Chow groups of the moduli spaces of weighted pointed stable curves of genus zero

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Abstract

The moduli space $\overline{M}_A$ of weighted pointed stable curves of genus zero is stratified according to the degeneration types of such curves. We show that the homology groups of $\overline{M}_A$ are generated by the strata of $\overline{M}_A$ and give all additive relations between them. We also observe that the Chow groups $A_i\overline{M}_A$ and the homology groups $H_{2i}\overline{M}_A$ are isomorphic. This generalizes Kontsevich–Manin’s and Losev–Manin’s theorems to arbitrary weight data $A$.
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1. Introduction

The stability conditions have been used to compactify moduli spaces which have the desired properties. Families of moduli spaces with respect to stability parameters have recently attracted attention in various contexts such as weighted maps [1,2,16,17], weighted pointed curves [5,12,13,15], weighted configuration spaces, vector bundles and applications to representation theory [7,8], and triangulated categories [3].

Moduli space of pointed stable curves of genus $g$ has been extensively studied in the literature. In [5], Hassett enriched the pointed curves by assigning a weight to each marked point and studied the moduli stack of weighted pointed stable curves. In particular, he studied variations of the
compactifications of moduli spaces as well as the corresponding chambers of stability conditions and wall crossing phenomena.

The homology groups of the moduli spaces $\overline{M}_{A}$ of pointed weighted stable curves of genus zero corresponding to particular chambers of stability data play an essential role in the study of mirror symmetry, quantum cohomology and Frobenius manifolds. The weight data $A = (1, 1, \ldots, 1)$ give the well known moduli space of pointed stable curves $\overline{M}_{0,n}$ studied in [6,9–11,14], and the weight data $A = (1, 1, \varepsilon, \ldots, \varepsilon), 0 < \varepsilon \ll 1$ give the moduli space $L_{0,n}$ that has been studied by Losev and Manin in [12,13,15].

In this note, we give a presentation of the homology groups of the moduli space $\overline{M}_{A}$ for arbitrary weight data $A$. We show that the homology groups are generated by the cycles of the strata of $\overline{M}_{A}$, as in the cases of Kontsevich–Manin and Losev–Manin. The additive relations are obtained from the additive relations in the homology groups of $\overline{M}_{0,n}$ by using the reduction morphisms i.e., our description generalizes Kontsevich–Manin’s and Losev–Manin’s theorems which give additive structures on $H_{*} \overline{M}_{0,n}$ and $H_{*} L_{0,n}$ to arbitrary weight data $A$. Since the homology groups are generated by the strata of $\overline{M}_{A}$ and the relations are obtained from the relations in $H_{*}(\overline{M}_{0,n})$, the homology groups $H_{2i}(\overline{M}_{A})$ are isomorphic to the Chow groups $A_{i}(\overline{M}_{A})$. This result directly transfers the chamber decomposition and wall crossing phenomena to the homology level.

It is important to note that the technique used in this paper is significantly different than Keel’s calculation of $A_{*} \overline{M}_{0,n}$ in [6] and Mustaţă and Mustaţă’s technique in [16,17]. Instead of using birational morphisms and blow-up formulas for homology/Chow groups, we directly use the stratification of the moduli space $\overline{M}_{A}$ and the spectral sequence of forgetful morphisms. This technique has its own intrinsic power and may be used beyond the cases where blow-up formulas are applicable; for instance, it has been used in [4] to calculate the homology of the moduli space of real curves where blow-up formulas can only calculate homology in $\mathbb{Z}/2\mathbb{Z}$ coefficients.

In this paper, all varieties are considered over the field $\mathbb{C}$ except when the contrary is stated. Therefore, we usually omit mentioning the base field.

1.1. Plan of this paper

In Section 2, we review some basic facts on weighted pointed curves of genus zero and their moduli space $\overline{M}_{A}$. In the following section, we give the combinatorial stratification of $\overline{M}_{A}$. In Section 4, we consider restrictions of forgetful morphisms to the strata of $\overline{M}_{A}$ and study their fibers. We calculate the relative homology of the strata inductively by using forgetful morphisms. Finally, in Section 5, we give a combinatorial presentation of the homology and Chow groups of $\overline{M}_{A}$ and prove the statement inductively by using the spectral sequence of the forgetful morphisms.

2. Weighted pointed stable curves and their moduli

This section reviews the basic facts on the moduli problem of weighted pointed stable curves of genus zero.

2.1. The moduli problem

A family of nodal curves of genus zero with $n$ labeled points over $B$ consists of
• a flat proper morphism $\pi : C \to B$ whose geometric fibers $\Sigma$ are nodal connected curves of arithmetic genus zero; and
• a set of sections $s = (s_1, \ldots, s_n)$ of $\pi$.

A weight datum $\mathcal{A}$ is an element $(m_1, \ldots, m_n) \in \mathbb{Q}^n$ such that $0 < m_i \leq 1$ for $i = 1, \ldots, n$, and $m_1 + \cdots + m_n > 2$.

A family of nodal curves of genus zero with $n$ labeled points $\pi : C \to B$ is stable (with respect to $\mathcal{A}$) if

• the sections $s_1, \ldots, s_n$ lie in the smooth locus of $\pi$, and for any subset $\{s_{i_1}, \ldots, s_{i_r}\}$ with non-empty intersection we have $m_{i_1} + \cdots + m_{i_r} \leq 1$;
• $K_\pi + m_1 s_1 + \cdots + m_n s_n$ is $\pi$-relatively ample.

An $\mathcal{A}$-pointed curve is a fiber $(\Sigma; s)$ of a family which is stable with respect to $\mathcal{A}$.

**Theorem.** (See Hassett, [5].) For any $\mathcal{A}$ with $n \geq 3$, there exists a Deligne–Mumford stack $\mathcal{M}_\mathcal{A}$, smooth and proper over $\mathbb{Z}$, representing the moduli problem of $\mathcal{A}$-pointed curves. The corresponding moduli scheme $\bar{M}_\mathcal{A}$ is projective over $\mathbb{Z}$.

2.2. Natural transformations

Reduction morphism: Let $\mathcal{A} = (m_1, \ldots, m_n)$ and $\mathcal{A}' = (m'_1, \ldots, m'_n)$ be a pair of weight data such that $m'_i \leq m_i$ for all $i$. In [5], Hassett showed that there exists a natural birational reduction morphism

$$\rho_{\mathcal{A}, \mathcal{A}'} : \bar{M}_\mathcal{A} \to \bar{M}_{\mathcal{A}'}.$$  \hspace{1cm} (1)

The image $\rho_{\mathcal{A}, \mathcal{A}'}(\Sigma; s)$ of $(\Sigma; s) \in \bar{M}_\mathcal{A}$ is obtained by successively collapsing components of $\Sigma$ along which $K_\Sigma + m'_1 s_1 + \cdots + m'_n s_n$ fails to be ample.

Forgetful morphism: Let $\mathcal{A}$ be a weight datum and $\mathcal{B} = ((m_i)_{m_i \in \mathcal{A}}, m_{n+1})$. In [5], Hassett also showed that there exists a natural forgetful morphism

$$\pi_{\mathcal{B}, \mathcal{A}} : \bar{M}_\mathcal{B} \to \bar{M}_\mathcal{A}.$$ \hspace{1cm} (2)

The image $\pi_{\mathcal{B}, \mathcal{A}}(\Sigma; s)$ of $(\Sigma; s) \in \bar{M}_\mathcal{B}$ is obtained by forgetting the labeled point $s_{n+1}$ and successively collapsing components of $\Sigma$ along which $K_\Sigma + m_1 s_1 + \cdots + m_n s_n$ fails to be ample.

It is important to note that Hassett’s result is in fact much more general; he proved the same statements for arbitrary genus.

3. Stratification of the moduli space $\bar{M}_\mathcal{A}$

In this section, we first introduce the combinatorial structures encoding the degeneration types of $\mathcal{A}$-pointed curves. Then, we give a stratification of the moduli space $\bar{M}_\mathcal{A}$ in terms of these combinatorial data.
3.1. Combinatorial types of weighted pointed curves

3.1.1. Graphs

A graph $\Gamma$ is a collection of finite sets of vertices $V_\Gamma$ and flags (or half edges) $F_\Gamma$ with a boundary map $\partial_\Gamma : F_\Gamma \to V_\Gamma$ and an involution $j_\Gamma : F_\Gamma \to F_\Gamma$ ($j_\Gamma^2 = \text{id}$). We call $E_\Gamma = \{(f_1, f_2) \in F_\Gamma^2 \mid f_1 = j_\Gamma f_2$ and $f_1 \neq f_2\}$ the set of edges, and $T_\Gamma = \{f \in F_\Gamma \mid f = j_\Gamma f\}$ the set of tails. For a vertex $v \in V_\Gamma$, let $F_\Gamma(v) = \partial_\Gamma^{-1}(v)$ and $|v| = |F_\Gamma(v)|$ be the valency of $v$.

A weighted graph is a graph $\Gamma$ endowed with a map $A_\Gamma : F_\Gamma \to \mathbb{Q} \cap [0, 1]$ such that $A_\Gamma(f) = 1$ for all flags that are part of an edge i.e., for which $j_\Gamma(f) \neq f$.

We think of a graph $\Gamma$ in terms of its geometric realization $\|\Gamma\|$: Consider the disjoint union of closed intervals $\bigcup_{f_i \in F_\Gamma} [0, 1] \times f_i$ and identify $(0, f_i)$ with $(0, f_j)$ if $\partial_\Gamma f_i = \partial_\Gamma f_j$, and identify $(t, f_i)$ with $(1 - t, j_\Gamma f_i)$ for $t \in [0, 1]$ and $f_i \neq f_j$. The geometric realization of $\Gamma$ has a piecewise linear structure.

A (weighted) tree is a (weighted) graph whose geometric realization is connected and simply-connected.

3.1.2. $A$-trees

Let $\gamma$ be a weighted tree. An $r$-structure is a function on $V_\gamma$ associating to each vertex $v \in V_\gamma$ an equivalence relation $\approx$ on the set of tails $T_\gamma(v)$ such that $\sum_{f_i \in [f]} A(f) \leq 1$ for each equivalence class $[f]$.

An $A$-tree is a weighted tree $\gamma$ that carries an $r$-structure and satisfies $\sum_{f \in F_\gamma(v)} A(f) > 2$ for each $v \in V_\gamma$.

We denote $A$-trees by $(\gamma, r)$, $(\tau, r)$ or by bold Greek characters $\gamma$, $\tau$. When it is necessary to indicate different $r$-structures on the same weighted tree, we use indices in parentheses (e.g., $\gamma_{(i)}$).

The $r$-structure of an $A$-tree $\gamma$ determines the weight structure

$$A_v = ((m_{[f]})_{[f] \in O_{r(v)}}, (1)_{f \in E_\gamma(v)}) \quad \text{for } \forall v \in V_\gamma$$

where $O_{r(v)}$ is the set of equivalence classes of $r(v)$, and $m_{[f]} = \sum_{f_i \in [f]} m_{f_i}$.

3.1.3. Morphisms of graphs

Let $\gamma$ and $\tau$ be trees with $n$ tails. A morphism between these trees $\phi : \gamma \to \tau$ is a pair of maps $\phi_F : F_\tau \to F_\gamma$ and $\phi_V : V_\gamma \to V_\tau$ satisfying the following conditions:

- $\phi_F$ is injective and $\phi_V$ is surjective.
- The following diagram commutes

$$\begin{array}{ccc}
F_\gamma & \xrightarrow{\partial_\gamma} & V_\gamma \\
\phi_F \uparrow & & \downarrow \phi_V \\
F_\tau & \xrightarrow{\partial_\tau} & V_\tau.
\end{array}$$

- $\phi_F \circ j_\tau = j_\gamma \circ \phi_F$.
- $\phi_T := \phi_F|_T$ is a bijection.
Each morphism induces a piecewise linear map on geometric realizations. An isomorphism $\phi: \gamma \to \tau$ is a morphism where $\phi_F$ and $\phi_V$ are bijections.

### 3.1.4. Dual trees of weighted pointed curves

Let $(\Sigma; s)$ be an $A$-pointed curve and $\eta: \hat{\Sigma} \to \Sigma$ be its normalization. Let $(\hat{\Sigma}_v; \hat{s}_v)$ be the following pointed curve: $\hat{\Sigma}_v$ is a component of $\hat{\Sigma}$, and $\hat{s}_v$ is the set of points consisting of the preimages of special (i.e., labeled and nodal) points on $\Sigma_v := \eta(\hat{\Sigma}_v)$. The points $\hat{s}_v = (s_{f_1}, \ldots, s_{f_{|v|}})$ on $\hat{\Sigma}_v$ are ordered by the elements $f_*$ in the set $\{f_1, \ldots, f_{|v|}\}$.

The dual tree of an $A$-pointed curve $(\Sigma; s)$ is an $A$-tree $\gamma$ consisting of following data:

- $V_\gamma$ is the set of components of $\hat{\Sigma}$.
- $F_\gamma$ is the set consisting of the preimages of special points.
- $\partial_\gamma : f \mapsto v$ if and only if $s_f \in \hat{\Sigma}_v$.
- $j_\gamma : f \mapsto f$ if and only if $s_f$ is a labeled point, and $j_\gamma : f_1 \mapsto f_2$ if and only if $s_{f_1} \in \hat{\Sigma}_{v_1}$ and $s_{f_2} \in \hat{\Sigma}_{v_2}$ are the preimages of the nodal point $\Sigma_{v_1} \cap \Sigma_{v_2}$.
- $A_\gamma(f) = m_f$ if $f \in T_\gamma$, and $A_\gamma(f) = 1$ if $f \in F_\gamma \setminus T_\gamma$.
- A pair of tails $f_1, f_2 \in T_\gamma(v)$ are equivalent if and only if $s_{f_1} = s_{f_2}$.

### 3.2. Combinatorics of degenerations

The degenerations of $A$-pointed curves are encoded by the morphisms of $A$-trees as follows.

#### 3.2.1. Contractions of edges

Let $(\Sigma; s)$ be an $A$-pointed curve and let $\gamma = (\gamma, \tau)$ be its dual tree. Consider the deformation of a nodal point of $(\Sigma; s)$. Such a deformation of $(\Sigma; s)$ gives a contraction of an edge $\gamma \mapsto \gamma/e$: Let $e = (f_e, f^e) \in E_\gamma$ be the edge corresponding to the nodal point and $\partial_\gamma(e) = \{v_e, v^e\}$, and consider the equivalence relation $\sim$ on the set of vertices, defined by: $v \sim v$ for all $v \in V_\gamma \setminus \{v_e, v^e\}$, and $v_e \sim v^e$. Then, there is a weighted tree $\gamma/e$ whose vertices are $V_\gamma/\sim$ and whose flags are $F_\gamma/\sim$ and whose special points are $\{f_e, f^e\}$. The involution, boundary and weight maps of $\gamma/e$ are the restrictions of $j_\gamma, \partial_\gamma$ and $A_\gamma$. The $\tau$-structure is the same as before the contraction.

#### 3.2.2. Identifications of tails

Let $C \to B$ be a family $A$-pointed curves whose dual tree is $\tau = (\tau, \tau_{old})$. Let $f_{i_1}, \ldots, f_{i_r}$ be the tails corresponding to a set of labeled points supported by the same component $\Sigma_v$ with $m_{i_1} + \cdots + m_{i_r} \leq 1$. Consider the limits of this family where corresponding sections $s_{f_{i_1}}, \ldots, s_{f_{i_r}}$ intersect. Such a degeneration gives a new $\tau$-structure $\tau_{new}$ on the set of tails of $\tau$: The new equivalence relation $\approx_{new}$ is given by using the old one $\approx_{old}$ as follows

- $f_i \approx_{new} f_j$ iff $f_i \approx_{old} f_j$ for each pair $f_i, f_j \in T_\tau$, and
- $f_{i_k} \approx_{new} f_{i_l}$ for all $f_{i_k}, f_{i_l} \in \{f_{i_1}, \ldots, f_{i_r}\}$.

We use the notation $\gamma < \tau$ to indicate that either $\tau$ is obtained by contracting some edges of $\gamma$, or $\gamma$ is obtained by identifying a set of tails of $\tau$. 

3.3. Stratification of the moduli space $\overline{M}_A$

The stratification of $\overline{M}_A$ according to the degeneration types of its elements is a direct consequence of Hassett’s theorem. The following statement is almost a tautology due to the definition of dual trees of $A$-pointed curves.

**Proposition 3.1.**

1. For any $A$-tree $\gamma$, there exists a quasi-projective subvariety $D_{\gamma} \subset \overline{M}_A$ of codimension $|E_{\gamma}| + (n - \sum_{v} |O_{\tau(v)}|)$ parameterizing $A$-pointed curves whose dual tree is $\gamma$. The subvariety $D_{\gamma}$ is isomorphic to $\prod_{v \in \gamma} M_{A_v}$ where $A_v$ is the weight structure at vertex $v$.

2. $\overline{M}_A$ is stratified by pairwise disjoint subvarieties $D_{\gamma}$. The closure $\overline{D}_{\gamma}$ of any stratum $D_{\gamma}$ is stratified by $\{ D_{\gamma'} \mid \gamma' \leq \gamma \}$.

**Example 3.2.** Consider the case $n = 4$. For any $A$ with $n = 4$, the principal stratum $M_A$ is isomorphic to $\mathbb{P}^1 \setminus \{0, 1, \infty\}$. Moreover, its compactification $\overline{M}_A$ is isomorphic to $\mathbb{P}^1$ obtained by adding three boundary divisors. However, the stratification is encoded by different sets of $A$-trees.

Fig. 1 depicts the $A$-trees used in the combinatorial stratification of the three separate cases:

(a) $A = (1, 1, 1, 1)$ i.e., $\overline{M}_A = \overline{M}_{0,4}$; (b) $A = (1, 1, \varepsilon, \varepsilon)$ where $0 < \varepsilon \ll 1$ i.e., $\overline{M}_A = L_{0,4}$; (c) $A = (1, \varepsilon, \varepsilon, \varepsilon)$ where $1/3 < \varepsilon \leq 1/2$. In this figure, the dotted red tails depict the tails lying in the same equivalence class.

4. Homology of the strata of $\overline{M}_A$

In this chapter, we calculate the homology of the strata of $\overline{M}_A$ relative to the union their substrata of codimension one and higher.

4.1. Fibers of the forgetful morphism

Let $\mathcal{A}$ be a weight datum and $B = ((m_i)_{i \in \mathcal{A}}, m_{n+1})$. Let $\pi : D_{\gamma^*} \to D_{\gamma}$ be the restriction of the morphism $\pi_{B, \mathcal{A}} : \overline{M}_B \to \overline{M}_A$ which forgets the labeled point $s_{n+1}$. Let $\phi^{s_{n+1}} : \gamma^* \to \gamma$
be the corresponding forgetful morphism of trees, and let \( v_\delta = \partial_{\gamma'}(s_{n+1}) \). In order to avoid the trivial cases here, we assume that \( s_{n+1} \neq s_f \) for all \( f \in T_{\gamma'}(v_\delta) \).

We will denote the fibers \( \pi^{-1}(\Sigma; s) \) of the forgetful morphism \( \pi \) simply by \( F_{\gamma'} \).

**Lemma 4.1.** Let \( (\Sigma; s) \in D_{\gamma} \). Then, the fiber \( F_{\gamma'} \) is

1. a projective line \( \mathbb{P}^1 \) minus the special points \( s_f \) where \( f \in F_{\gamma'}(v_\delta) \setminus \{ s_{n+1} \} \) if \( \sum_{f \in F_{\gamma'}(v_\delta) \setminus \{ s_{n+1} \}} B(f) > 2 \);
2. the product \( \prod_{v \in V_{\gamma'}}(\phi_v^{n+1} - 1)(\gamma) M_{B_v} \) if \( \sum_{f \in F_{\gamma'}(v_\delta) \setminus \{ s_{n+1} \}} B(f) \leq 2 \).

**Proof.** Pick a curve \( (\Sigma; s) \in D_{\gamma} \). Let \( (\Sigma^*, s^*) \) be points in the fiber \( F_{\gamma'} \).

(1) If \( (\Sigma^*, s^*) \in D_{\gamma'} \) does not require the contraction of its component \( \Sigma_v^* \) after forgetting the labeled point \( s_{n+1} \), then the special points on \( \Sigma_v^* \) must satisfy the inequality \( \sum_{f \in F_{\gamma'}(v_\delta) \setminus \{ s_{n+1} \}} B(f) > 2 \). For all such \( (\Sigma^*, s^*) \), the curve \( \Sigma_v^* \) is isomorphic to \( \Sigma_v^* \) (and \( \Sigma_v^* \) is isomorphic to \( \mathbb{P}^1 \)). All special points are fixed (by choosing \( (\Sigma^*, s^*) \)) except \( s_{n+1} \). The labeled point \( s_{n+1} \) can be any point in \( \Sigma_v^* \setminus \{ s_f \mid f \in F_{\gamma'}(v_\delta) \setminus \{ s_{n+1} \} \} \). If we assume that \( s_f \neq s_{n+1} \) for \( f \in F_{\gamma'}(v_\delta) \setminus \{ s_{n+1} \} \), the special points supported by \( \Sigma_v^* \) can realize all possible configurations. Therefore, the fiber \( F_{\gamma'} \) is \( \prod_{v} M_{B_v} \) where the product runs over all contracted components.

4.2. Homology of the two-dimensional fibers of the forgetful morphisms

Let \( \pi : D_{\gamma'} \to D_{\gamma} \) be the map forgetting the labeled point \( s_{n+1} \) which is discussed above. Assume that \( \sum_{f \in F_{\gamma'}(v_\delta) \setminus \{ s_{n+1} \}} A(f) > 2 \) i.e., the fibers are projective lines with punctures (see case (1) of Lemma 4.1).

Let us identify the fiber \( F_{\gamma'} \) with \( \mathbb{P}^1 \setminus s \), and let \( z = [z : 0] \) be an affine coordinate on it. Let \( z_f \) denote the positions of special points \( s_f \) in these coordinates.

Then, the cohomology of a fiber is generated by the logarithmic differentials

\[
H^0(F_{\gamma'}) = \mathbb{Z}, \quad H^1(F_{\gamma'}) = \bigoplus_f \mathbb{Z} \omega_{f,s_{n+1}}
\]

where

\[
\omega_{f,s_{n+1}} = \frac{1}{2\pi \sqrt{-1}} d \log(z - z_f)
\]

for \( f \in F_{\gamma'}(v_\delta) \setminus \{ s_{n+1} \} \).

The homology with closed support \( H^i_1(F_{\gamma'}) \) is isomorphic to the cohomology group \( H^1_{\text{closed}}(F_{\gamma'}) \) via Poincaré duality. Moreover, the definition of homology with closed support implies that \( H^1_1(F_{\gamma'}) \) is isomorphic to the relative homology group \( H_1(\overline{F}_{\gamma'}, s) \) of the closure of the fiber. The group \( H_1(\overline{F}_{\gamma'}, s) \) is clearly generated by the homotopy classes of arcs connecting the pairs of punctures \( s_{f_1}, s_{f_2} \). These arcs are the duals of the cohomology classes \( \omega_{f_1,s_{n+1}} - \omega_{f_2,s_{n+1}} \). We
denote them by $\mathcal{R}_{s_{n+1}, f_1 f_2}$. The homology groups $H_2^c(F_{\mathcal{Y}^*})$ and $H_2(\overline{F}_{\mathcal{Y}^*}, s)$ are isomorphic to $H^0(F_{\mathcal{Y}^*}) = \mathbb{Z}$. Hence

$$H_2(\overline{F}_{\mathcal{Y}^*}, s) = \mathbb{Z}, \quad H_1(\overline{F}_{\mathcal{Y}^*}, s) = \left( \bigoplus_f \mathbb{Z} \mathcal{R}_{s_{n+1}, f_1 f_2} \right) / \mathcal{I}_{\mathcal{Y}^*},$$

where $f_i \in F_{\mathcal{Y}^*}(v_i) \setminus \{s_{n+1}\}$, and the subgroup of relations $\mathcal{I}_{\mathcal{Y}^*}$ is generated by

$$\mathcal{R}_{s_{n+1}, f_1 f_2} + \mathcal{R}_{s_{n+1}, f_2 f_3} + \mathcal{R}_{s_{n+1}, f_3 f_1}. \quad (4)$$

4.3. Homology of the strata

In this section, we give the generators of the homology of the closed strata $D_{\mathcal{Y}}$ relative to the union of their lower-dimensional strata $Q_{\mathcal{Y}} := \bigcup_{\tau < \gamma} D_{\tau}$.

**Lemma 4.2.** Let $\pi : D_{\mathcal{Y}^*} \to D_{\mathcal{Y}}$ be the forgetful morphism discussed Section 4.1. Then,

$$H^0_c(D_{\mathcal{Y}^*}; \mathbb{Z}) = \bigoplus_{p+q=d} H^p_c(D_{\mathcal{Y}}; \mathbb{Z}) \otimes H^q_c(F_{\mathcal{Y}^*}; \mathbb{Z}).$$

**Proof.** The strata $D_{\mathcal{Y}^*}$ and $D_{\mathcal{Y}}$ are given by the products

$$\prod_{v \in \mathcal{V}_{\mathcal{Y}^*}} M_{B_v}, \quad \prod_{v \in \mathcal{V}_{\mathcal{Y}}} M_{A_v},$$

due to Proposition 3.1. The forgetful map $\pi$ preserves the components $(\Sigma^*, \mathbf{p}^*)$ of $(\Sigma^*, \mathbf{p}^*) \in D_{\mathcal{Y}^*}$ except the contracted ones. Hence, it gives the identity map on the factors $M_{B_v} \to M_{A_v}$ for $v \in (\phi_{n+1}^V)^{-1}(\mathcal{V}_{\mathcal{Y}})$. On the other hand, it gives a fibration

$$\pi_{\text{res}} : \prod_{v \in \mathcal{V}_{\mathcal{Y}} \setminus (\phi_{n+1}^V)^{-1}(\mathcal{V}_{\mathcal{Y}})} M_{B_v} \to \text{point} \quad \text{when} \quad \sum_{f \in F_{\mathcal{Y}^*}(v_i) \setminus \{s_{n+1}\}} B(f) \leq 2, \quad (5)$$

$$\pi_{\text{res}} : M_{B_v} \to M_{A_v} \quad \text{when} \quad \sum_{f \in F_{\mathcal{Y}^*}(v_i) \setminus \{s_{n+1}\}} B(f) > 2, \quad (6)$$

with the same fibers $F_{\mathcal{Y}^*}$ of $\pi : D_{\mathcal{Y}^*} \to D_{\mathcal{Y}}$.

In the case of $\sum_{f \in F_{\mathcal{Y}^*}(v_i) \setminus \{s_{n+1}\}} B(f) \leq 2$, the stratum $D_{\mathcal{Y}^*}$ is clearly $D_{\mathcal{Y}} \times F_{\mathcal{Y}^*}$. Hence, the claim follows from the Künneth formula.

In the case of $\sum_{f \in F_{\mathcal{Y}^*}(v_i) \setminus \{s_{n+1}\}} B(f) > 2$, the spaces $M_{B_v}$ and $M_{A_v}$ are diffeomorphic to the products of $\mathbb{P}^1 \setminus \{s_1, s_2, s_3\}$ minus all diagonals. The map $\pi_{\text{res}}$ forgets the coordinate subspace $\mathbb{P}^1$ corresponding to the labeled point $s_{n+1}$ i.e., it is

$$(\mathbb{P}^1 \setminus \{s_1, s_2, s_3\})^{\mathbb{P}^1_-} \setminus \Delta \to (\mathbb{P}^1 \setminus \{s_1, s_2, s_3\})^{\mathbb{P}^1_-} \setminus \Delta.$$
The logarithmic differentials \( d \log(z - z_f) \) give global cohomology classes on \( M_{B_{vs}} \). On the other hand, we have seen that the restriction of these logarithmic forms to each fiber generate the cohomology of the fiber (see Section 4.2). By using the Leray–Hirsch theorem, we obtain

\[
H^d(D_{\gamma^*}) = \bigoplus_{p+q=d} H^p(D_{\gamma}) \otimes H^q(F_{\gamma^*}).
\]

The duality between cohomology and homology with closed supports gives us isomorphisms which are needed to complete the proof.

Since the strata of \( \overline{M_A} \) are the products given in Proposition 3.1, their homology groups are obtained from the homology of their factors \( \overline{M}_{A_i} \) by using Künneth formula. Here, we give only the relative homology for the strata of one-vertex trees:

Now, let \( \gamma \) be a one-vertex \( A \)-tree, and let \( Q_{\gamma} \) be the union of the codimension one and higher strata of the closed stratum \( \overline{D}_{\gamma} \).

**Proposition 4.3.** The relative homology group \( H_{\dim(D_{\gamma})-d}(\overline{D}_{\gamma}, Q_{\gamma}; \mathbb{Z}) \) is generated by

\[
\mathcal{R}_{s_{j_1}, j_1, s_{k_1}} \otimes \cdots \otimes \mathcal{R}_{s_{j_d}, j_d, s_{k_d}}
\]

where \( j_n, k_n < i_n \) and \( i_1 < \cdots < i_d < n \). In particular,

\[
H_{\dim(D_{\gamma})}(\overline{D}_{\gamma}, Q_{\gamma}; \mathbb{Z}) = \mathbb{Z}\langle \overline{D}_{\gamma} \rangle
\]

where \( \overline{D}_{\gamma} \) is the fundamental class of the stratum \( D_{\gamma} \).

**Proof.** The homology with closed support is defined by

\[
H^c_n(D_{\gamma}) = \lim_{\longrightarrow} H_n(D_{\gamma}, D_{\gamma} \setminus K)
\]

where \( K \) ranges over all closed subsets of \( D_{\gamma} \). The group \( H_n(D_{\gamma}, D_{\gamma} \setminus K) \) is isomorphic to \( H_n(\overline{D}_{\gamma}, \overline{D}_{\gamma} \setminus \overline{K}) \) where \( \overline{K} \) ranges over all closed subsets of \( \overline{D}_{\gamma} \) which do not intersect \( Q_{\gamma} \). In the limit, \( \overline{D}_{\gamma} \setminus \overline{K} \) gives the union of substrata \( Q_{\gamma} \). Hence, the homology with closed support is isomorphic to the relative homology of \( \overline{D}_{\gamma} \).

On the other hand, Lemma 4.2 implies that

\[
H_d(\overline{D}_{\gamma^*}, Q_{\gamma^*}; \mathbb{Z}) = \bigoplus_{p+q=d} H_p(\overline{D}_{\gamma}, Q_{\gamma}; \mathbb{Z}) \otimes H_q(F_{\gamma^*}, s; \mathbb{Z}),
\]

where \( F_{\gamma^*} \) is the fiber of the map \( \pi : D_{\gamma^*} \to D_{\gamma} \) which forgets \( s_{n+1} \).

We obtain the result by applying the forgetful morphism successively and using the generators of the relative homologies the fibers given in Section 4.2. In order to simplify the notation, we omit the factors coming from the generators of the second homology of the fibers.

It is clear that the top-dimensional relative homology is generated by the relative fundamental class \( \overline{D}_{\gamma} \). \( \square \)
5. Homology groups of $\overline{M}_A$

In this section, we give the homology groups of the moduli space $\overline{M}_A$ in terms of generators and relations.

5.1. Relations between strata of $\overline{M}_A$

Here, we introduce a set of interesting relations between the strata of $\overline{M}_A$ which will play a crucial role in our description of the homology of $\overline{M}_A$.

Consider an $A$-tree $\mathbf{y}$ such that $\dim D_{\mathbf{y}} = d + 1$, and a vertex $v \in V_{\mathbf{y}}$ with $|O_{\mathbf{y}}(v)| \geq 4$. Let $f_1, f_2, f_3, f_4 \in F_{\mathbf{y}}(v)$ and let their equivalence classes be pairwise disjoint. Put $F = F_{\mathbf{y}}(v) \setminus \{f_1, f_2, f_3, f_4\}$ and let $F_1, F_2$ be a partition of $F$ such that $F_1 \cap \{f_1, f_2, f_3, f_4\}$ is either empty or equal to $\{f_1, f_2, f_3, f_4\}$ for all $f \in F$.

We define two $A$-trees $\mathbf{y}_1, \mathbf{y}_2$ such that $\dim D_{\mathbf{y}_i} = d$:

**The $A$-tree $\mathbf{y}_1$**: In order to define $\mathbf{y}_1$, we first introduce a weighted tree $\hat{\gamma}_1$. It is obtained by inserting a new edge $e = (f_e, f^e)$ into $\mathbf{y}$ at $v$ and its flags are given as follows: Let $\delta_{\hat{\gamma}_1}(e) = \{v_e, v^e\}$. The distribution of flags is given by $F_{\hat{\gamma}_1}(v_e) = F_1 \cup \{f_1\} \cup \{f_2\} \cup \{f_e\}$ and $F_{\hat{\gamma}_1}(v^e) = F_2 \cup \{f_3\} \cup \{f_4\} \cup \{f^e\}$.

Consider the following $r$-structure on $\hat{\gamma}_1$: Let the equivalence relations on $T_{\hat{\gamma}_1}(v_e), T_{\hat{\gamma}_1}(v^e)$ be the restrictions of the equivalence relation on $T_{\mathbf{y}}(v)$ and the equivalence relations for $T_{\hat{\gamma}_1}(v) = T_{\mathbf{y}}(v)$ for $v \in V_{\hat{\gamma}_1} \setminus \{v_e, v^e\}$ remain the same as before. Denote this $r$-structure by $r_1$. Then, we define the $A$-tree $\mathbf{y}_1$ in two separate cases:

- **Stable case**: If $\hat{\gamma}_1$ and the $r$-structure $r_1$ provide an $A$-tree (i.e., $\sum_{f \in F_{\hat{\gamma}_1}(v_e)} A(f) > 2$ for $v \in \{v_e, v^e\}$), then we put $\mathbf{y}_1 := (\hat{\gamma}_1, r_1)$.
- **Unstable case**: If $\hat{\gamma}_1$ and the $r$-structure $r_1$ do not provide an $A$-tree (i.e., $\sum_{f \in F_{\hat{\gamma}_1}(v_e)} A(f) \leq 2$ or $\sum_{f \in F_{\hat{\gamma}_1}(v^e)} A(f) \leq 2$), then $\mathbf{y}_1$ is obtained from $\mathbf{y}$ by identifying the tails in $\{f_1\}, \{f_2\}$ or $\{f_3\}, \{f_4\}$ whichever are adjacent to the unstable vertex (i.e., the one satisfying the weight inequality $\sum_{f \in F_{\hat{\gamma}_1}(v)} A(f) \leq 2$) in $\hat{\gamma}_1$.

**The $A$-tree $\mathbf{y}_2$**: The $A$-tree $\mathbf{y}_2$ is obtained the same as $\mathbf{y}_1$ after swapping $f_2$ and $f_3$: We first introduce a weighted tree $\hat{\gamma}_2$. It is obtained by inserting a new edge $e = (f_e, f^e)$ into $\mathbf{y}$ at $v$ and its flags are given as follows: Let $\delta_{\hat{\gamma}_2}(e) = \{v_e, v^e\}$. The distribution of flags is given by $F_{\hat{\gamma}_2}(v_e) = F_1 \cup \{f_1\} \cup \{f_3\} \cup \{f_e\}$ and $F_{\hat{\gamma}_2}(v^e) = F_2 \cup \{f_2\} \cup \{f_4\} \cup \{f^e\}$.

Consider the following $r$-structure on $\hat{\gamma}_2$: Let the equivalence relations on $T_{\hat{\gamma}_2}(v_e), T_{\hat{\gamma}_2}(v^e)$ be the restrictions of the equivalence relation on $T_{\mathbf{y}}(v)$, and the equivalence relations for $T_{\hat{\gamma}_2}(v) = T_{\mathbf{y}}(v)$ for $v \in V_{\hat{\gamma}_2} \setminus \{v_e, v^e\}$ remain the same as before. Denote this $r$-structure by $r_2$. Then, we define the $A$-tree $\mathbf{y}_2$ in two separate cases:

- **Stable case**: If $\hat{\gamma}_2$ and the $r$-structure $r_2$ provide an $A$-tree (i.e., $\sum_{f \in F_{\hat{\gamma}_2}(v_e)} A(f) > 2$ for $v \in \{v_e, v^e\}$), then we put $\mathbf{y}_2 := (\hat{\gamma}_2, r_2)$.
- **Unstable case**: If $\hat{\gamma}_2$ and the $r$-structure $r_2$ do not provide an $A$-tree (i.e., $\sum_{f \in F_{\hat{\gamma}_2}(v_e)} A(f) \leq 2$ or $\sum_{f \in F_{\hat{\gamma}_2}(v^e)} A(f) \leq 2$), then $\mathbf{y}_2$ is obtained from $\mathbf{y}$ by identifying the tails in $\{f_1\}, \{f_3\}$ or $\{f_2\}, \{f_4\}$ whichever are adjacent to the unstable vertex in $\hat{\gamma}_2$.  


Principal relations between strata: We define $d$-dimensional class

$$\mathcal{R}(\mathbf{y}; v, f_1, f_2, f_3, f_4) := \sum_{\gamma_1} [\overline{D}_{\gamma_1}] - \sum_{\gamma_2} [\overline{D}_{\gamma_2}]$$  \hspace*{1cm} (7)$$

where sum is taken over the isomorphism classes of all possible $\gamma_1$'s and $\gamma_2$'s for a fixed set of flags $\{f_1, f_2, f_3, f_4\}$.

**Lemma 5.1.** $\mathcal{R}(\mathbf{y}; v, f_1, f_2, f_3, f_4)$ is rationally equivalent to zero.

**Remark 5.2.** If we consider only the fully stable cases (i.e., $A(f_1) + A(f_2) > 1$, $A(f_3) + A(f_4) > 1$, $A(f_1) + A(f_3) > 1$ and $A(f_2) + A(f_3) > 1$), such as the cases where $A_1 = (1, 1, \ldots, 1)$ or where all of the $f_i$ are parts of edges, the homology relations given in Lemma 5.1 reduce to the additive relations in $H_*(\overline{M}_{0,n})$ given by Kontsevich and Manin in [10]. They appear as a consequence of Keel's work [6]. In [10], Kontsevich and Manin proved that they in fact generate all additive relations in $H_*(\overline{M}_{0,n})$.

**Proof of Lemma 5.1.** All additive relations in the Chow group $A_d(\overline{M}_{0,n})$ are generated by

$$R(\mathbf{y}; v, f_1, f_2, f_3, f_4) := \sum_{f_1,f_2} [\overline{D}_{\gamma_1}] - \sum_{f_1,f_2} [\overline{D}_{\gamma_2}] \equiv 0$$  \hspace*{1cm} (8)$$

where $\gamma_i, i = 1, 2$ are $A_1$-trees and where $A_1 = (1, 1, \ldots, 1)$ (see, for example [14]). The notation $f_1,f_2 \gamma f_3,f_4$ indicates the flags $f_1,f_2$ and $f_3,f_4$ are adjacent to two vertices $\{v,v'\} = \partial \gamma(e)$ of $\gamma$.

Consider the reduction morphism $\rho_{A_1; A_0} : \overline{M}_{0,|\Gamma_\mathbf{y}(v)|} \rightarrow \overline{M}_{0,n}$ for the vertex $v$ of the $A$-tree $\mathbf{y}$. The push-forward $(\rho_{A_1; A_0}, *) R(\mathbf{y}; v, f_1, f_2, f_3, f_4)$ gives us the relations $\mathcal{R}(\mathbf{y}; v, f_1, f_2, f_3, f_4) \equiv 0$. \hspace*{1cm} \Box

5.2. Homology groups of $\overline{M}_A$

**Theorem 1.** The homology groups of $\overline{M}_A$ are

$$H_{2d}(\overline{M}_A) = \left( \bigoplus_{\gamma : \dim \gamma = 2d} H_{2d}(\overline{D}_\gamma, Q_\gamma) \right) / \mathcal{I}_d,$$

$$H_{2d-1}(\overline{M}_A) = 0$$

where the subgroup of relations $\mathcal{I}_d$ is generated by $\mathcal{R}(\mathbf{y}; v, f_1, f_2, f_3, f_4)$, see (7).

**Proof.** First, we note that the statement directly follows when $n = 4$. In this case, the main stratum is $\mathbb{P}^1 \setminus \{s_1, s_2, s_3\}$ and the classes of codimension one strata are the boundary divisors $s_i$ which are pairwise rationally equivalent. The relation between the pairs of classes of codimension one strata is given by (7).

We prove the statement for $n > 4$ by induction on $n$.

Let $\pi : \overline{M}_B \rightarrow \overline{M}_A$ be the map forgetting the labeled point $s_{n+1}$. Here, we use the notations introduced in Section 4.1.
Let \( B_d \) denote the union of \( d \)-dimensional (closed) strata of \( \overline{M}_A \). Let \( \overline{M}_A \) be filtered by \( \emptyset = B_{-1} \subset B_0 \subset \cdots \subset B_{2n-6} = \overline{M}_A \).

The forgetful map \( \pi \) induces a filtration of \( \overline{M}_B \):

\[
\emptyset = E_{-1} \subset E_0 \subset \cdots \subset E_{2n-6} = \overline{M}_B
\]

where \( E_d = \pi^{-1}(B_d) \). Then, the spectral sequence obtained from this filtration gives us

\[
E_1^{p,q} = H_{p+q}(E_p, E_{p-1}) \implies H_{p+q}(\overline{M}_B; \mathbb{Z}).
\]

We prove the theorem by writing down this spectral sequence explicitly. As a first step, we calculate the homology groups \( H_{p+q}(E_p, E_{p-1}) \).

From now on, we assume that the statement of the theorem holds for \( \overline{M}_A \).

**Step 1.** Since all strata are even-dimensional, we only need to consider \( (E_{2p}, E_{2p-1}) \) pairs. We can write homology of \( (E_{2p}, E_{2p-1}) \) as a direct sum of the homology of its pieces:

\[
H_{2p+q}(E_{2p}, E_{2p-1}) = \bigoplus_{\gamma: \dim_{\mathbb{R}} D_\gamma = 2p} H_{2p+q}(\pi^{-1}(\overline{D}_\gamma), \pi^{-1}(Q_{\gamma})).
\]

Assume that the maximum of the dimension of the fibers \( F_{\gamma^*} \) of \( \pi: D_{\gamma^*} \to D_{\gamma} \) is 2k. Consider the following filtration of \( \pi^{-1}(\overline{D}_\gamma) \):

\[
\emptyset \subset Y_0 \subset Y_1 \subset \cdots \subset Y_{2k} = \pi^{-1}(\overline{D}_\gamma)
\]

where \( Y_i \) is the union of the strata \( \overline{D}_{\gamma^*} \) of \( \overline{M}_B \) of dimension 2p + i such that \( \pi(D_{\gamma^*}) = D_{\gamma} \). Clearly, \( Y_{2i} = Y_{2i+1} \) due to the absence of odd-dimensional strata.

The spectral sequence of this filtration gives

\[
\mathbb{Y}_{i,j}^1 = H_{i+j}(Y_i, Y_{i-1} \cup (Y_i \cap \pi^{-1}(Q_{\gamma}))) \implies H_{i+j}(E_{2p}, E_{2p-1}).
\]

Here, \( Y_i \cap \pi^{-1}(Q_{\gamma}) \) contains the substrata of \( \overline{D}_{\gamma^*} \) that maps to \( B_{p-1} \) (i.e., substrata of codimension one or higher in \( \overline{D}_{\gamma^*} \)). Hence, we have

\[
\mathbb{Y}_{2i,j}^1 = \bigoplus_{\gamma^*} H_{2i+j}(\overline{D}_{\gamma^*}, Q_{\gamma^*}), \quad \mathbb{Y}_{2i+1,j}^1 = 0.
\]

By using Lemma 4.2 (and isomorphism between relative homology and homology with closed supports), we can have the groups \( \mathbb{Y}_{i,j} \) as products of homology groups. We consider \( \mathbb{Y}_{*,*}^1 \) as a direct sum of stable and unstable pieces:

If the restriction of forgetful morphism \( \pi: D_{\gamma^*} \to D_{\gamma} \) doesn’t require stabilization, then the fibers \( F_{\gamma^*} \) are punctured \( \mathbb{P}^1 \)'s. Then, we have
If the restriction of forgetful morphism $\pi : D_{\tau^{*}} \to D_{\gamma}$ requires stabilization, we have

$$K^{1}_{2,j} = \bigoplus_{\gamma^{*}} H_{j} (\overline{D}_{\gamma}, Q_{\gamma}) \otimes H_{2} (\overline{F}_{\gamma^{*}}, s),$$

$$K^{1}_{2,j-1} = \bigoplus_{\gamma^{*}} H_{j} (\overline{D}_{\gamma}, Q_{\gamma}) \otimes H_{1} (\overline{F}_{\gamma^{*}}, s).$$

The union of unstable fibers is $\bigsqcup_{\tau^{*}} \overline{M}_{B_{\text{vs}}}$ where $|V_{\tau^{*}}| = |V_{\gamma^{*}}| + 1$ and all tails supported by $v_s$ are distinct (i.e., there is no subset of tails that are identified). This follows from the fact that all other strata (corresponding to $B$-trees with more vertices or identified tails) are contained in these strata (see Proposition 3.1). Moreover, these strata are pairwise disjoint since they are uniquely determined by the set $F_{\tau^{*}}(v_s)$. Hence,

$$L^{1}_{i,j} = \bigoplus_{\tau^{*} | |V_{\tau^{*}}| = |V_{\gamma^{*}}| + 1} H_{j} (\overline{D}_{\gamma}, Q_{\gamma}) \otimes H_{i} (\overline{M}_{B_{\text{vs}}}),$$

and

$${\mathcal{Y}}_{i,j} = K^{1}_{i,j} \oplus L^{1}_{i,j}. \quad (10)$$

Then, the differential $d_{1} : {\mathcal{Y}}^{1}_{i,j} \to {\mathcal{Y}}^{1}_{i-1,j}$ is zero since $H_{\text{odd}} (\overline{M}_{B_{\text{vs}}})$ is zero.

Finally, the differential $d_{2} : {\mathcal{Y}}^{1}_{2,j} \to {\mathcal{Y}}_{0,j+1}$ is given by the differentials

$$\partial_{*} : H_{1} (\overline{F}_{\gamma^{*}}, s) \to H_{0} (\overline{F}_{\tau^{*}})$$

where $|V_{\tau^{*}}| = |V_{\gamma^{*}}| + 1$ and $|O_{\tau(v_s)}| = 3$. For each pair of points lying in the same component of $\overline{F}_{\gamma^{*}}$, there is a generator in $H_{1} (\overline{F}_{\gamma^{*}}, s)$ whose image under $\partial_{*}$ gives the difference of these points (see Section 4.2). Therefore, the strata $\overline{D}_{\tau^{1}_{1}}, \overline{D}_{\tau^{2}_{2}}$, which are zero-dimensional fibrations over $\overline{D}_{\gamma}$, are homologous relative to $\pi^{-1} (Q_{\gamma})$.

It is important to note that the kernel of the differential $d_{2} : {\mathcal{Y}}^{1}_{2,j} \to {\mathcal{Y}}_{0,j+1}$ is trivial. This follows from the fact that the same is true for $\partial_{*}$ given in (11) due to the relations of the homology of the fibers given in (4).
The subgroups $I_2i$ are generated by a subset of relations $R(γ^*; v, f_1, f_2, f_3, f_4) \equiv 0$: If $\sum_{f \in F, \gamma^*(v) \{sn+1\}} B(f) \leq 2$, they arise from the homology relations of $\overline{M}_{BS_n}$. If $\sum_{f \in F, \gamma^*(v) \{sn+1\}} B(f) > 2$ and $f_i \neq sn+1$, then we have again $R(γ^*; v, f_1, f_2, f_3, f_4) \equiv 0$. These relations are obtained by pulling back the relations from the base $\overline{M}_A$. Finally, in addition to these, we have

$$[\overline{D}_\tau] - [\overline{D}_\tau_2] = 0$$

because these are in the image of $d_2$ as described above.

**Step 2.** The calculation in Step 1 implies that the $E_1^{s,s}$ are generated by the fundamental classes of the strata. Moreover, they admit the relations that are imposed in the statement of the theorem: For each relation (12) in relative homology $H_*(E_p, E_{p-1})$, there is a relation in $H_*(\overline{M}_B)$. We obtain the missing relations: $R(γ^*; f_1, f_2, f_3, sn+1) \equiv 0$. This completes the set of relations given in the statement.

**Step 3.** We have a complete description of generators and relations in $E_1^{s,s}$. We need to calculate the higher differentials. Since all strata of $\overline{M}_B$ are even-dimensional (which means $E_{k,l} = 0$ unless both $k$ and $l$ are even), the higher differentials

$$d_i : E_{k,l} \to E_{k-i,l+i-1}, \quad i \geq 1$$

must be zero. This follows from the simple fact that both $k - i$ and $l + i - 1$ cannot be zero modulo 2 when $k, l = 0 \mod 2$. □

**Corollary 2.** The Chow groups of $\overline{M}_A$ are

$$A_d(\overline{M}_A) = \left( \bigoplus_{\gamma | \dim \mathcal{C}_D, D_\gamma = d} \mathbb{Z}[\overline{D}_\gamma] \right) / \mathcal{I}_d$$

where the subgroup of relations $\mathcal{I}_d$ is generated by $R(\gamma^*; v, f_1, f_2, f_3, f_4)$ defined in (7).

**Proof.** The homology groups of complex points of the moduli space is generated by the fundamental cycles of its strata. Obviously, the strata of $\overline{M}_A$ are algebraic cycles. Moreover, the relations between strata are given by rational equivalence in Lemma 5.1 i.e., the Chow groups $A_i(\overline{M}_A)$ are isomorphic to the homology groups $H_{2i}(\overline{M}_A)$. □

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