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Convergence Theorems of the Ishikawa Type Iterative Sequences with Errors for Generalized Quasi-Contractive Mappings in Convex Metric Spaces

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Abstract—In this paper, some convergence theorems of Ishikawa type iterative sequence with errors for nonlinear generalized quasi-contractive mapping in convex metric spaces are proved. The results presented in this paper not only extend and improve the main results in [1–8] but also give an affirmative answer to the open question of Rhoades-Naimpally-Singh in convex metric spaces.

Keywords—Convex metric space, Fixed point, Generalized Ishikawa (Mann) iterative sequence with errors, Generalized quasi-contractive mapping.

1. INTRODUCTION AND PRELIMINARIES

Recently concerning the problem of the Ishikawa iterative sequence $\{x_n\}$ defined by

$$\begin{aligned} x_0 &\in C, \\ x_{n+1} &= (1-\alpha_n)x_n + \alpha_n T y_n, \qquad n \geq 0, \\ y_n &= (1-\beta_n)x_n + \beta_n T x_n, \end{aligned} \tag{1.1}$$

converging strongly to a fixed point of T or to a solution of the equation $Tx = f$ has been considered by many authors (see, for example, [1,2,6,8–12]), where C is a nonempty closed convex subset of a Banach space E, $T : C \to C$ is a nonlinear pseudocontractive mapping or accretive mapping, and $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in [0,1].

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On the other hand, in 1974, Ciric [1] proved the following theorem.

THEOREM. *(See [1].) Let (E, d) be a complete metric space, $T : E \to E$ be a quasi-contractive mapping, i.e., there exists a constant $k \in [0,1)$ such that*

$$d(Tx, Ty) \le k \max\{d(x,y), d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\}, \tag{1.2}$$

for all $x, y \in E$. Then T has a fixed point x^* in E and for any given $x_0 \in E$, the Picard iterative sequence $\{T^n x_0\}$ converges to this fixed point x^*.

In 1976 and 1983, Rhoades [7] and Naimpally-Singh [5] suggest the following open question.

OPEN QUESTION. Can the Ishikawa iterative procedure be extended to nonlinear quasi-contractive mapping in a metric space?

This question is in fact solved in the affirmative (see [3,4,8]) for the Hilbert or Banach space setting.

The purpose of this paper is to prove some new convergence theorems for Ishikawa type iterative sequence with errors in a convex metric space. The results presented in this paper not only extend and improve the main results in [1–8], but also give an affirmative answer to the open question mentioned above in convex metric spaces.

For the purpose of this paper, we first give some definitions and notations.

DEFINITION 1.1. *Let (E, d) be a metric space and $I = [0,1]$. For any positive integer $n \ge 2$, denote by $E^n = \underbrace{E \times E \times \cdots \times E}_{n}$, $I^n = \underbrace{I \times I \times \cdots \times I}_{n}$. A mapping $W : E^n \times I^n \to E$ is said to be a convex structure of E, if it satisfies the following conditions: for any $u, x_1, x_2, \dots, x_n \in E$ and for any $\alpha_1, \alpha_2, \dots, \alpha_n \in I$ with $\sum_{i=1}^n \alpha_i = 1$; that is,*

$$\begin{aligned} &(1) \qquad W(x_1, x_2, \dots, x_n; 0, 0, \dots, \alpha_i, 0, \dots, 0) = x_i, \qquad i = 1, 2, \dots, n; \\ &(2) \qquad d(u, W(x_1, x_2, \dots, x_n; \alpha_1, \alpha_2, \dots, \alpha_n)) \le \sum_{i=1}^n \alpha_i d(u, x_i). \end{aligned} \tag{1.3}$$

E together with a metric d and a convex structure W is called a convex metric space and is denoted by (E, d, W).

It should be pointed out that each linear normed space is a special example of convex metric space, but there exist some convex metric spaces which cannot be embedded into any normed space (see [13]).

DEFINITION 1.2. *A function $\Phi : [0,\infty) \to [0,\infty)$ is said to satisfy the condition (C_Φ), if it is nondecreasing, continuous from right, $\Phi(t) < t$, $\forall\, t > 0$, and $\Phi(0) = 0$.*

It is easy to prove the following proposition.

PROPOSITION 1.1. *If function $\Phi : [0,\infty) \to [0,\infty)$ satisfies the condition (C_Φ) and $t \le \Phi(t)$, $t \in [0,\infty)$, then $t = 0$.*

DEFINITION 1.3. *Let (E, d) be a metric space and $T : E \to E$ be a mapping. If there exists a function $\Phi : [0,\infty) \to [0,\infty)$ satisfying the condition (C_Φ) such that*

$$d(Tx, Ty) \le \Phi(\max\{d(x,y), d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\}), \qquad \forall\, x, y \in E, \tag{1.4}$$

then T is said to be a generalized quasi-contractive mapping.

If $\Phi(t) = kt$, $k \in [0,1)$, then (1.4) is equivalent to (1.2), i.e., T is a quasi-contractive mapping.

DEFINITION 1.4. *Let (E, d, W) be a convex metric space with a convex structure $W : E^3 \times I^3 \to E$ satisfying condition (1.3) for $n = 3$. Let $T : E \to E$ be a generalized quasi-contractive mapping, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\xi_n\}$, $\{\eta_n\}$, and $\{\delta_n\}$ be six sequences in $[0,1]$ with $\alpha_n + \beta_n + \gamma_n = 1$, $\xi_n + \eta_n + \delta_n = 1$, $n = 0, 1, 2, \dots$, and $\sum_{n=0}^{\infty} \beta_n = \infty$. For any given $x_0 \in E$, define a sequence $\{x_n\}$ as follows:*

$$\begin{aligned} x_{n+1} &= W(x_n, Ty_n, u_n; \alpha_n, \beta_n, \gamma_n), \\ y_n &= W(x_n, Tx_n, v_n; \xi_n, \eta_n, \delta_n), \qquad n \geq 0, \end{aligned} \tag{1.5}$$

where $\{u_n\}, \{v_n\}$ are two sequences in E satisfying the following conditions.

For any nonnegative integers n, m with $0 \leq n < m$, if $\delta(A_{n,m}) > 0$, then

$$\max_{n \leq i,j \leq m} \{d(x,y) : x \in \{u_i, v_i\},\ y \in \{x_j, y_j, Tx_j, Ty_j, u_j, v_j\}\} < \delta(A_{n,m}), \tag{1.6}$$

where

$$A_{n,m} = \{x_i, y_i, Tx_i, Ty_i, u_i, v_i : n \leq i \leq m\}$$

and

$$\delta(A_{n,m}) = \sup_{x,y \in A_{n,m}} d(x,y).$$

Then $\{x_n\}$ is called the Ishikawa type iterative sequence with errors of T.

Especially, if $\eta_n = 0$, $\delta_n = 0$, $\forall\, n \geq 0$, it follows, from (1.3), that $y_n = x_n$. Hence, from (1.5), we have

$$x_{n+1} = W(x_n, Tx_n, u_n; \alpha_n, \beta_n, \gamma_n). \tag{1.7}$$

The sequence defined by (1.7) is called the Mann type iterative sequence with errors of T.

It should be pointed out that if E is a linear normed space, then E is a convex metric space with a convex structure $W(x, y; 1-\lambda, \lambda) = (1-\lambda)x + \lambda y$, $\forall\, x, y \in E$, $\lambda \in I$. Therefore, the Ishikawa iterative sequence (1.1) is a special case of (1.5) with $\gamma_n = 0$, $\delta_n = 0$, and $u_n = v_n = 0$, $\forall\, n \geq 0$.

2. MAIN RESULTS

We are now in a position to prove the main results of this paper.

THEOREM 2.1. *Let (E, d, W) be a complete convex metric space with a convex structure $W : E^3 \times I^3 \to E$ of E, T a generalized quasi-contractive mapping satisfying condition (1.4), and $\{x_n\}$ the Ishikawa type iterative sequence with errors of T defined by (1.5). Then the sequence $\{x_n\}$ converges to a unique fixed point p of T in E.*

PROOF. Let N be the set of all nonnegative integers. For any $n, m \in N$, $0 \leq n < m$, we denote

$$A_{n,m} = \{x_i, y_i, Tx_i, Ty_i, u_i, v_i : n \leq i \leq m\}$$

and

$$\delta(A_{n,m}) = \sup_{x,y \in A_{n,m}} d(x,y).$$

Then we have

$$\delta(A_{n,m}) = \max\{D_1, D_2, D_3, D_4, D_5, D_6\},$$

where

$$\begin{aligned} D_1 &= \max\{d(x_n, Tx_i), d(x_n, Ty_i) : n \leq i \leq m\}, \\ D_2 &= \max\{d(Tx_i, Tx_j), d(Tx_i, Ty_j), d(Ty_i, Ty_j) : n \leq i, j \leq m\}, \\ D_3 &= \max\{d(x_i, Tx_j), d(x_i, Ty_j) : n < i \leq m,\ n \leq j \leq m\}, \\ D_4 &= \max\{d(y_i, Tx_j), d(y_i, Ty_j) : n \leq i, j \leq m\}, \\ D_5 &= \max\{d(x_i, x_j), d(x_i, y_j), d(y_i, y_j) : n \leq i, j \leq m\}, \\ D_6 &= \max\{d(x,y) : x \in \{u_i, v_i\},\ y \in \{x_j, y_j, Tx_j, Ty_j, u_j, v_j\} : n \leq i, j \leq m\}. \end{aligned}$$

First, we prove that

$$\delta\left(A_{n,m}\right)=D_1. \tag{2.1}$$

For this purpose, we consider the following four steps.

(I) It follows from (1.4) that

$$D_2 \leq \Phi\left(\delta\left(A_{n,m}\right)\right). \tag{2.2}$$

(II) It follows from (1.5) and (1.3) that, if $n<i\leq m,\ n\leq j\leq m$, then we have

$$\begin{aligned} d\left(x_i,Tx_j\right) &= d\left(W\left(x_{i-1},Ty_{i-1},u_{i-1};\alpha_{i-1},\beta_{i-1},\gamma_{i-1}\right),Tx_j\right)\\ &\leq \alpha_{i-1}d\left(x_{i-1},Tx_j\right)+\beta_{i-1}d\left(Ty_{i-1},Tx_j\right)+\gamma_{i-1}d\left(u_{i-1},Tx_j\right)\\ &\leq \max\left\{d\left(x_{i-1},Tx_j\right),\Phi\left(\delta\left(A_{n,m}\right)\right),D_6\right\}. \end{aligned}$$

If $i-1>n$, then in the same way, we can prove that

$$d\left(x_{i-1},Tx_j\right)\leq \max\left\{d\left(x_{i-2},Tx_j\right),\Phi\left(\delta\left(A_{n,m}\right)\right),D_6\right\}.$$

By induction, for $n<i\leq m,\ n\leq j\leq m$, we can prove that

$$\begin{aligned} d\left(x_i,Tx_j\right) &\leq \max\left\{d\left(x_{i-1},Tx_j\right),\Phi\left(\delta\left(A_{n,m}\right)\right),D_6\right\}\\ &\leq \max\left\{d\left(x_{i-2},Tx_j\right),\Phi\left(\delta\left(A_{n,m}\right)\right),D_6\right\}\\ &\ \ \vdots\\ &\leq \max\left\{d\left(x_n,Tx_j\right),\Phi\left(\delta\left(A_{n,m}\right)\right),D_6\right\}. \end{aligned}$$

Similarly, for $n<i\leq m,\ n\leq j\leq m$, we can also prove that

$$d\left(x_i,Ty_j\right)\leq \max\left\{d\left(x_n,Ty_j\right),\Phi\left(\delta\left(A_{n,m}\right)\right),D_6\right\}.$$

This implies that

$$\begin{aligned} D_3 &= \max\left\{d\left(x_i,Tx_j\right),d\left(x_i,Ty_j\right): n<i\leq m,\ n\leq j\leq m\right\}\\ &\leq \max\left\{d\left(x_n,Tx_j\right),d\left(x_n,Ty_j\right),\Phi\left(\delta\left(A_{n,m}\right)\right),D_6: n\leq j\leq m\right\}\\ &= \max\left\{D_1,\Phi\left(\delta\left(A_{n,m}\right)\right),D_6\right\}. \end{aligned} \tag{2.3}$$

(III) For $n\leq i,j\leq m$, by (2.2) and (2.3), we have

$$\begin{aligned} d\left(y_i,Tx_j\right) &= d\left(W\left(x_i,Tx_i,v_i;\xi_i,\eta_i,\delta_i\right),Tx_j\right)\\ &\leq \xi_i d\left(x_i,Tx_j\right)+\eta_i d\left(Tx_i,Tx_j\right)+\delta_i d\left(v_i,Tx_j\right)\\ &\leq \max\left\{d\left(x_i,Tx_j\right),\Phi\left(\delta\left(A_{n,m}\right)\right),D_6\right\}\\ &\leq \max\left\{D_1,\Phi\left(\delta\left(A_{n,m}\right)\right),D_6\right\}. \end{aligned}$$

Similarly, we can prove that

$$d\left(y_i,Ty_j\right)\leq \max\left\{D_1,\Phi\left(\delta\left(A_{n,m}\right)\right),D_6\right\}.$$

Hence, we have

$$\begin{aligned} D_4 &= \max\left\{d\left(y_i,Tx_j\right),d\left(y_i,Ty_j\right): n\leq i,j\leq m\right\}\\ &\leq \max\left\{D_1,\Phi\left(\delta\left(A_{n,m}\right)\right),D_6\right\}. \end{aligned} \tag{2.4}$$

(IV) Since

$$D_5=\max\left\{d\left(x_i,x_j\right),d\left(x_i,y_j\right),d\left(y_i,y_j\right): n\leq i,j\leq m\right\}.$$

(a) We first make an estimation for $\max\{d(x_i,x_j): n \le i,j \le m\}$.
Let

$$A_1 = \max\{d(x_i,x_j): n \le i,j \le m\}.$$

Then there exist $k,l: n \le k < l \le m$ such that $A_1 = d(x_k,x_l)$ and

$$d(x_k,x_{l-1}) < d(x_k,x_l) = A_1. \tag{2.5}$$

Hence, we have

$$\begin{aligned} A_1 &= d(x_k,x_l) \\ &= d(x_k, W(x_{l-1},Ty_{l-1},u_{l-1};\alpha_{l-1},\beta_{l-1},\gamma_{l-1})) \\ &\le \alpha_{l-1}d(x_k,x_{l-1}) + \beta_{l-1}d(x_k,Ty_{l-1}) + \gamma_{l-1}d(x_k,u_{l-1}) \\ &\le \alpha_{l-1}d(x_k,x_{l-1}) + \beta_{l-1}D_1 + \gamma_{l-1}D_6. \end{aligned} \tag{2.6}$$

If $\alpha_{l-1} = 0$, from (2.6), we have $A_1 \le \max\{D_1, D_6\}$. If $\alpha_{l-1} \ne 0$, from (2.5) and (2.6), we have

$$\begin{aligned} A_1 &< \alpha_{l-1}d(x_k,x_l) + \beta_{l-1}D_1 + \gamma_{l-1}D_6 \\ &\le \max\{A_1, D_1, D_6\} \\ &= \max\{D_1, D_6\}. \end{aligned} \tag{2.7}$$

(b) Next, we make an estimation for $\max\{d(x_i,y_j): n \le i,j \le m\}$.
Let

$$A_2 = \max\{d(x_i,y_j): n \le i,j \le m\}.$$

Since $y_j = W(x_j,Tx_j,v_j;\xi_j,\eta_j,\delta_j)$, using (2.7) and (2.3), we have

$$\begin{aligned} A_2 &= \max\{d(x_i, W(x_j,Tx_j,v_j;\xi_j,\eta_j,\delta_j)): n \le i,j \le m\} \\ &\le \max\{\xi_j d(x_i,x_j) + \eta_j d(x_i,Tx_j) + \delta_j d(x_i,v_j): n \le i,j \le m\} \\ &\le \max\{D_1, D_6, \Phi(\delta(A_{n,m}))\}. \end{aligned} \tag{2.8}$$

(c) Finally, we make an estimation for $\max\{d(y_i,y_j): n \le i,j \le m\}$.
Let

$$A_3 = \max\{d(y_i,y_j): n \le i,j \le m\}.$$

By using (2.8) and (2.4), we have

$$\begin{aligned} A_3 &= \max\{d(y_i, W(x_j,Tx_j,v_j;\xi_j,\eta_j,\delta_j)): n \le i,j \le m\} \\ &\le \max\{\xi_j d(y_i,x_j) + \eta_j d(y_i,Tx_j) + \delta_j d(y_i,v_j): n \le i,j \le m\} \\ &\le \max\{d(y_i,x_j), d(y_i,Tx_j), d(y_i,v_j): n \le i,j \le m\} \\ &\le \max\{D_1, \Phi(\delta(A_{n,m})), D_6\}. \end{aligned} \tag{2.9}$$

It follows, from (2.7)–(2.9), that

$$\begin{aligned} D_5 &= \max\{d(x_i,x_j), d(x_i,y_j), d(y_i,y_j): n \le i,j \le m\} \\ &\le \max\{D_1, \Phi(\delta(A_{n,m})), D_6\}. \end{aligned} \tag{2.10}$$

Combining (2.2)–(2.4) with (2.10), we have

$$\begin{aligned} \delta(A_{n,m}) &= \max\{D_1, D_2, D_3, D_4, D_5, D_6\} \\ &\le \max\{D_1, \Phi(\delta(A_{n,m})), D_6\}. \end{aligned} \tag{2.11}$$

If $D_1 < \Phi(\delta(A_{n,m}))$ or $D_1 < D_6$, then $\delta(A_{n,m}) > 0$. Since Φ satisfies the condition (C_Φ), by using (1.6), we have that

$$D_1 < \Phi(\delta(A_{n,m})) < \delta(A_{n,m}),$$

and

$$D_1 < D_6 < \delta(A_{n,m}).$$

Therefore, from (2.11), we have

$$\delta(A_{n,m}) < \delta(A_{n,m}),$$

which is a contradiction. Thus, we have $D_1 \geq \Phi(\delta(A_{n,m}))$ and $D_1 \geq D_6$. Hence, from (2.11), we have

$$\delta(A_{n,m}) \leq D_1.$$

However, it is obvious that

$$D_1 \leq \delta(A_{n,m}).$$

Therefore, we have

$$D_1 = \delta(A_{n,m}).$$

Taking $n = 0$ in (2.1), we have

$$\begin{aligned}\delta(A_{0,m}) &= \max\{d(x_0, Tx_j), d(x_0, Ty_j) : 0 \leq j \leq m\}\\ &\leq d(x_0, Tx_0) + \max\{d(Tx_0, Tx_j), d(Tx_0, Ty_j) : 0 \leq j \leq m\}\\ &\leq d(x_0, Tx_0) + \Phi(\delta(A_{0,m})),\end{aligned}$$

and so we have

$$\delta(A_{0,m}) \leq (I - \Phi)^{-1}(d(x_0, Tx_0)), \qquad \forall m \geq 0. \tag{2.12}$$

This implies that the sequence $\{\delta(A_{0,m})\}$ is bounded.

On the other hand, for any positive integers $n, m : 1 \leq n < m$, it follows, from (2.1), that

$$\begin{aligned}\delta(A_{n,m}) &= \max\{d(x_n, Tx_j), d(x_n, Ty_j) : n \leq j \leq m\}\\ &= \max_{n\leq j\leq m}\{d(W(x_{n-1}, Ty_{n-1}, u_{n-1}; \alpha_{n-1}, \beta_{n-1}, \gamma_{n-1}), Tx_j),\\ &\quad d(W(x_{n-1}, Ty_{n-1}, u_{n-1}; \alpha_{n-1}, \beta_{n-1}, \gamma_{n-1}), Ty_j)\}\\ &\leq \max_{n\leq j\leq m}\{\alpha_{n-1} d(x_{n-1}, Tx_j) + \beta_{n-1} d(Ty_{n-1}, Tx_j)\\ &\quad + \gamma_{n-1} d(u_{n-1}, Tx_j), \alpha_{n-1} d(x_{n-1}, Ty_j) + \beta_{n-1} d(Ty_{n-1}, Ty_j) + \gamma_{n-1} d(u_{n-1}, Ty_j)\}\\ &\leq \alpha_{n-1}\delta(A_{n-1,m}) + \beta_{n-1}\Phi(\delta(A_{n-1,m})) + \gamma_{n-1}\delta(A_{n-1,m})\\ &= (1 - \beta_{n-1})\delta(A_{n-1,m}) + \beta_{n-1}\Phi(\delta(A_{n-1,m}))\\ &= (I - \beta_{n-1}(I - \Phi))(\delta(A_{n-1,m})).\end{aligned}$$

By induction and using (2.12), we can prove that

$$\begin{aligned}\delta(A_{n,m}) &\leq \prod_{j=0}^{n-1}(I - \beta_j(I - \Phi))(\delta(A_{j,m}))\\ &\leq \prod_{j=0}^{n-1}(I - \beta_j(I - \Phi))(\delta(A_{0,m}))\\ &\leq \prod_{j=0}^{n-1}(I - \beta_j(I - \Phi))(t_0),\end{aligned}$$

where $t_0 = (I - \Phi)^{-1}(d(x_0, Tx_0))$.

Since the function $\Phi : [0,\infty) \to [0,\infty)$ satisfies the condition (C_Φ), for any $t > 0$, $\Phi(t) < t$, i.e., $(I - \Phi)(t) > 0$, $\forall t > 0$. Again, since $\sum_{j=0}^{\infty} \beta_j = \infty$. Therefore, we have that

$$\prod_{j=0}^{\infty} \left(I - \beta_j \left(I - \Phi\right)\right)(t_0) = 0.$$

Hence, we have

$$\lim_{n,m\to\infty} \delta\left(A_{n,m}\right) = 0.$$

This implies that $\{x_n\}$ is a Cauchy sequence in E and

$$\lim_{n\to\infty} d\left(x_n, Tx_n\right) = 0.$$

Since E is complete, there exists a $p \in E$ such that $x_n \to p$. Hence, we have

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} Tx_n = p.$$

It follows, from (1.4), that

$$d\left(Tx_n, Tp\right) \le \Phi\left(\max\left\{d\left(x_n, p\right), d\left(x_n, Tx_n\right), d\left(p, Tp\right), d\left(x_n, Tp\right), d\left(p, Tx_n\right)\right\}\right).$$

Letting $n \to \infty$ on the both sides of the above expression and taking the right limit, we have

$$\begin{aligned} d\left(p, Tp\right) &\le \Phi\left(\max\left\{0, 0, d\left(p, Tp\right), d\left(p, Tp\right), 0\right\}\right) \\ &\le \Phi\left(d\left(p, Tp\right)\right). \end{aligned}$$

Therefore, it follows, from Proposition 1.1, that $d(p,Tp) = 0$; i.e., p is a fixed point of T in E.

Finally, we prove that p is a unique fixed point of T in E. Suppose the contrary. Let q also be a fixed point of T in E. Then we have

$$\begin{aligned} d\left(p, q\right) &= d\left(Tp, Tq\right) \\ &\le \Phi\left(\max\left\{d\left(p, q\right), d\left(p, Tp\right), d\left(q, Tq\right), d\left(p, Tq\right), d\left(q, Tp\right)\right\}\right) \\ &= \Phi\left(d\left(p, q\right)\right). \end{aligned}$$

Hence, $d(p,q) = 0$, i.e., $p = q$, and so p is the unique fixed point of T in E. This completes the proof of Theorem 2.1.

Taking $\Phi(t) = kt$ in Theorem 2.1, where $k \in [0,1)$ is a constant and $t \in [0,\infty)$, we can obtain the following result immediately.

THEOREM 2.2. *Let (E,d,W) be the same as in Theorem 2.1, T a quasi-contractive mapping satisfying condition (1.2), and $\{x_n\}$ the Ishikawa type iterative sequence with errors of T defined by (1.5). Then the sequence $\{x_n\}$ converges to a unique fixed point p of T in E.*

REMARK. Theorems 2.1 and 2.2 are two new convergence theorems of the Ishikawa type sequences with errors for the generalized pseudocontractive mappings in convex metric spaces. These two theorems not only provide an affirmative answer to the open question of Rhoades and Naimpally-Singh in the setting of convex metric spaces, but also improve and extend the corresponding results in [1–8].

For the Mann type iterative sequence $\{x_n\}$ with errors, we also have the following result from Theorem 2.1.

THEOREM 2.3. *Let (E,d,W) be the same as in Theorem 2.1, T a generalized quasi-contractive mapping satisfying condition (1.4), and $\{x_n\}$ the Mann type iterative sequence with errors of T defined by (1.7). Then the sequence $\{x_n\}$ converges to a unique fixed point p of T in E.*

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