COMBINATORIAL CHARACTERIZATION OF HEXAGONAL SYSTEMS

Antonije D. JOVANOVIC

Faculty of Electrical Engineering, University of Belgrade, and Dept. Electrical & Computer Engineering, Drexel University, Philadelphia, PA 19104, USA

Received 21 October 1985
Revised 17 October 1986

This paper provides a combinatorial characterization for the class of graphs that model molecular structures of aromatic benzenoid hydrocarbons. Based on invariant characteristics of these graphs, cata-, peri-, and corona-fused benzenoid hydrocarbons, and helicanes are simply distinguished. A canonical coding algorithm for cata- and peri-fused systems is given. Other applications are briefly discussed.

1. Introduction

The class of graphs under consideration in this paper has been initially recognized by the fact that these graphs represent structural models for the molecules of a class of organic compounds which are known under the names 'aromatic benzenoid hydrocarbons', 'fused benzenoid hydrocarbons' (FBH), or 'condensed benzenoid hydrocarbons'. Eight types of structures which can be found in FBH molecules are represented by simple graphs (see Figs. 4 and 5). The corresponding class of graphs has been intensively studied in the past under different names: hexagonal animals [6], hexagonal p-minos, polyominoes [8], polihexes [2], honeycomb lattice graphs [1]. These graphs are referred to in this paper as hexagonal systems (HS), a term proposed by Cvetkovic [5]. Studies of HS have developed to meet an increasing need for a consistent method of classifying, naming, and describing fused benzenoid hydrocarbons. Two recent papers [2,4] and other papers referenced in them, provide extensive accounts of these needs and of previous results. These accounts reveal that attempts to find an appropriate method of characterizing HS have failed to clearly discriminate between some classes of FBH, and have proposed codes that are more complex than necessary. The absence of a suitable graph-theoretical characterization for the whole class of HS is also an open problem. A survey of recent papers addressing these problems is presented below. Combinatorial definitions for the class of HS as a whole, and for all eight subclasses of HS separately, are presented in Section 3.

The various solutions proposed for the coding of HS, use one or more steps from the following general three-step procedure:
Step 1. Imbed the HS in the planar hexagonal lattice (also referred to in literature as ‘honeycomb’ or ‘graphite’ lattice).

Step 2. Use the notation of some coordinate system for the hexagons in the hexagonal lattice to label the hexagons of the imbedded HS.

Step 3. Organize the set of HS’s hexagon labels into a cannonical code for the HS.

A method for HS enumeration proposed by Lunnon in [8] applies this general three-step procedure. The integer triples used for labeling hexagons in a three-axe coordinate system, are redundant for the planar hexagonal lattice; any two of them are sufficient. Defining the boundary hexagon of a HS by the six lines which coincide in three directions with the outermost lines passing through the centers of all hexagons contained in the HS under consideration is consistent with the coordinate system. This boundary hexagon corresponds to more than one HS. Therefore, the coordinate system and boundary hexagon, which are essential to Lunnon’s enumeration procedure [8] do not seem to be well suited to test HS isomorphism, or to code a HS.

Coding fused benzenoid hydrocarbons using the procedure proposed by Cioslowsky and Turek [4], also involves using a larger portion of the hexagonal lattice than is actually occupied by hexagons of the imbedded HS. The applied bounding parallelogram, with four sides parallel to two axes at angles of \( \pi/3 \), is simpler than Lunnon’s bounding hexagon. Like Lunnon, Cioslowski and Turek also mark the hexagons of the bounding parallelogram with ones and zeros depending on whether they belong to the imbedded HS or not. Treating the zero/one pattern obtained as a matrix and ordering the elements of that matrix by rows, they derive a binary code for the HS which is not unique because of the 12 possible symmetries of the imbedding. To obtain the cannonical code, matrices of all possible imbeddings must be considered according to the rules of that procedure. The requirement of needing to find all possible imbeddings and compare their matrices is a significant drawback of the algorithm.

Bonchev and Balaban [2] consider the same problem. Retaining the structure of the general three-step procedure, they recommend a circular labeling of hexagons around the one which is chosen as the center. This produces lengthy labels and codes because the number of hexagons in circuits around the center increases in arithmetical progression. Before imbedding the HS, the hexagon to be imbedded at the center must be determined. This requires a complex procedure to determine the generalized center of the HS’s ‘dual graph’. When the generalized center has more than one vertex, all imbeddings corresponding to the vertices in it must be considered to obtain the canonical code. Together with some other drawbacks, it seems that the results of this paper share the same general objections as does the previous work.

Bonchev et al. [3] present a different approach to the problem of discriminating the molecular structures of isomers by constructing topological indices. Knowing that no set of graph invariants is sufficient to test efficiently for isomorphism between even planar graphs, the authors of that paper do not attempt to solve the
coding problem by means of invariants. Instead, they try to find a good approximate solution. Their solution consists of a single number, the topographical information superindex. This superindex is calculated starting from a set of carefully chosen graph invariants. Theoretical limitations to the discriminative power of such solutions, together with considerable computational complexity, are primary drawbacks to this approach.

The combinatorial characterization of HS presented here relies entirely on invariant graph properties characteristic of the class of HS. A set of these invariants can be therefore proven sufficient to test HS isomorphism and even to code HS. This characterization does not require HS imbeddings in hexagonal lattice, although some results derived directly from invariant codes can be given such an interpretation.

The combinatorial characterization of cata-, peri- and corona-fused benzenoid hydrocarbons and for helicanes given in Section 3 are used in Section 4 as the basis for proposing a solution to the coding problem for these four fundamental subclasses. Enumeration algorithms are also briefly considered.

2. Underlying graph-theoretical concepts

Molecular structures of fused benzenoid hydrocarbons (FBH) can be represented by a class of graphs called hexagonal systems (HS). The fundamental role in the combinatorial characterization of HS is played by the meshes of planar graphs. The set of meshes, $M$, of a two-connected planar graph $G(V,E)$ is

$$M = \{m_i | m_i \text{ is a boundary circuit of a face in any planar imbedding of } G \}.$$ 

For the class of two-connected planar graphs, to which the HS belong, $|M| = n_c - n_v + 2$, where $n_c = |E|$ and $n_v = |V|$. We shall consider that the length of the mesh $m_i$ is equal to the number of edges included in $m_i$ and that two meshes of $M$ are adjacent if they both contain the same edge $e_j \in E$. Every mesh is also considered to be adjacent to itself.
The structure of the HS, whose graphical representation imbedded in the hexagonal lattice is shown in Fig. 1, was tailored more toward demonstrating the properties of HS than to depict a real molecule. Nevertheless, its structural characteristics correspond to the class of mixed cata + peri-fused benzenoid hydrocarbons. As applied here, different subclasses of the HS are referred to as fusion types using the terminology by which different FBH subclasses are identified in [2].

The HS from Fig. 1 is partly characterized by having a unique long HS-mesh and \( n_c - n_v + 1 \) hexagonal meshes. The unique long HS-mesh forms the perimeter of this HS. In the graphical representation of all HS imbedded in the hexagonal lattice, the perimeter is represented by the contour enclosing the drawing of the HS. Later, when other underlying concepts are developed, a combinatorial characterization of the perimeter for the whole class of HS will become feasible without using geometrical concepts.

Fig. 1 suggests that for every HS from the cata/peri subclass there is a unique cyclically-ordered sequence of the perimeter-vertex degree-values \( D = d_1, d_2, \ldots, d_{np}, d_i \in \{2, 3\} \). Subtracting 2 from every element of \( D \) converts it into the binary-coded sequence \( B = b_1, b_2, \ldots, b_{np}, b_i \in \{0, 1\} \). The sequence \( B \), which is also unique for any given HS, is called the B-code of a HS. The particular B-code of the HS from Fig. 1 can be written as

\[
B = 000011000100111100010001010100011.
\]

The following theorem shows that the converse relation, between a HS and its B-code also exists.

**Theorem.** The B-code of a cata/pari-fused HS determines that unique HS.

**Proof.** This can be verified by considering a tour around the graphical representation of the perimeter when the HS is imbedded in the hexagonal lattice. Whenever a vertex is passed on such a tour, a change in the direction of motion takes place. When the degree of the vertex is 2, the change is \( +\pi/3 \), and when the degree is 3, the change is \( -\pi/3 \), assuming a counter-clockwise tour. Thus, the sequence of the perimeter-vertex degrees completely determines the graphical representation of a HS imbedded in the hexagonal lattice, and, consequently, it also determines the HS. \( \square \)

The B-code provides a complete characterization for a broader subclass of HS than cata/peri but the most complex FBH molecular structures require additional concepts for their characterization. Note that the B-code is not restricted to perimeter-meshes. The B-code describes all structural characteristics stemming from the presence of long HS-meshes. To each such mesh, there is a corresponding B-code.

The concept of H-code is required to discriminate between several long HS meshes of the same HS, among which only one is the perimeter. It is also required
to discriminate between some subclasses of HS. The $H$-code is another cyclical code derived from the $B$-code. Its form is

$$H = h_1, h_2, \ldots, h_{n_H}, \quad h_i = (x_i, y_i), \quad x_i, y_i \in \mathbb{Z}.$$  

Each element of the $H$-code corresponds to a hexagonal mesh adjacent to the long HS-mesh from whose $B$-code the $H$-code is derived. Conversely, the number of elements in the $H$-code, $n_H$, and the number of elements in the $B$-code, $n_B$, are in the relation $n_H < n_B/2$.

Closed formulae for the calculation of the $H$-code elements are given in an earlier paper [7]. A new, computationally faster procedure is provided here. Given a $B$-code, one can calculate the $H$-code elements as follows.

**Preparation:**

(a) Find $B_x$, one of the longest subsequences of adjacent elements of the $B$-code having the value 0.

(b) Find the cyclical permutation of the record of $B$ which places $B_x$ on the beginning of the record.

(c) Group the $B$-code elements into the groups of Table 1 by the order of precedence in Table 1.

After the preparation step the $B$-code record of the HS from Fig. 1 becomes

$$B = 00\;00\;11\;00\;01\;00\;11\;11\;00\;01\;00\;01\;01\;01\;00\;00\;111.$$  

The groups formed in $B$ by the preparation step can be partitioned into two classes. The $X$-class, with at least one element of the value 1 in the group, and the $Y$-class, including all others. The $X$-class groups generate an element of the $H$-code; the $Y$-class groups do not. The $Y$-class groups contribute to the values of the components, $x_j$ and $y_j$ of the code element, $h_j$ introduced by the next group in the sequence.

<table>
<thead>
<tr>
<th>Group</th>
<th>$n_s$</th>
<th>$k_b$</th>
</tr>
</thead>
<tbody>
<tr>
<td>00001111</td>
<td>4</td>
<td>+3</td>
</tr>
<tr>
<td>000111</td>
<td>3</td>
<td>+2</td>
</tr>
<tr>
<td>0011</td>
<td>2</td>
<td>+1</td>
</tr>
<tr>
<td>01</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>111</td>
<td>3</td>
<td>-1</td>
</tr>
<tr>
<td>11</td>
<td>2</td>
<td>-1</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>000</td>
<td>0</td>
<td>+3</td>
</tr>
<tr>
<td>00</td>
<td>0</td>
<td>+2</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>+1</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$k$</th>
<th>$x(k)$</th>
<th>$y(k)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>-2</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>-1</td>
<td>-1</td>
</tr>
<tr>
<td>5</td>
<td>1</td>
<td>-1</td>
</tr>
</tbody>
</table>
The initial step in the procedure for generating the $H$-code elements assigns the values 0 to $h_1$ and $k_1$ arbitrarily,

$$h_1 = (0,0), \quad k_1 = 0.$$  

The components $x_i$ and $y_i$ of the $H$-code elements, $h_i$ for $i=2,3,\ldots,n_H$ are then calculated using the following steps:

**Step 1.** Calculate the parameters $k_f$ and $k_i$ according to the following formulae

$$k_f = k_{i-1} + k_{bn} + k_{bg}, \quad k_i = k_f - n_s + 1$$

where the values for $k_{bn}$ and $k_{bg}$ can be obtained from Table 1 as follows:
- $k_{bg}$ belongs to the group with $n_s \neq 0$ generating $h_i$,
- $k_{bn}$ belongs to the group with $n_s = 0$, which immediately precedes in $B$ the group generating $h_{i-1}$; if such a group does not exist, $k_{bn} = 0$.

**Step 2.** Calculate the components $x_i$ and $y_i$ using Table 2 according to the following formulas

$$x_i = x_{i-1} + \sum_{k=k_f}^{k_i} x(k), \quad y_i = y_{i-1} + \sum_{k=k_f}^{k_i} y(k).$$

When applied to the $B$-code of the HS from Fig. 1, the above procedure produces the following $H$-code

$$H = (0,0), (-3,1), (-4,0), (-4,-2), (-7,-1), (-8,-2), (-6,-2), (-4,-2), (-2,-2).$$

As mentioned earlier, the $H$-code can be interpreted geometrically. Fig. 2 illustrates the scheme by which the hexagons of the hexagonal lattice must be coded by the integer-valued rectangular coordinates. Fig. 3 shows the HS from Fig. 1 imbedded in the hexagonal lattice with the elements of its calculated $H$-code equal to the coordinates of the hexagons occupied by the meshes adjacent to the perimeter.

The hexagonal mesh, whose vertices have entered the subsequence $B_x$, is imbed-
Fig. 3. A canonical imbedding of the HS from Fig. 1.

A comparison of the $B$ and $H$ codes is now appropriate. Elements of the $B$-code are related to the vertices in the long HS-mesh: every vertex gives one element of the $B$-code. Elements of the $H$-code are related to the hexagonal meshes adjacent to the long HS-mesh: every element of the $H$-code belongs to one hexagonal mesh. But, as can be seen from the sample HS's $H$-code and Fig. 3, there are hexagonal meshes adjacent to the long HS-mesh which do not generate an element of the $H$-code. This characteristic has been obtained by designing the $H$-code to include only the selected essential information on the 'outermost hexagonal meshes'.

A combinatorial characterization of the perimeter, valid for all subclasses of HS, is now stated in the following form:

- If a HS has a single long HS-mesh, this long HS-mesh is its perimeter.
- If a HS has $n$ long HS-meshes $m_1, m_2, \ldots, m_n$, the perimeter is the mesh $m_j$ whose $H$-code, $H_j$, contains two elements $h_{jr}$ and $h_{js}$ such that

$$|x_{jr} - x_{js}| = \max \{|x_{ip} - x_{iq}|: i = 1, 2, \ldots, n; p, q \in \{1, 2, \ldots, n_H\}\}$$

where $n_H$ designates the number of elements in the $H$-code $H_j$.

In the case when there is more than one long HS-mesh, the above characterization of the perimeter relies on the fact that the perimeter's contour has to include contours of all other long HS-meshes, when a graphical representation of a HS is imbedded in the hexagonal lattice. It should be noted that the simpler concept of the longest HS-mesh could not be used for the definition of the perimeter, although the coincidence of that characteristic and the one used in the characterization may exist.

3. Combinatorial characterization of fused benzenoid hydrocarbons

Relying on the concepts introduced in Section 2, the eight subclasses of the class of HS can be individually characterized by satisfying or contradicting subsets of the following seven conditions.

1. The set of meshes of the HS contains at most one long HS-mesh, all other meshes being hexagonal meshes.
2. The set of meshes of the HS contains \( n_l, \ n_l > 1 \), long HS-meshes and \( n_e - n_e + 2 - n_l \) hexagonal meshes.

3. The \( B \)-code of the long HS-mesh of the HS contains a single subsequence of adjacent ones, consisting of more than four ones.

4. \( B \)-codes of the long HS-meshes of the HS do not contain a subsequence of more than 3 consecutive zeros.

5. All the vertices of the HS are in the long HS-meshes.

6. Vertices of degree 3 in the long HS-meshes are adjacent in the HS only if they are adjacent in the long HS-meshes.

7. In the \( H \)-code of a long HS-mesh, every pair of elements \((h_i, h_k)\) which do not belong to two adjacent hexagonal meshes satisfies the relation:

\[
|y_i - y_k| > 1 \quad \text{or} \quad |y_i - y_k| > 1.
\]

These conditions are sufficient for the combinatorial characterization of the molecular structures of eight subclasses of FBH modeled by the corresponding subclasses from the class of HS. Fig. 4 shows graphical representations of graphs from the four fundamental subclasses of HS which represent the simplest FBH molecular structures.

Fig. 5 shows examples from the other four HS subclasses. These subclasses represent more complex molecular structures, characterized by the mixed topological properties of the fundamental subclasses.

The subsets of necessary and sufficient conditions which define the eight different subclasses of HS are summarized in Table 3. A plus sign means that a condition is satisfied; a minus sign means that a condition is contradicted by the subclass.

![Fig. 4. Examples of the four fundamental HS subclasses: (a) cata, (b) peri, (c) corona, and (d) helicane.](image)
Fig. 5. Examples of the four combined-condensation HS subclasses: (a) cata + peri, (b) cata + corona, (c) peri + corona, and (d) cata + peri + corona.

having the sign in its row. The signs (+) or (−) mean that the satisfaction or contradiction is implied by another necessary condition.

Condition 3 differentiates the helicane-subclass from all other graphs which do not satisfy condition 7 for different reasons than the satisfaction of the condition 3. Considering graphical representations of graphs imbedded in the hexagonal lattice, it should be noted that violations of condition 7, or satisfactions of condition 3, while other conditions for HS are met, result in the overlapping of two or more lines which represent the edges of these graphs. Structural characteristics of FBH molecules allow such overlappings only for the special configuration of the helicane-subclass molecules.

<table>
<thead>
<tr>
<th>Fusion Type</th>
<th>Conditions</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1</td>
</tr>
<tr>
<td>cata</td>
<td>+</td>
</tr>
<tr>
<td>peri</td>
<td>+</td>
</tr>
<tr>
<td>corona</td>
<td>(−)</td>
</tr>
<tr>
<td>peri + cata</td>
<td>+</td>
</tr>
<tr>
<td>corona + peri</td>
<td>(−)</td>
</tr>
<tr>
<td>corona + cata</td>
<td>(−)</td>
</tr>
<tr>
<td>corona + peri + cata</td>
<td>( )</td>
</tr>
</tbody>
</table>
| helicane     | + | (−)| (−)| + | (−)| (−)| (−)
Apart from the fundamental difference between cata- and corona-fusion subclasses, expressed by conditions 1 and 2, the corona-fusion subclass must satisfy conditions 4 and 5 to be differentiated from the mixed-type fusion cata + corona and cata + peri + corona subclasses.

The peri-fusion subclass is characterized by having vertices which are not included in the long HS-meshes, thereby contradicting condition 5. Mixed-type fusion peri + cata and peri + corona subclasses are distinguished from the fundamental peri-type by contradicting condition 6, while the fundamental peri-type subclass must satisfy condition 6.

A mixed cata + peri + corona fusion type is distinguished from all the fundamental and mixed fusion types by contradicting both conditions 4 and 5.

At the outset of the research that eventually led to the results which are presented here, the diversity of molecular structures modeled by the different subclasses of HS indicated difficulties in finding a concise definition for the whole class of HS. The initial idea was, therefore, to find characterizations for each of the eight HS subclasses separately, and to define the whole HS class as the class comprising the defined subclasses. After this was accomplished, it became also possible to formulate a combinatorial characterization for the whole class of HS as follows.

HS are two-connected planar graphs which are characterized by the following four invariant properties:
- HS vertex degrees take values from the set \{2, 3\}.
- HS can have only two types of meshes:
  (a) hexagonal meshes having length \(l_i = 6\),
  (b) long HS-meshes having length \(l_i = 2k, k = 5, 6, \ldots\).
- Each edge of a HS is included in at least one hexagonal mesh.
- A long HS-mesh of the HS may violate condition 7 only when conditions 1, 3 and 5 are simultaneously met.

4. Canonical coding of HS and other applications

Combinatorial concepts sufficient to characterize HS are intimately related to the nature of their structures. It is easy to visualize their applications in the solutions to other practical problems like coding, isomorphism testing, enumeration and generation of HS graphs. All these problems are complex. Space in this paper permits but a brief discussion.

A solution to the problem of the canonical coding can be found easily for the four fundamental subclasses of HS represented in Fig. 4 because the \(B\)-codes of their perimeters contain the complete information describing their structures. Breaking the cyclical structure of the perimeter's \(B\)-code at a characteristic point, and adopting the so obtained acyclic binary sequence as a canonical \(C\)-code, is a natural way to obtain a canonical code for them.

The simplest choices for the characteristic point are those which yield the mini-
mum or maximum value binary sequence for the \( C \)-code. The minimum value sequence seems to be a better choice because perimeters always contain six zeros more than ones. The probability that the \( B \)-code is going to contain a unique maximum-length subsequence is therefore greater for the consecutive zero – than for the consecutive one-subsequences. Because a unique maximum-length subsequence represents the solution to the problem of finding a characteristic point of the \( B \)-code and because the number of such subsequences in the \( B \)-code is going to be smaller, the algorithm for coding will be faster in applications if minimum-value binary sequences are adopted as \( C \)-codes.

As an example of the proposed minimum-value \( C \)-code, we take the \( C \)-code of the HS from Fig. 1, whose \( B \)-code was given in Section 2 in the form of a \( C \)-code. Further examples of \( C \)-codes are found in Fig. 4. The \( C \)-code for the HS from Fig. 4b is quite straightforward because its \( B \)-code contains the single 3-zero sequence, while all other zero-sequences are shorter. \( C \)-codes for the HS from Fig. 4a and 4c need a second pass through the \( B \)-code to find the appropriate point for breaking the \( B \)-code, while they both contain two four-zero sequences.

The order of complexity of this algorithm for the canonical coding of HS is \( O(n, \log n) \), because two linear searches of the \( B \)-code are sufficient for finding the \( C \)-code, and the \( B \)-code can be found using an \( O(n, \log n) \) algorithm.

Mixed-fusion subclasses of HS from Fig. 5, with the exception of the cata + peri subclass, present more difficult coding problems.

The algorithm to test the isomorphism of graphs from the HS subclasses represented in Fig. 4 becomes trivial when canonical \( C \)-codes for them are available. Considering that a simple comparison of the \( C \)-codes of two such HS determines whether they are isomorphic, it follows that they can be tested for isomorphism by using an \( O(n, \log n) \) algorithm.

HS can also be enumerated and generated without referring to their imbeddings in the hexagonal lattice. The necessary tool for the solution of these problems is Table 1. Permuting appropriate combinations of zero/one groups from Table 1 to produce the \( C \)-codes reduces this problem to two simpler problems:

- finding the set of appropriate combinations of groups from Table 1,
- permuting groups inside the combinations.

If Table 3 is used as an additional tool, straightforward enumeration and generation algorithms for separate subclasses of HS can be developed.

5. Conclusion

A characterization of molecular structures of the eight structurally different subclasses of the fused benzenoid hydrocarbons is given for the first time in terms of the invariant properties of a class of graphs which model these molecular structures. Differential characterization of the eight corresponding graph subclasses is based on seven conditions which are satisfied or contradicted in different combinations by the
graphs from different subclasses. The fundamental concept in this characterization is the set of meshes of a planar graph. Descriptions of specific molecular structures are obtained in the form of cyclical binary codes, which completely define corresponding graphs. The length of these codes is at most $n_v$, where $n_v$ denotes the number of vertices of the graph and is equal to the number of atoms in the molecule. Complete descriptions of molecules are readily obtained from these codes by incorporating the information on atoms that occupy different positions in the molecular structure.

Algorithms for the canonical coding of graphs from the different subclasses that model structurally simpler molecules are discussed. A simple $O(n_v \log n_v)$ algorithm capable of producing canonical codes for five of the eight subclasses is proposed.

Comparing the described canonical codes is the most efficient algorithm to test the isomorphism of graphs which belong to the class of hexagonal systems because such algorithms retain the low order of complexity of the algorithm for canonical coding.

Procedures for enumeration and generation of HS are also discussed. Reducing these problems to a pair of simpler problems is made possible by the concepts developed for characterization. The separate subclasses of these graphs can be enumerated or generated by selecting and permuting combinations of the ten fundamental binary sequences from which the codes of all hexagonal systems are constructed.

References