quad = \sum_{i \geqslant_u v} \mathrm{med}\left\{ \sum_{j \geqslant_u i} x^k_{u,j} \colon k = 1,2,3 \right\} - \sum_{i >_u v} \mathrm{med}\left\{ \sum_{j \geqslant_u i} x^k_{u,j} \colon k = 1,2,3 \right\} \\
&\quad = \mathrm{med}\left\{ \sum_{j \geqslant_u v} x^k_{u,j} \colon k = 1,2,3 \right\},
\end{aligned}$$

where the last equality follows by simply cancelling identical terms. So (16) has been established.

To prove (17) we consider two cases. First, we consider when $\{i \in V \colon i >_u v\} = \emptyset$. Then (17) holds trivially since both sides of the equation are null. For the second case, let $v' \equiv \min_u\{i \in V \colon i >_u v\}$. Then (17) coincides with (16) when v' replaces v. $\square$

We now show that the median of three fractional stable matchings is also a fractional stable matching.

Theorem 5.5. *Let* x^1, x^2, x^3 *belong to* $FSM(G; P)$. *Then* $\text{med}\{x^1, x^2, x^3\} \in FSM(G; P)$.

Proof. Let $v \in V$. By Lemma 5.4,

$$\sum_{i \in N(v)} [\text{med}\{x^1, x^2, x^3\}_{v,i}] = \text{med}\left\{ \sum_{i \in N(v)} x^k_{v,i} \colon k = 1,2,3 \right\} \leqslant 1,$$

where the last inequality holds since each x^k satisfies (5) for v. Thus, $\text{med}\{x^1, x^2, x^3\}$ verifies constraints (5).

Next, let $\{u, v\}$ be an edge in E. Then, for each $k = 1, 2, 3$,

$$\sum_{i \geqslant_u v} x^k_{u,i} \geqslant \sum_{i >_u v} x^k_{u,i}$$

implies

$$\text{med}\left\{ \sum_{i \geqslant_u v} x^k_{u,i} \colon k = 1,2,3 \right\} \geqslant \text{med}\left\{ \sum_{i >_u v} x^k_{u,i} \colon k = 1,2,3 \right\}.$$

Thus,

$$\begin{aligned}
&[\text{med}\{x^1, x^2, x^3\}]_{u,v} \\
&\quad = \text{med}\left\{ \sum_{i \geqslant_u v} x^k_{u,i} \colon k = 1,2,3 \right\} - \text{med}\left\{ \sum_{i >_u v} x^k_{u,i} \colon k = 1,2,3 \right\} \geqslant 0
\end{aligned}$$

and, therefore, $\text{med}\{x^1, x^2, x^3\}$ satisfies the nonnegativity constraints (6).

Finally, each x^k verifies the stability constraints (7). Hence, for $k = 1, 2, 3$,

$$\sum_{i \geqslant_u v} x^k_{i,u} + \sum_{j \geqslant_v u} x^k_{j,v} \geqslant 1$$

and therefore, by Lemma 5.2,

$$\text{med}\left\{ \sum_{i \geqslant_u v} x^k_{u,i} \colon k = 1,2,3 \right\} + \text{med}\left\{ \sum_{j \geqslant_v u} x^k_{v,j} \colon k = 1,2,3 \right\} \geqslant 1.$$

By, combining this inequality with Eqs. (16) and (17) of Lemma 5.2, we see that

$$\sum_{i \geqslant_u v} [\text{med}\{x^1, x^2, x^3\}]_{u,i} + \sum_{j \geqslant_v u} [\text{med}\{x^1, x^2, x^3\}]_{v,j} \geqslant 1.$$

Hence, $\text{med}\{x^1, x^2, x^3\}$ verifies the stability constraint (7) for $\{u, v\}$ and the proof that $\text{med}\{x^1, x^2, x^3\} \in FSM(G;P)$ is completed. $\square$

6. Extreme fractional stable matchings

We have seen in the earlier sections that the stable matching polytope $SM(G;P)$ is contained in the fractional stable matching polytope $FSM(G;P)$, and that the extreme points of the latter are always half-integral. In the current section we refine these results. First we characterize the half-integral vectors in $FSM(G;P)$ which are in $SM(G;P)$, and second, we characterize the extreme points of $FSM(G;P)$.

We continue to let $(G;P)$ be a stable matching problem where $G = (V,E)$. For $x \in FSM(G;P)$, we define the mappings $\bar{\sigma}_x$ and $\underline{\sigma}_x$ from the set V into itself by

$$\underline{\sigma}_x(v) = \min_v\{u \in N(v)\colon x_{u,v} > 0\},$$

$$\bar{\sigma}_x(v) = \max_v\{u \in N(v)\colon x_{u,v} > 0\},$$

where we remind the reader that according to the definition given in Section 2, $\min_v \emptyset = \max_v \emptyset = v$. We recall from Theorem 4.2 that V is partitioned into V^0 and V^1, where, for $\delta \in \{0,1\}$, $V^\delta = \{v \in V\colon \sum_{j \in N(v)} x_{j,v} = \delta$ for all $x \in FSM(G;P)\}$.

We next show that $\underline{\sigma}_x$ and $\bar{\sigma}_x$ are the inverse of each other. The result extends parts of Lemmas 2 and 3 of [20] which concern the stable marriage problem.

Lemma 6.1. *Let $x \in FSM(G;P)$. Then, for $u, v \in V$,*

$$v = \underline{\sigma}_x(u) \text{ if and only if } u = \bar{\sigma}_x(v).$$

Proof. First, consider $v \in V^0$. Then $v = \bar{\sigma}_x(v)$, and the assertion holds trivially. Next, consider $v \in V^1$. Now, if $v = \underline{\sigma}_x(u)$, then $x_{u,v} > 0$ and $u \in V^1$. Thus, $\sum_{i \geqslant_u v} x_{u,i} = \sum_{i \in N(u)} x_{u,i} = 1$. Also, by Theorem 4.5 and the assertion $x_{u,v} > 0$, implies that

$$\sum_{i >_u v} x_{u,i} + \sum_{j >_v u} x_{v,j} + x_{u,v} = 1.$$

Hence, $\sum_{j >_v u} x_{v,j} = 0$ and therefore $u \geqslant_v \bar{\sigma}_x(v)$. As $x_{u,v} > 0$, we conclude that $u = \bar{\sigma}_x(v)$, establishing one direction of the lemma. For the other direction, suppose $u = \bar{\sigma}_x(v)$. Then $\sum_{j >_u v} x_{v,j} = 0$ and the stability constraint (7) for $\{u,v\}$ implies $\sum_{i >_u v} x_{u,i} + x_{u,v} \geqslant 1$. Hence, by (5) and the assertion $x_{u,v} > 0, v = \bar{\sigma}_x(u)$. $\square$

We next define for each $x \in FSM(G;P)$ the set of edges $T(x) \subset E$ given by

$$T(x) \equiv \{\{u,v\} \in E\colon v \geqslant_u \underline{\sigma}_x(u) \text{ and } u \geqslant_v \underline{\sigma}_x(v)\}. \tag{20}$$

Of course, it immediately follows that

$$E_+(x) \subseteq T(x). \tag{21}$$

The next result shows that for each $x \in FSM(G;P)$, the vertices that are contained in the edges of $T(x)$ are precisely those in V^1.

Lemma 6.2. *Let $x \in FSM(G;\, P)$. Then $V_{T(x)} = V^1$.*

Proof. We observe that (21) implies $V^1 \subseteq V_{T(x)}$. Next, suppose $v \in V^0$. Then $\sum_{u \in N(v)} x_{u,v} = 0$. It follows that for each $u \in N(v)$, the stability constraint (7) for the pair $\{u, v\}$ implies $\sum_{i >_u v} x_{u,i} = 1$, thus, $v <_u \underline{\sigma}_x(u)$ and, by (20), $\{u, v\} \notin T(x)$. Hence, $v \notin V_{T(x)}$. □

Let $T \subseteq E$ be a subset of the edges of G and let $G_T \equiv (V_T, T)$ be the subgraph of G spanned by T. We observe that T specifies a stable matching problem $(G_T; P_T)$, where P_T denotes the restriction of P on G_T. For $v \in V_T$, we denote by $N_T(v)$ the set of neighbours of v in G_T, i.e. $N_T(v) = \{u \in V\colon \{u, v\} \in T\}$. The definition of V_T implies that $N_T(v) \neq \emptyset$ for all $v \in V_{T(x)}$. Also, as $N_T(v) \subseteq N(v)$, $\max_v N_T(v)$ and $\min_v N_T(v)$ are well defined. Finally, for $x \in R^E$ we denote by x_T the subvector of x consisting of the coordinates indexed by T, i.e., x_T is the orthogonal projection of x on R^T.

Let $x \in FSM(G;\, P)$. The *stable matching problem determined by* x is defined to be $(G_{T(x)}; P_{T(x)})$, where $T(x)$ is given by (20). In particular, Lemma 6.2 shows that $G_{T(x)} = (V_{T(x)}, T(x)) = (V^1, T(x))$. The next result shows that for each vertex $v \in V^1$, $\min_v N_{T(x)}(v)$ and $\max_v N_{T(x)}(v)$ are equal to $\underline{\sigma}_x(v)$ and $\bar{\sigma}_x(v)$, respectively.

Lemma 6.3. *Let $x \in FSM(G;P)$ and let $v \in V_{T(x)} = V^1$. Then*

$$\min_v N_{T(x)}(v) = \underline{\sigma}_x(v) \text{ and } \max_v N_{T(x)}(v) = \bar{\sigma}_x(v). \tag{22}$$

Proof. As $v \in V^1, x_{v,\underline{\sigma}_x(v)} > 0$ and, by (21), we conclude that $\underline{\sigma}_x(v) \in N_{T(x)}(v)$ and $\bar{\sigma}_x(v) \in N_{T(x)}(v)$. Now if $w <_v \underline{\sigma}_x(v)$ then (20) implies $\{v, w\} \notin T(x)$. Further, if $w >_v \bar{\sigma}_x(v)$ then $\sum_{j \geqslant_w w} x_{u,j} = 0$ and the stability constraint (7) for $\{v, w\}$ implies that $\sum_{j >_w v} x_{w,j} \geqslant 1$, thus $v <_w \underline{\sigma}_x(v)$ and, by (20), $\{v, w\} \notin T(x)$. As $\underline{\sigma}_x(v)$ and $\bar{\sigma}_x(v)$ are in $N_{T(x)}(v)$ and each w satisfying $w <_v \underline{\sigma}_x(v)$ or $w >_v \bar{\sigma}_x(v)$ is not in $N_{T(x)}$, we have that (22) follows. □

Corollary 6.4. *Let $x \in FSM(G;\, P)$ and let $e \in E_1(x)$. Then $G_{\{e\}}$ is a connected component of $G_{T(x)}$.*

Proof. Let $e = \{u, v\}$. Then $\underline{\sigma}_x(v) = \bar{\sigma}_x(v) = u$ and Lemma 6.3 implies that $N_{T(x)}(v) = u$. Similarly, $N_{T(x)}(u) = v$. □

Lemmas 6.1 and 6.3 give the following corollary.

Corollary 6.5. *Let $x \in FSM(G;\, P)$ and let $\{u, v\} \in T(x)$. Then*

$$u = \min_v N_{T(x)}(v) \text{ if and if only } v = \max_v N_{T(x)}(u).$$

For a fractional stable matching x, the definition of $T(x)$ via (20) and the above results show that the stable matching problem determined by x is a *stable table* for $(G; P)$, as defined by Gusfield and Irving [8]. We note that stable tables play a major role in the polynomial algorithm Irving developed to solve the stable matching problem (see [8]). Stable tables are further studied with the tools of polyhedral combinatorics in Abeledo [1], but are not used formally in the forthcoming development.

Let $x \in FSM(G; P)$. We next consider the polytope $FSM(G_{T(x)}; P_{T(x)}) \subseteq R^{T(x)}$, defined by the linear system (5)–(7) applied to the stable matching problem $(G_{T(x)}; P_{T(x)})$. The next result shows that all vectors in $FSM(G_{T(x)}; P_{T(x)})$ satisfy constraints (5), applied to $(G_{T(x)}; P_{T(x)})$, as equalities.

Corollary 6.6. *Let $x \in FSM(G; P)$ and let $y \in FSM(G_{T(x)}; P_{T(x)})$. Then, for each $v \in V_{T(x)} = V^1$, $\sum_{j \in N_{T(x)}(v)} y_{j,v} = 1$.*

Proof. Let $v \in V^1$ and let $w \equiv \min_v N_{T(x)}(v) \neq v$. By Corollary 6.5, $v = \max_w N_{T(x)}(w)$. Hence, constraint (7), applied to $(G_{T(x)}; P_{T(x)})$, for $\{v, w\} \in T(x)$, can be written as $\sum_{\{j \in N_{T(x)}(v): j \geqslant_v w\}} y_{j,v} \geqslant 1$. Since y satisfies constraint (5) for v, applied to $(G_{T(x)}; P_{T(x)})$, we have $\sum_{j \in N_{T(x)}(v)} y_{j,v} \leqslant 1$. Combining these two inequalities implies, since $\{j \in N_{T(x)}(v): j \geqslant_v w\} \subseteq N_{T(x)}(v)$, that $\sum_{j \in N_{T(x)}(v)} y_{j,v} = 1$, establishing the assertion. $\square$

For $x \in FSM(G;P)$, the following lemma provides a representation of the fractional stable matching polytope of $(G_{T(x)}; P_{T(x)})$.

Lemma 6.7. *Let $x \in FSM(G; P)$, let $y \in R^E$ satisfy $E_+(y) \subseteq T(x)$. Then $y \in FSM(G; P)$ if and only if $y_{T(x)} \in FSM(G_{T(x)}; P_{T(x)})$.*

Proof. As $E_+(y) \subseteq T(x)$, y trivially satisfies (5) for $v \in V \backslash V_{T(x)} = V^0$. Also, since $E_+(y) \subseteq T(x)$, we trivially have that $y_{T(x)}$ satisfies (5)–(7) with respect to $(G_{T(x)}; P_{T(x)})$ if and only if y satisfies, with respect to $(G; P)$, (5) for all $v \in V_{T(x)} = V^1$, (6) for all pairs $\{u, v\} \in E$ and (7) for all pairs $\{u, v\} \in T(x)$. So it remains to show that if $y_{T(x)} \in FSM(G_{T(x)}; P_{T(x)})$ then y satisfies (7) for $\{u, v\} \in E \backslash T(x)$.

So, assume that $y_{T(x)} \in FSM(G_{T(x)}; P_{T(x)})$ and let $\{u, v\} \in E \backslash T(x)$. By (20) and a possible exchange of the roles of u and v, we may assume that $w \equiv \underline{\sigma}_x(v) >_v u$. Then $\sum_{j \in N(v)} x_{j,v} > 0$ and therefore, by Theorem 4.2, $v \in V^1$. From Lemma 6.3, $w = \min_v N_{T(x)}(v)$ and, by Corollary 6.6, $\sum_{\{j \in N_{T(x)}: j \geqslant_v w\}} \hat{y}_{j,v} = 1$. Thus, $\sum_{\{j \in N(v): j \geqslant_v w\}} y_{j,v} = 1$ and, since $w >_v u$, it immediately follows that y satisfies the stability constraint (7) for $\{u, v\}$. $\square$

By restricting the conclusion of Lemma 6.7 to integral vectors we obtain the next result, proved originally by Gusfield and Irving [8] in the context of stable tables.

Corollary 6.8. *Let $x \in FSM(G; P)$. Then any stable matching of $(G_{T(x)}; P_{T(x)})$ is a stable matching of $(G; P)$.*

Corollary 6.9. *Let $x \in FSM(G;P)$ and let $\{T^1, T^2, \dots T^r\}$ be the partition of $T(x)$ such that $G_{T^1}, G_{T^2}, \dots, G_{T^r}$ are the connected components of $G_{T(x)}$. Then x is an extreme point of $FSM(G;P)$ if and only if x_{T^i} is an extreme point of $FSM(G_{T^i}; P_{T^i})$.*

Proof. We first show that x is an extreme point of $FSM(G;P)$ if and only if $x_{T(x)}$ is a extreme point of $FSM(G_{T(x)}; P_{T(x)})$. Assume that $x_{T(x)}$ is an extreme point of $FSM(G_{T(x)}; P_{T(x)})$ and that x has a representation $x = (1-\alpha)y + \alpha z$ where $0 < \alpha < 1$ and $y, z \in FSM(G;P)$. Then the nonnegativity of x, y and z assures that $E_+(y) \subseteq E_+(x) \subseteq T(x)$ and $E_+(z) \subseteq E_+(x) \subseteq T(x)$; hence by Lemma 6.7 $y_{T(x)}$ and $z_{T(x)}$ are in $FSM(G_{T(x)}; P_{T(x)})$. As $x_{T(x)} = (1-\alpha)y_{T(x)} + \alpha z_{T(x)}$, we conclude from the extremality of $x_{T(x)}$ that $x_{T(x)} = y_{T(x)} = z_{T(x)}$, implying that $x = y = z$. Alternatively, assume x is an extreme point of $FSM(G;P)$ and $x_{T(x)} = (1-\alpha)y' + \alpha z'$, for $0 < \alpha < 1$ and $y', z' \in FSM(G_{T(x)}; P_{T(x)})$. Let y and z be the extension of y' and z' to R^E obtained by setting the additional coordinates to zero. Then, as $E_+(x) \subseteq T(x)$, we conclude that $x = (1-\alpha)y + \alpha z$. So, the extremality of x implies $x = y = z$, assuring that $x_{T(x)} = y_{T(x)} = y'$ and $x_{T(x)} = z_{T(x)} = z'$.

We next observe that no pairs of variables corresponding to different components of $G_{T(x)}$ are in the same equation defining $FSM(G_{T(x)}; P_{T(x)})$. Hence, $FSM(G_{T(x)}; P_{T(x)})$ is the cartesian production of $FSM(G_{T^i}; P_{T^i})$, for $i = 1, \dots, r$. It immediately follows that $x_{T(x)}$ is an extreme point of $FSM(G_{T(x)}; P_{T(x)})$ if and only if, for $i = 1, \dots, r$, x_{T^i} is an extreme point of $FSM(G_{T^i}; P_{T^i})$. □

The next lemma is key to this section.

Lemma 6.10. *Let $x \in FSM(G;P) \cap \{0, 1/2, 1\}^E$ have the representation $x = \sum_{k \in K} \lambda_k y^k$, where $\sum_{k \in K} \lambda_k = 1$ and, for $k \in K$, $y^k \in FSM(G;P)$ and $\lambda_k > 0$. Then, for each $k \in K$ and for each vertex v which is contained in an odd cycle of $G_{T(x)}$.*

$$x_{v, \bar{\sigma}_x(v)} = x_{v, \underline{\sigma}_x(v)} = y^k_{v, \bar{\sigma}_x(v)} = y^k_{v, \underline{\sigma}_x(v)} = 1/2.$$

Proof. Let C be an odd cycle of $G_{T(x)}$ and let V_c be the vertex set of C. Then $V_C \subseteq V_{T(x)} = V^1$. Now, for $v \in V_C$ the connected component of $G_{T(x)}$ containing v contains V_C; hence that component does not consist of a single edge and Corollary 6.4 implies $x_{u,v} \neq 1$ for all $u \in N(v)$. Then $x \in \{0, 1/2, 1\}^E$ implies $x_{u,v} \in \{0, 1/2\}$ for all $u \in N(v)$. As $v \in V^1$, we conclude that $x_{v, \bar{\sigma}_x(v)} = x_{v, \underline{\sigma}_x(v)} = 1\backslash 2$ and $x_{u,v} = 0$ for all $u \in N(v) \backslash \{\bar{\sigma}_x(v), \underline{\sigma}_x(v)\}$.

As $x = \sum_{k \in K} \lambda_k y^k$ and the λ_k's are positive we have that $E_+(y^k) \subseteq E_+(x)$ for each $k \in K$. So, for $v \in V_C$, $y^k_{u,v} = x_{u,v} = 0$ for all $u \in N(v) \backslash \{\bar{\sigma}_x(v), \underline{\sigma}_x(v)\}$. Also, as $y^k \in FSM(G;P)$ and $v \in V_C \subseteq V^1$, we conclude that for $v \in V_C$, $y^k_{v, \bar{\sigma}_x(v)} + y^k_{v, \underline{\sigma}_x(v)} = \sum_{j \in N(v)} x_{v,j} = 1$. Thus, if suffices to show that $y^k_{v, \bar{\sigma}_x(v)} = 1/2$, for all $v \in V_C$ and for all $k \in K$.

Let $k \in K$. We will show that if u and v are adjacent vertices on the cycle C, then

$$y^k_{v, \bar{\sigma}_x(v)} + y^k_{u, \bar{\sigma}_x(u)} \geqslant 1. \tag{23}$$

We consider two cases. First assume that $x_{u,v} > 0$. Then $v \in \{\bar{\sigma}_x(u), \underline{\sigma}_x(u)\}$, Now, if $v = \bar{\sigma}_x(u)$, Lemma 6.1 implies that $u = \underline{\sigma}_x(v)$ and, therefore,

$$y^k_{v,\bar{\sigma}_x(v)} + y^k_{u,\bar{\sigma}_x(u)} = y^k_{v,\bar{\sigma}_x(v)} + y^k_{v,\underline{\sigma}_x(v)} = 1,$$

establishing (23). Alternatively, if $v = \underline{\sigma}_x(u)$, then Lemma 6.1 implies that $u = \bar{\sigma}_x(v)$ and a symmetric argument establishes (23). It remains to consider the case where $x_{u,v} = 0$. In this case, since $\{u,v\} \in T(x)$, Lemma 6.3 implies that $\underline{\sigma}_x(u) <_u v <_u \bar{\sigma}_x(u)$ and $\underline{\sigma}_x(v) <_u u <_v \bar{\sigma}_x(v)$. Hence, the stability constraint (7) applied to $\{u,v\}$ shows that

$$y^k_{u,\bar{\sigma}_x(u)} + y^k_{v,\bar{\sigma}_x(v)} = \sum_{i >_u v} y^k_{u,i} + \sum_{j >_v u} + y^k_{v,j} + y^k_{u,v} \geqslant 1,$$

establishing (23).

We next argue that

$$\sum_{u \in V_C} y^k_{u,\bar{\sigma}_x(u)} \geqslant \frac{|V_C|}{2}, \tag{24}$$

with equality holding if and only if $y^k_{u,\bar{\sigma}_x(u)} = 1/2$ for every $u \in V_C$. This conclusion is trite if $y^k_{u,\bar{\sigma}_x(u)} \geqslant 1/2$ for every $u \in V_C$. So, assume that for some $u \in V_C$, $y^k_{u,\bar{\sigma}_x(u)} < 1/2$ and we will show that (24) holds at strict inequality. Now, enumerate the vertices in V_C so that u is the first vertex and each pair of consecutive vertices are adjacent in C. In particular, the last vertex, say v, is adjacent to u and (23) implies that $y^k_{v,\bar{\sigma}_x(v)} > 1/2$. As $|V_C|$ is odd the vertices of $V_C \backslash \{v\}$ can be partitioned into disjoint pairs of adjacent vertices. As (23) holds for each such pair, we conclude that (24) holds as strict inequality.

Finally, by multiplying (24) by λ_k and summing over K, we see that

$$\frac{|V_C|}{2} = \sum_{v \in V_C} x_{v,\bar{\sigma}_x(v)} = \sum_{v \in V_C} \left(\sum_{k \in K} \lambda_k y^k_{v,\bar{\sigma}_x(v)} \right) = \sum_{k \in K} \lambda_k \left(\sum_{v \in V_C} y^k_{v,\bar{\sigma}_x(v)} \right) \geqslant \sum_{k \in K} \lambda_k \frac{|V_C|}{2} = \frac{|V_C|}{2}.$$

We conclude from the positivity of the λ_k's that (24) must hold as equality for all k. As we have seen, this conclusion means that $y^k_{v,\bar{\sigma}_x(v)} = 1/2$ for all $v \in V_C$ and our proof is completed. □

The next theorem characterizes the half-integral stable matchings in $FSM(G; P)$ that belong to the stable matching polytope $SM(G; P)$.

Theorem 6.11. *Let $x \in FSM(G; P) \cap \{0, 1/2, 1\}^E$. Then $x \in SM(G; P)$ if and only if $G_{T(x)}$ is bipartite. Further, in this case x can be represented as $x = 1/2(y + z)$, where y, z are the incidence vectors of two stable matchings.*

Proof. If $G_{T(x)}$ is not bipartite it contains an odd cycle and Lemma 6.10 implies that x cannot be expressed as a convex combination of integral vectors of $FSM(G; P)$, i.e, $x \notin SM(G; P)$.

Conversely, suppose $G_{T(x)} = (V^1, T(x))$ is bipartite. Let $M, W \subset V^1$ be two sets of vertices that give a bipartition for $G_{T(x)}$, i.e., $V^1 = M \cup W, M \cap W = \emptyset$ and all edges in $T(x)$ contain a vertex in M and a vertex in W. Let μ_M, μ_W denote the following subsets of $T(x)$:

$$\mu_M = \{\{m, \bar{\sigma}_x(m)\}: m \in M\},$$

$$\mu_W = \{\{w, \bar{\sigma}_x(w)\}: w \in W\}.$$

Then μ_M and μ_W are subsets of $E_+(x) \subseteq T(x)$. Further, Lemma 6.1 shows that if $w = \bar{\sigma}_x(m_1) = \bar{\sigma}_x(m_2)$, then $m_1 = \underline{\sigma}_x(w) = m_2$; thus μ_M is a matching. To prove that μ_M is a stable matching for $(G_{T(x)};P_{T(x)})$, we observe that any corresponding blocking pair in $T(x)$ must involve a vertex $m \in M$, but for each such vertex m, Lemma 6.3 implies that $\mu_M(m) = \max_m N_{T_x}(m) \geqslant_m w$ for every $w \in N_{T(x)}(m)$. Thus no vertex in M can belong to a blocking pair and, therefore, μ_M is a stable matching. Exchanging the roles of M and W, we also conclude that μ_W is a stable matching. Hence, by Corollary 6.8, μ_M and μ_W are stable matchings for $(G; P)$.

Let $y, z \in SM(G; P) \cap \{0, 1\}^E$ denote the incidence vectors of μ_M and μ_W, respectively. It suffices to prove, for each $\{u, v\} \in E$, that

$$x_{u,v} = 1/2(y_{u,v} + z_{u,v}). \tag{25}$$

We observe that if $\{u, v\} \in E \setminus T(x)$ then $x_{u,v} = y_{u,v} = z_{u,v} = 0$ and (25) holds. For the pairs $\{m, w\} \in T(x)$ we consider three cases. First, if $x_{m,w} = 0$ then (25) holds since $\mu_M, \mu_W \subseteq E_+(x)$. Next, if $x_{m,w} = 1/2$, then the assertion $x \in \{0, 1/2, 1\}^E$ implies that one and only one of the following holds: either $w = \bar{\sigma}_x(m) >_m \underline{\sigma}_x(m)$ or $w = \underline{\sigma}_x(m) <_m \bar{\sigma}_x(m)$. In the first case $y_{m,w} = 1$ and $z_{m,w} = 0$, and in the other case Lemma 6.1. implies $y_{m,w} = 0$ and $z_{m,w} = 1$, and both cases verify (25). Finally, if $x_{m,w} = 1$ then, $\bar{\sigma}_x(m) = \underline{\sigma}_x(m) = w$ and $\underline{\sigma}_x(w) = \bar{\sigma}_x(w) = m$, implying that $y_{m,w} = 1$ and $z_{m,w} = 1$. Therefore, (25) is established and, thus $x \in SM(G; P)$. □

We arrive to our characterization of the extreme points of $FSM(G; P)$.

Theorem 6.12. *Let $(G; P)$ be a stable matching problem and let $x \in FSM(G; P)$. Then x is an extreme point of $FSM(G; P)$ if and only x is half-integral and each component of $G_{T(x)}$ with edges in $E_{1/2}(x)$* contains an odd cycle.

Proof. By Corollary 6.9 we may assume, without loss of generality, that $G_{T(x)}$ is a connected graph. We first observe that if $E_1(x) \neq \emptyset$, the assumption that $G_{T(x)}$ is a connected graph and Corollary 6.4. imply that $T(x) = E_1(x)$ and that this set consists of a unique edge e. We conclude that $FSM(G; P)$ consists of a single point and the two assertions of our theorem's statement are trivially satisfied by this point.

Alternatively, assume that $E_1(x) = \emptyset$. We first prove necessity. Let x be an extreme point of $FSM(G; P)$. Then, by Theorem 3.3, x is half-integral and, since $E_1(x) = \emptyset$, it follows that $x \in \{0, 1/2\}^E$. Suppose $G_{T(x)}$ does not contain an odd cycle. Then $G_{T(x)}$ is bipartite and Theorem 6.11 implies that x has a representation $x = 1/2(y + z)$, where $y, z \in FSM(G;\, P) \cap \{0, 1\}^E$, contradicting the extremality of x.

To show sufficiency, let x be half-integral and let $G_{T(x)}$ contain an odd cycle C with vertex set V_C. Then $E_1(x) = \emptyset$ implies $x \in \{0, 1/2\}^E$ and as $V_{T(x)} = V^1$, we have for each $v \in V_{T(x)}$ that $x_{v,\underline{\sigma}_x(v)} = x_{v,\bar{\sigma}_x(v)} = 1/2$. To see that x is extreme for $FSM(G;\, P)$, suppose $y^k \in FSM(G;P)$ and $\lambda_k > 0$, for $k \in K$, are such that $x = \sum_{k \in K} \lambda_k y^k$ and $\sum_{k \in K} \lambda_k = 1$. It follows that for $k \in K$, $E_+(y^k) \subseteq E_+(x) = E_{1/2}(x)$. As $y^k \in FSM(G;\, P)$ we conclude that for each $v \in V_{T(x)} = V^1$

$$y^k_{v,\underline{\sigma}_x(v)} + y^k_{v,\bar{\sigma}_x(v)} = \sum_{u \in N(v)} y^k_{v,u} = 1\,. \tag{26}$$

We call a vertex $v \in V_{T(x)}$ determined if $y^k_{v,\underline{\sigma}_x(v)} = y^k_{v,\bar{\sigma}_x(v)} = 1/2$ for all $k \in K$, i.e., all y^k's coincide with x on the edges incident to v. To establish the theorem we next show that all vertices are determined. By Lemma 6.10, the set of determined vertices is nonempty as the vertices in V_C are determined.

We next show that if u is a determined vertex and $\{u, v\} \in T(x)$, v is also determined. We consider two cases. First assume that $x_{u,v} > 0$. Then $x_{u,v} = 1/2$ and as u is determined $y^k_{u,v} = x_{u,v} = 1/2$ for all $k \in K$. Further, either $u = \underline{\sigma}_x(v)$ or $u = \bar{\sigma}_x(v)$ and therefore we conclude from (26) that $y^k_{v,\underline{\sigma}_x(v)} = y^k_{v,\bar{\sigma}_x(v)} = 1/2$ for all $k \in K$. So, v is indeed determined. We next consider the case where $x_{u,v} = 0$. We then conclude from (20) that $v >_u \underline{\sigma}_x(u)$ and $u >_v \underline{\sigma}_x(v)$. As $y^k \in FSM(G;\, P)$ for all $k \in K$, the stability constraint (7) for y^k and $\{u, v\}$ implies that

$$y^k_{u,\underline{\sigma}_x(u)} + y^k_{v,\bar{\sigma}_x(v)} \geqslant 1\,. \tag{27}$$

Since u is determined we have $y^k_{u,\bar{\sigma}_x(u)} = 1/2$, hence (27) means that $y^k_{v,\bar{\sigma}_x(v)} \geqslant 1/2$ for all $k \in K$. We conclude that

$$1/2 = \left(\sum_{k \in K} \lambda_k \right)^{1/2} \leqslant \sum_{k \in K} \lambda_k y^k_{v,\bar{\sigma}_x(v)} = x_{v,\bar{\sigma}_x(v)} = 1/2\,;$$

hence the positivity of the λ_k's implies that $y^k_{v,\bar{\sigma}_x(v)} = 1/2$ for all $k \in K$. So, v is indeed determined.

We concluded that the set of determined vertices is nonempty and connected. As $G_{T(x)}$ is connected it follows that every vertex in $V_{T(x)}$ is determined, implying that $x = y^k$ for each $k \in K$, and thereby establishing the extremality of x. □

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