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Computation of Characters of the Higman-Sims Group and its Automorphism Group

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1. INTRODUCTION

A new simple group H of order $1100 \cdot 8!$ was discovered in 1967 by D.G. Higman and C.C. Sims, represented as a rank three group of degree 100 with subdegrees 1, 22, 77, in which the stabilizer of a point is isomorphic with the Mathieu group $M = M_{22}$ of order $11 \cdot 8!$ [5]. Using this information, together with the characters of M and general properties of group characters, we shall derive the character tables of H and of the automorphism groups M' of M and H' of H. The subgroups M of M' and H of H' have index 2. Examination of the character table of H suggests the existence of a doubly transitive permutation representation of H of degree 176, with point stabilizer G of order 252 000 containing the simple group $U = PSU_3(5)$ of order 126 000 [2]. Indeed Graham Higman [6] arrived independently at the simple group H through this subgroup G, which contains the alternating group A_7 with index 100.

Principal tools used in constructing the character tables include (1) the Frobenius induce-restrict relations between irreducible characters of a group and those of a subgroup, (2) the decomposition of Kronecker powers and products, (3) the Brauer theory of p-blocks of defect 0 or 1, (4) congruence relations between the character values of an element and those of its powers, (5) orthogonality relations, and (6) the use of the square root enumerating function ζ_t for discovering classes of elements in H or M' but not in M, or in H' but not in H, whose squares are in known classes. This function is also useful in determining how certain pairs of characters of H fuse in H'. The mean value of $\zeta_t(\zeta_t - 1)$ over a group is shown to be one less than the number of self inverse classes, and this is a useful check on characters.

Only 11 of the 24 classes of H are represented among the 12 classes of M, of which the two 7-classes fuse into the one 7-class of H. All but one of the remaining classes of H are represented among the classes of G. The two 11-classes of M fuse into a single 11-class of M', and the 10 classes of M' not in M contain elements of orders 2, 2, 4, 4, 6, 8, 10, 12, 14, 14. All but the last pair represent distinct classes of H', and the remaining nine classes of H' not in H or M contain elements of orders 4, 6, 6, 8, 10, 20, 20, 20, 30.

Irreducible characters of M and H will be denoted by their degrees with subscripts m or h, except that the letters i, j, k or n will be used to avoid duplications when two or more characters have the same degree. The same symbols will denote irreducible characters of M' and H' that are positive for the class C_1' of involutions that are 7th powers of elements of order 14, whereas their products with the alternating character $1_m'$ or $1_h'$ will be indicated by primes. Certain permutation characters will be indicated by subscripts p, and certain reducible characters by subscripts r.

2. Orders of Centralizers in the M-Classes of H

Let h_k of order $°h_k$ be an element of class C_s^M of M contained in the class C_k^H of H. Let $°N_s^M$ and $°N_k^H$ denote the orders of the centralizers (or normalizers) of h_k in M and H, respectively. Then the value 100_{pk} for C_k^H of the permutation character 100_p is

$$100_{pk} = \sum_s °N_k^N/°N_s^M \qquad \text{for all} \quad C_s^M \subset C_k^H. \tag{2.1}$$

Whenever C_k^H contains exactly one C_s^M,—as is true for ten classes of H,—we have $°N_k^H = 100_{pk}°N_s^M$.

We determine first the degrees of the irreducible constituents of 100_p. These are positive integers $1, d, 99\text{-}d$ such that $100(22)(77)/d(99\text{-}d)$ is a rational integral square [1]. Hence d is 22 or 77, and H has the irreducible characters 1_h, 22_h, and 77_h. We can split these uniquely in the subgroup M, since the only degrees of irreducible characters of M less than 77 are 1, 21, 45, and 55. Thus in M we have

$$1_h \overset{M}{=} 1_m, \qquad 22_h \overset{M}{=} 1_m + 21_m, \qquad 77_h \overset{M}{=} 1_m + 21_m + 55_m. \tag{2.2}$$

Thus the characters 1_h, 22_h, and 77_h are determined for all M-classes from the characters of M, and 100_p is given for all classes of H by

$$100_p = 1_m{}^H = 3(1_m) + 2(21_m) + 55_m = 1_h + 22_h + 77_h . \qquad (2.3)$$

We list the eleven M-classes of H in a convenient order to display congruence relations. We denote the classes by generalized permutation class symbols $1^a 2^b 3^c \cdots$ [4], based on the integral-valued characters of 22_h and 77_h. Given the characters 1_m, 21_m and 55_m of M and given ${}^ch_k$ and ${}^cN_k{}^M$, we determine 100_{pk} and ${}^cN_k{}^N$.

M-characters			${}^ch_k$	${}^cN_k{}^M$	100_{pk}	${}^cN_k{}^H$	Class symbols		
1_m	21_m	55_m	1	$11 \cdot 8!$	100	$1100 \cdot 8!$	1^{22}	1^{77}	C_1
1	-1	0	11	11	1	11	$(11)^2$	$(11)^7$	C_2
1	-1	0	11	11	1	11	$(11)^2$	$(11)^7$	C_3
1	0	-1	7, 7	7, 7	2	7	$1 \cdot 7^3$	7^{11}	C_4
1	1	0	5	5	5	25	$1^2 5^4$	$1^2 5^{15}$	C_5
1	3	1	3	36	10	360	$1^4 3^6$	$1^5 3^{24}$	C_6
1	-1	1	6	12	2	24	$2^2 3^2 6^2$	$12^2 3^4 6^{10}$	C_7
1	5	7	2	384	20	7680	$1^6 2^8$	$1^{13} 2^{32}$	C_8
1	1	3	4	32	8	256	$1^2 2^2 4^4$	$1^5 2^4 4^{16}$	C_9
1	1	-1	4	16	4	64	$1^2 2^2 4^4$	$12^6 4^{16}$	C_{10}
1	-1	1	8	8	2	16	$2 \cdot 4 \cdot 8^2$	$12^2 4^2 8^8$	C_{11}

(2.4)

3. Characters for the M-Classes of H

Starting with the characters 22_h and 77_h which are known by (2.2) for the M-classes of H, we deduce others by forming Kronecker powers and products, splitting these in M, and seeing how the pieces must be combined to form irreducible H-characters. We also induce characters $d_m{}^H$ of H from irreducible characters d_m of M, and note by the Frobenius reciprocity theorem that if f_h is an irreducible character of H then the restriction $f_h{}^M$ of f_h to M contains d_m with the same multiplicity v_{df} that $d_m{}^H$ contains f_h.

$$d_m{}^H = \sum_f v_{df} f_h , \qquad f_h{}^M = \sum_d d_m v_{df} . \qquad (3.1)$$

The matrix $W = VV^T$ can be computed from the characters of M, and the inclusion relations between classes of M and H. Each row of W indicates the splitting in M of the restriction to M of an induced character $d_m{}^H$.

The W-matrix of scalar products of multiplicities

d_m												
1_m	3	2	0	0	1	0	0	0	0	0	0	0
21_m	2	6	0	0	3	1	3	3	1	0	0	1
45_m	0	0	2	2	0	1	0	1	1	4	4	4
45_n	0	0	2	2	0	1	0	1	1	4	4	4
55_m	1	3	0	0	7	4	4	3	4	1	1	5
99_m	0	1	1	1	4	7	4	4	5	5	5	9
154_m	0	3	0	0	4	4	14	9	10	6	6	13
210_m	0	3	1	1	3	4	9	15	10	13	13	16
231_m	0	1	1	1	4	5	10	10	19	11	11	21
280_m	0	0	4	4	1	5	6	13	11	22	21	23
280_n	0	0	4	4	1	5	6	13	11	21	22	23
385_m	0	1	4	4	5	9	13	16	21	23	23	36

(3.2)

The first row of the W-matrix gives the decomposition of the induced character $1_m{}^H$ in (2.3). The second and fifth show the splitting of $21_m{}^H$ and $55_m{}^H$ which may be written as follows, by multiplying 21_m and 55_m by 100_p .

$$21_m{}^H = (22_h - 1_h)(1_h + 22_h + 77_h) = 22_h \times 22_h - 1_h - 77_h + 22_h \times 77_h$$
$$\overset{M}{=} 2(1_m) + 6(21_m) + 3(55_m + 154_m + 210_m) + 99_m + 231_m + 385_m \quad (3.3)$$

$$55_m{}^H = (77_h - 22_h)(1_h + 22_h + 77_h) = 77_h \times 77_h - 22_h \times 22_h + 77_h - 22_h$$
$$\overset{M}{=} 1_m + 7(55_m) + 3(21_m + 210_m) + 4(99_m + 154_m + 231_m) + 5(385_m)$$
$$+280_m + 280_n \quad (3.4)$$

To split $21_m{}^H$ in H we first split $22_h \times 22_h - 1_h$ into symmetric and alternating components, split these in M, and combine them in H.

$$22_h^{[2]} - 1_h \overset{M}{=} 1_m + 2(21_m) + 55_m + 154_m \quad (3.5)$$

$$22_h^{[1^2]} \overset{M}{=} 21_m + 210_m . \quad (3.6)$$

Since each irreducible H-component of $21_m{}^H$ contains 21_m , character (3.6) is irreducible and (3.5) splits into two components of which one is the component 22_h or 77_h of $1_m{}^H$. Degree divisibility checks rule out $22 + 230$ and leave $77 + 175$. Hence we have two new irreducible H-characters

$$175_h = 22_h^{[2]} - 1_h - 77_h \overset{M}{=} 21_m + 154_m \quad (3.7)$$

$$231_h = 22_h^{[1^2]} \overset{M}{=} 21_m + 210_m \quad (3.8)$$

Extracting these characters and 22_h and 77_h from $21_m{}^H$, we have

$$\begin{aligned} &22_h \times 77_h - 22_h - 77_h \\ &\quad = 21_m{}^H - 22_h - 77_h - 175_h - 231_h = a_h + b_h \\ &\quad \overset{M}{=} 2(21_m + 55_m + 154_m + 210_m) + 99_m + 231_m + 385_m . \end{aligned} \tag{3.9}$$

This splits into two characters called a_h and b_h, both containing 21_m in M. Their degree sum is $a + b = 1595$.

Using (3.4), we can partially split $55_m{}^H$ in H as follows:

$$55_m{}^H - 77_h = (77_h^{[1^2]} - 22_h^{[1^2]}) + (77_h^{[2]} - 22_h^{[2]} - 22_h). \tag{3.10}$$

Since $(22_h + 55_m)^{[1^2]}$ contains $22_h^{[1^2]}$, we have in M

$$\begin{aligned} 77_h^{[1^2]} - 22_h^{[1^2]} \overset{M}{=} 21_m &+ 2(55_m + 99_m + 210_m + 231_m + 385_m) \\ &+ 154_m + 280_m + 280_n . \end{aligned} \tag{3.11}$$

This splits into two irreducible H-characters, each containing 55_m. One of these is a common component of $21_m{}^H$ and $55_m{}^H$ not in $1_m{}^H$, so it is either a_h or b_h. Call it b_h and the other c_h. Combining (3.9) and (3.11), we have

$$c_h - a_h \overset{M}{=} -21_m - 154_m + 99_m + 231_m + 280_m + 280_n + 385_m . \tag{3.12}$$

Thus the characters a_h, b_h, c_h split in M as

$$\begin{aligned} a_h &\overset{M}{=} 21_m + 55_m + 154_m + x_r , \\ b_h &\overset{M}{=} 21_m + 55_m + 154_m + y_r , \\ c_h &\overset{M}{=} 55_m + 99_m + 231_m + 280_m + 280_n + 385_m + x_r , \end{aligned} \tag{3.13}$$

where $x + y = 1135$ and

$$x_r + y_r = 2(210_m) + 99_m + 231_m + 385_m . \tag{3.14}$$

Since x is a sum of numbers 210, 210, 99, 231, 385 such that $230 + x$, $1365 - x$, and $1330 + x$ are all divisors of the group order, the only possible solution is found to be $x = 595$, $a = 825$, $b = 770$, $c = 1925$. Thus in M-classes we have

$$\begin{aligned} 825_h &= a_h \overset{M}{=} 21_m + 55_m + 154_m + 210_m + 385_m , \\ 770_h &= b_h \overset{M}{=} 21_m + 55_m + 99_m + 154_m + 210_m + 231_m , \\ 1925_h &= c_h \overset{M}{=} 55_m + 99_m + 210_m + 231_m + 280_m + 280_n + 2(385_m). \end{aligned} \tag{3.15}$$

In all classes of H we have

$$825_h + 770_h = 22_h \times 77_h - 22_h - 77_h\,, \tag{3.16}$$

$$1925_h + 770_h = 77_h^{[1^2]} - 22_h^{[1^2]}. \tag{3.17}$$

Consider next the decomposition of the Kronecker cube of 22_h. From (3.7) and (3.8) we obtain

$$22_h \times 231_h = 22_h^{[21]} + 22_h^{[1^3]}, \tag{3.18}$$

$$22_h \times 175_h = 22_h^{[21]} + 22_h^{[3]} - 22_h(1_h + 77_h). \tag{3.19}$$

All H-components of (3.18) are contained in $21_m{}^H + 210_m{}^H$, and must contain either 21_m or 210_m. In M we have

$$22_h^{[1^3]} \overset{M}{=} 2(210_m + 280_m + 280_n), \tag{3.20}$$

$$\begin{aligned} 22_h^{[21]} - 22_h \overset{M}{=} {} & 2(21_m + 55_m + 231_m) + 3(154_m + 210_m + 385_m) \\ & + 99_m + 280_m + 280_n\,. \end{aligned} \tag{3.21}$$

The former splits into two H-characters 770_i and 770_j which agree on M. They are later found to be conjugate complex characters of H which fuse into a single character 1540_h of H'.

$$770_i \overset{M}{=} 770_j \overset{M}{=} 210_m + 280_m + 280_n \text{ in } M\text{-classes}. \tag{3.22}$$

The character $22_h^{[21]}$ contains the pair of characters 825_h and 770_h common to $21_m{}^H$ and $55_m{}^H$. Removing them we have a single irreducible H-character left, of degree 1925, called 1925_k.

$$\begin{aligned} 1925_k &= 22_h^{[21]} - 22_h - 825_h - 770_h \\ &\overset{M}{=} 154_m + 210_m + 231_m{}' + 280_m + 280_n + 2(385_m). \end{aligned} \tag{3.23}$$

From (3.16), (3.19), and (3.23), we obtain

$$22_h^{[3]} = 22_h \times 175_h + 22_h + 77_h - 1925_k \tag{3.24}$$

$$\begin{aligned} 22_h^{[3]} - 2(22_h) - 77_h \overset{M}{=} {} & 2(21_m + 55_m + 210_m) + 99_m \\ & + 4(154_m) + 231_m + 385_m\,. \end{aligned} \tag{3.25}$$

The two characters 825_h and 770_h in (3.15) that contain 21_m and 55_m split off in (3.24) leaving $2(154_m)$ in M. This yields a character of degree 308 which turns out to be irreducible in H', but splits into two characters 154_i and 154_j of H that agree in M.

$$154_i + 154_j = 22_h(175_h - 77_h) + 77_h - 1925_k \qquad (3.25)$$
$$= 22_h^{[3]} - 22_h(1_h + 77_h),$$
$$154_i \overset{M}{=} 154_j \overset{M}{=} 154_m \text{ in } M\text{-classes of } H. \qquad (3.26)$$

4. Blocks of Defect 1

We now apply the Brauer theory of p-blocks of defect 1 to obtain additional "$pd1$"-characters for the primes $p = 3, 5, 7, 11$. Each such block in H (or M) gives rise to a pair of blocks of H' (or M'), obtained one from the other by multiplying by the alternating character $1_h'$ (or $1_m'$). We determine later, but indicate now by primes, which characters in a block are negative for the involution class C_1'.

First, for $p = 3$, the "$3d1$"-chain for M is

$$\begin{aligned} 21_m' - 231_m + 210_m &= 0 \text{ in 3-regular classes of } M', \\ 21_m' = -231_m &= 210_m \text{ in 3-singular classes of } M'. \end{aligned} \qquad (4.1)$$

The corresponding chain in H contains the two characters 231_h and 825_h, found in $21_m{}^H$, whose degrees are each $\equiv 6 \pmod 9$. The middle character of the chain is a new character 1056_h which we list between 231_h and 825_h for easy checking. We have

$$\begin{aligned} 231_h' - 1056_h + 825_h &= 0 \text{ in 3-regular classes of } H', \\ 231_h' = -1056_h &= 825_h \text{ in 3-singular classes of } H'. \end{aligned} \qquad (4.2)$$

This relation serves to determine 825_h from $231_h'$ in 3-singular non-M classes of H', and thus to calculate 770_h and 1925_h from (3.16) and (3.17) in those classes. The decomposition of 1056_h in M is found to be

$$1056_h \overset{M}{=} 2(231_m) + 55_m + 154_m + 385_m . \qquad (4.3)$$

Thus 1056_h is a constituent of $55_m{}^H$, in which 77_h and the characters 825_h, 770_h, and 1925_h of (3.15) have already been found. From (3.4), (3.10), and (3.11) we obtain

$$77_h^{[2]} - 22_h^{[22]} - 22_h - 1056_h - 825_h = 847_r$$
$$\overset{M}{=} 2(55_m + 99_m) + 154_m + 385_m . \qquad (4.4)$$

This reducible character 847_r must have just two components, since it lies in $55_m{}^H$, and each must contain 55_m and 99_m. Since $539 = 7^2.11$ is not a factor of ${}^\circ H$, the two characters have degrees 693 and 154.

$$693_h \overset{M}{=} 55_m + 99_m + 154_m + 385_m\,, \tag{4.5}$$

$$154_h \overset{M}{=} 55_m + 99_m\,. \tag{4.6}$$

We note the decompositions in H:

$$77_h^{[2]} - 22_h^{[2]} = 22_h + 1056_h + 825_h + 693_h + 154_h\,, \tag{4.7}$$

$$55_m{}^H = 77_h + 1056_h + 825_h + 770_h + 1925_h + 693_h + 154_h\,. \tag{4.8}$$

We also split $22_h \times 154_h$ in M and combine the pieces, which include three 55_m's, to obtain the check formula

$$22_h \times 154_h = 770_h + 1925_h + 693_h\,. \tag{4.9}$$

Next, for $p = 5$, the $5d1$-chain for M is found to be

$$\begin{aligned} &1_m - 99_m + 231_m - 154_m' + 21_m' = 0 \text{ in 5-regular classes,}\\ &1_m = -99_m = 231_m = -154_m' = 21_m' \text{ in 5-singular classes.}\end{aligned} \tag{4.10}$$

Since $55_m{}^H$ is a sum of 5-indecomposable characters of H which contains just two $5d1$-characters 825_h and 1925_h, these two yield an indecomposable character of H. Besides these and the characters 175_h and 1925_k, the $5d1$-block of H contains a new fifth character of degree $175 + 1925 + 1925 - 825 = 3200$. The $5d1$-chain for H is

$$\begin{aligned} &175_h - 825_h + 1925_h - 3200_h + 1925_k = 0 \text{ in 5-regular classes,}\\ &175_h = -825_h = 1925_h = -3200_h = 1925_k \text{ in 5-singular classes.}\end{aligned} \tag{4.11}$$

This is later seen to be a $5d1$-chain for H' also. In M, we decompose 3200_h as follows:

$$3200_h \overset{M}{=} 99_m + 154_m + 210_m + 2(231_m + 280_m + 280_n) + 3(385_m). \tag{4.12}$$

This character is the fifth of seven H-characters in $99_m{}^H$. We calculate in M.

$$\begin{aligned} &99_m{}^H - 770_h - 1925_h - 3200_h - 693_h - 154_h\\ &= 3158_r \overset{M}{=} 45_m + 45_m + 45_n + 154_m + 210_m + 231_m\\ &\qquad + 2(99_m + 280_m + 280_n) + 3(385_m). \end{aligned} \tag{4.13}$$

This reducible character 3158_r splits into exactly two irreducible H-characters, each containing 99_m. They are real of different even degrees, since 3158 is not the sum of two odd divisors of $°H$. Since $3158 \equiv 1 \pmod{77}$ the degrees have remainders 0 or 1 both (mod 7) and (mod 11), so the remainders (mod 154) are either 0 and 78 or 22 and 56. The only solution with summands dividing $°H$ is

$$3158_r = 1408_h + 1750_h . \tag{4.14}$$

The H-character which contains $99_m + 45_m + 45_n$ has degree divisible by 7, so it is 1750_h. It is an indecomposable (mod 5), since 5^3 divides 1750. It is also a constituent of $45_m{}^H$ which is a sum of 5-indecomposables of H of degree 4500, whose decomposition in M is

$$45_m{}^H \overset{M}{=} 2(45_m + 45_n) + 99_m + 210_m + 231_m + 4(280_m + 280_n + 385_m) \tag{4.15}$$

provided that the 7-classes of M fuse in H. Otherwise a third 45_m replaces one 45_n in (4.15). Subtracting 1750_h from $45_m{}^H$ we obtain a 5-indecomposable character 2750_h which is irreducible if the 7-classes of M fuse in H, or which splits into two characters with complex values on the two 7-classes otherwise.

To settle this question we examine the 7-blocks of M and H. For M the 7-chain is

$$1_m - 55_m + 99_m - 45_{mn} = 0 \text{ in 7-regular classes} \tag{4.16}$$

The $7d1$-characters in $1_m{}^H$ are 1_h and 22_h, in $55_m{}^H$ are 825_h and 1056_h, and in $99_m{}^H$ are 3200_h and 1408_h. This requires that the remaining $7d1$-character have degree

$$1 + 22 - 825 - 1056 + 3200 + 1408 = 2750. \tag{4.17}$$

Hence the degree is not 1375, and the 7-classes of M fuse in H. From the 7-indecomposable 21_m we see that $22_h + 825_h$ is a 7-indecomposable of H. Similar reasoning gives the 7-chain for H and H':

$$1_h - 1056_h + 3200_h - 2750_h + 1408_h - 825_h + 22_h = 0 \text{ in 7-regular classes.} \tag{4.18}$$

This determines the difference $2750_h - 1408_h$, which equals the difference $45_m{}^H - 3158_r$, but does not separate out the character 1750_h.

For additional information we consider the $11d1$-blocks of M and H. In M, we have

$$1_m - 21_m + 210_m - 280_{mn} + (45_m + 45_n) = 0 \text{ for 11-regular classes.} \tag{4.19}$$

Here the conjugate characters 280_m and 280_n are equal (mod 11) and their sum $280_m + 280_n$ joins with each of the characters 210_m, 45_m, and 45_n to form 11-indecomposables of degrees 770 and 605 in M.

Hence in (4.19) each of the characters 45_m and 45_n in the indecomposable 2750_h of H brings with it a pair $280_m + 280_n$, and if 210_m is not in 1750_h, it brings to both 11-indecomposables 1408_h and 2750_h of H the 11-indecomposable $(210_m + 280_m + 280_n)$ of M. The character 1408_h must be divisible by 2^7 on the class C_8 of $3.5^2.7.11$ involutions that lie in the center of a Sylow 2-group. Hence it is 0 for C_8. Since $99_m + 154_m = 13$, $385_m = 1$ and $(210_m + 280_m + 280_n) = -14$ on class C_8, we obtain the following decompositions:

$$45_m{}^H = 2750_h + 1750_h, \tag{4.20}$$

$$\begin{aligned} 1408_h &\overset{M}{=} 99_m + 154_m + 210_m + 280_m + 280_n + 385_m, \\ 1750_h &\overset{M}{=} 45_m + 45_n + 99_m + 231_n + 280_m + 280_n + 2(385_m), \\ 2750_h &\overset{M}{=} 45_m + 45_n + 210_m + 3(280_m + 280_n) + 2(385_m). \end{aligned} \tag{4.21}$$

To determine the $11d1$-block of H, we first compute the product $22_h \times 154_i$ and split it in M.

$$\begin{aligned} 22_h \times 154_i \overset{M}{=} {} & 21_m + 55_m + 99_m + 4(154_m) + 2(210_m + 231_m) \\ & + 280_m + 280_n + 3(385_m). \end{aligned} \tag{4.22}$$

Since $22_h \times 175_h$ contains 154_i it follows that $22_h \times 154_i$ contains 175_h, and we deduct $21_m + 154_m$ to obtain a reducible character of degree 3213. The only irreducible H-characters whose restrictions to M contain both 99_m and 154_m, but not 21_m, are 3200_h, 1408_h and 693_h. Since 3213−3200 is too small we reject 3200. If 1408_h is in 3213_r, then there must also be a component of $55_m{}^H$ that contains neither 21_m nor 99_m. This must be 1056_h. The remaining character of degree 749 contains 154_m, 210_m, and 385_m, but no two of these have a sum that divides $°H$; so we reject 1408_h. If 693_h is extracted, the remaining character is of degree 2520.

$$22_h \times 154_i = 22_h \times 154_j = 175_h + 693_h + 2520_h, \tag{4.23}$$

$$2520_h \overset{M}{=} 2(154_m + 210_m + 231_m + 385_m) + 280_m + 280_n. \tag{4.24}$$

The degree of this character divides $^\circ H$ and is congruent to 1 (mod 11). Either it is irreducible or it splits into two conjugate characters of degree 1260.

So far we have found the $11d1$-characters 1_h, 175_h, 1750_h, 3200_h, and possibly 2520_h or 1260_i, 1260_j. Since $22_h \times 22_h$ contains $1_h + 175_h$ and $22_h \times 154_i$ contains $175_h + 2520_h$, the $11d1$-chain for H is

$$1_h - 175_h + 2520_h - 3200_h + 1750_h - 896_{ij} = 0 \text{ in 11-regular classes.} \tag{4.25}$$

It is not possible to replace 2520_h by 1260, since the degrees of the modular irreducibles would then be 1, 174, 1086, 2114 and -364, and a negative degree is not possible. Hence 2520_h is irreducible in H, and there is a pair of conjugate characters 896_i and 896_j which agree for 11-regular classes, but assume complex values $(-1 \pm i\sqrt{11})/2$ in the two 11-singular classes.

Since the $7d1$-block and the $11d1$-block have been completed, all remaining irreducible H-characters have degrees divisible by 77, and they vanish in classes C_2, C_3, and C_4. In class C_5 of 5-elements, the sum of squares of characters, 24, lacks 1 from the centralizer order $^\circ N_m{}^H = 25$, so a missing character has the value ± 1. The scalar product with the identity class determines the degree to be 1386. It is not in the representation of H induced by any character of M of degree less than 154, so it contains 231_m, and 385_m. Completion of the inducing table requires

$$1386_h \overset{M}{=} 210_m + 231_m + 280_m + 280_n + 385_m . \tag{4.26}$$

A check shows that the sum of squares of degrees is $^\circ H$, so all characters are accounted for.

5. The Square Root Counting Function ζ_t

The value of

$$c_i = \frac{1}{^\circ G} \sum_{g \in G} \chi^i(g^2) = \sum_t \zeta_t \chi_t{}^i / {}^\circ N_t{}^G \tag{5.1}$$

was shown by Frobenius to be 1, 0, or -1 according as the irreducible character χ^i belongs to a real representation, a representation with complex character, or a "symplectic type" representation which has real character of even degree but is not similar to any real representation. Here ζ_t denotes the number of square roots in G of an element g_t of class C_t, and is computable by the formula

$$\zeta_t = \sum_s \zeta_s \sum_i \chi_s{}^i \chi_t{}^i / {}^\circ N_s{}^G = \sum_i c_i \chi_t{}^i. \tag{5.2}$$

THEOREM. *If a finite group G has r self-inverse ("real") classes, then*

$$\sum_t \zeta_t(\zeta_t - 1)/{}^\circ N_t^G = r - 1. \tag{5.3}$$

Proof. We count the number of solutions $x, y \in G$, $y \neq 1$ of the equivalent equations

$$(xy)^2 = x^2, \tag{5.4}$$

$$x^{-1}yx = y^{-1}. \tag{5.5}$$

In (5.4) there are ${}^\circ G/{}^\circ N_t^G$ choices for x^2 in class C_t, and for each x^2 there are ζ_t values of x and $\zeta_t - 1$ values of $y \neq 1$. In (5.5) there are $r - 1$ choices for the class C_t of y, and then ${}^\circ G/{}^\circ N_t^G$ choices for y and ${}^\circ N_t^G$ choices for x. We equate the two counts and divide by ${}^\circ G$ to get (5.3).

6. Character Values for the Non-M-Classes of H

Since H has 24 irreducible characters, it has 24 conjugacy classes, of which 11 are M-classes, six more contain elements whose squares are in $M = M_{22}$. The subgroup M has the two pairs of conjugate complex characters 45_m, 45_n, and 280_m, 280_n, but has no irreducible symplectic characters. A character of a group H induced from a real character of a subgroup M can contain irreducible symplectic components only with even multiplicity. Thus from the induce-restrict table from M to H we see that the only candidate for a symplectic character of H is 2520_h. The characters 896_i and 896_j are conjugate complex, and the pairs 770_i, 770_j and 154_i, 154_j are either real or complex. All other irreducible characters of H are of real type.

The centralizer orders ${}^\circ N_k^H$ for $k = 2, 3, 4, 5$ are the odd numbers 11, 11, 7, and 25, so each element in these classes has exactly one square root (an element of odd order) and $\zeta_k = 1$. The character sums for these classes are 1 if the two complex characters 896_i and 896_j for which $c_i = 0$ are omitted. Hence c_i must be 1 for each of the three characters 2520_h, 154_i, and 154_j. These belong to real representations. In class C_{11} there are $256/16 = 16$ elements of order 8 whose squares are a fixed element of C_9, so $\zeta_9 \geq 16$. Here the character sum is only 12 if the two -2's in characters 770_i and 770_j are counted, but it is 16 if they are omitted. Hence these characters are a conjugate complex pair with $c_i = 0$. All characters except 770_i, 770_j, 896_i, and 896_j are of real type with $c_i = 1$.

Let ζ_k' of the square roots of an element of class C_k lie in M-classes of H, and $\zeta_k'' = \zeta_k - \zeta_k'$ in non-M-classes of H. Noting that the squares of elements in classes C_7, C_8, C_9, C_{10}, C_{11} lie, respectively, in C_6, C_1, C_8, C_8 and C_9, we compute $\zeta''_k/{}^{\circ}N_k{}^H$ as follows:

k	1	2	3	4	5	6	7	8	9	10	11
${}^{\circ}N_k{}^H$	$1100 \cdot 8!$	11	11	7	25	360	24	7680	256	64	16
ζ_k	$1 + 55(385)$	1	1	1	1	26	2	152	16	8	0
ζ_k'	$1 + 15(385)$	1	1	1	1	16	0	150	16	0	0
ζ_k''	$40(385)$	0	0	0	0	10	2	2	0	8	0
$\zeta_k''/{}^{\circ}N_k{}^H$	1/2880	0	0	0	0	1/36	1/12	1/3840	0	1/8	0

(6.1)

Thus there are non-M-classes of elements of orders 2, 6, 12, 4, and 8 whose squares are in M-classes. The elements of order 2 and 4 have centralizer orders divisible by 5, as do the elements of order 3 in class C_6. Hence there are elements of order 5 not in class C_5 (where ${}^{\circ}N_5{}^H = 25$) that commute with elements of orders 2, or 4, or 3 producing product elements of orders 10, 20, and 15, respectively. One such class contains the center elements of a Sylow 5-subgroup, and for this class 5^3 divides ${}^{\circ}N_k{}^H$.

We form the sums $\sum'$ and $\sum''$ of $1/{}^{\circ}N_k{}^H$ over the M-classes and non-M-classes with squares in M, respectively, and subtract from 1 to find the sum $\sum'''$ over the remaining "type 3" classes.

$$\textstyle\sum' (1/{}^{\circ}N_k{}^H) = 16/125 + 3/8 - 3/256 \text{ (for } M\text{-classes)}, \tag{6.2}$$

$$\textstyle\sum'' (1/{}^{\circ}N_k{}^H) = 1/2880 + 1/36 + 1/12 + 1/3840 + 1/8, \tag{6.3}$$

$$\textstyle\sum''' (1/{}^{\circ}N_k{}^H) = 1/500 + 1/300 + 4/15. \tag{6.4}$$

Since the permutation character $100_p = 1_h + 22_h + 77_h$ vanishes in the non-M-classes of H, many characters can be expressed in terms of the value of 22_h. We record a few relations for easy reference:

$$\begin{aligned}
77_h &= -1_h - 22_h + 100_p, \\
231_h &= 22_h^{[1^2]}, \\
175_h &= 22_h^{[2]} - 1_h - 77_h \equiv -231_h - 22_h \times 77_h \pmod{100_p}, \\
825_h + 770_h &= 22_h \times 77_h + 1_h - 100_p, \\
1925_h + 770_h &= 77_h^{[1^2]} - 231_h, \\
1925_k &= 22_h^{[22]} - 22_h \times 77_h + 77_h, \\
770_i + 770_j &= 22_h^{[1^3]}, \\
154_i + 154_j &= 22_h^{[3]} - 22_h \times 77_h - 22_h.
\end{aligned} \tag{6.5}$$

In addition, we recall the relations (4.9) and (4.23), and note that $22_h \times 154_i = 22_h \times 154_j$; so the characters 154_i and 154_j can differ only in classes for which $22_h = 0$. Similarly, the characters 770_i and 770_j can differ only in classes for which $77_h = 0$, since $77_h \times 770_i = 77_h \times 770_j$. From (4.20) we also obtain

$$2750_h + 1750_h = 0 \text{ in non-}M\text{-classes.} \tag{6.6}$$

The non-M-classes of H may be described as follows:

C_{12} *and* C_{13}. Let class C_{12} contain elements of order 8 whose squares are in C_{10}. These are of type $2 \cdot 4 \cdot 8^2$ in 22_h or $1^{-1}2 \cdot 4^3 8^8$ in 77_h. Such elements form either a single class with $°N_{12}^H = 8$ or two classes with $°N_{12}^H = °N_{13}^H = 16$. In either case, the character values for the eight characters of odd degree are ± 1 and the rest are 0 or ± 2. If $22_h = 0$, the relations (6.5) and (4.9) determine the odd values in order to be $(1, -1, 1, -1, 1, 1, -1, -1)$. We check that $\zeta_{12} = 0$, and check orthogonality with all the M-classes. If there are two classes of such 8-elements, the remaining nonzero χ_{12}^i must be 2 and -2 in two characters of the same degree which are equal for all M-classes. These must be the characters 154_i and 154_j, and we verify later they cannot vanish on these classes, since they are equal in all classes for which $22_h \neq 0$.

C_{14}. Since $°N_6{}^H = 360$ is divisible by 5, the 3-elements in class 6 commute with elements of order 5 not in C_5 to form products of order 15 forming class C_{14}. Since $\chi_{14}^i \equiv \chi_6{}^i \pmod 5$ we take least residues (mod 5) of χ_{14}^i and find that 15 values are ± 1 and the rest 0. The sum of squares is $°N_{14}^H = 15$, and scalar products with the 13 known rows of the table vanish. Type symbols are $1^{-1}3 \cdot 5 \cdot 15$ and $3^{-1}5 \cdot 15^5$.

C_{15}. This class contains elements of order 5 that are cubes of elements of order 15 in C_{14}. Since $\chi_{15}^i \equiv \chi_{14}^i \pmod 3$ and $\chi_{15}^i \equiv \chi_1{}^i \pmod 5$, the character values are determined (mod 15). If we take least residues, the sum of squares gives $°N_{15}^H = 300$, which is a multiple of 75 as it should be. Scalar products with the known classes vanish. Type symbols are $1^2 5^4$ and $1^{-3}5^{16}$.

C_{16}. Since $\zeta_{15} = 16$, there are 15 elements of order ten whose squares are a fixed element of C_{15}. Since $15/300 = 1/20$, these elements account for $1/20$ of the elements of H. Since the 15-elements in C_{14} do not commute with involutions in H to produce elements of order 30, N_{15}^H does not contain elements of order 30, so $°N_{16}^H$ is not divisible by 3. It divides $300/3$, but is at least 20. Since $\chi_{16}^i \equiv \chi_{15}^i \pmod 2$ there are 12 odd values of χ_{16}^i. If we try $77_h = -1$ and $22_h = 0$, we obtain values whose scalar product with C_{12} does not vanish. If $77_h = 1$, then $22_h = -2$, and the values obtained satisfy $\chi_{16}^i \equiv \chi_{15}^i \pmod 4$. Scalar products check out, and $\zeta_{16} = 0$. Type symbols are $1^{-2}2^2 10^2$ and $1 \cdot 2^{-2}10^8$.

C_{17}. The fifth powers of elements in C_{16} are involutions not in C_8, since $\chi_{16}^i \not\equiv \chi_8{}^i \pmod 5$. These must be the missing involutions not in M-classes, and we expect that ${}^\circ N_{17}^H = 2880$. Using the two congruences $\chi_{17}^i \equiv \chi_{16}^i \pmod 5$ and $\chi_{17}^i \equiv \chi_1{}^i \pmod 4$, we may determine the characters (mod 20), but least residues do not produce a sum of squares of 2880. The values must be divisible by 16 for the characters 3200_h, 1408_h, 896_i and 896_j whose degrees are divisible by 128. These conditions together with formulas (6.5) and others do determine the characters, and scalar products check. Type symbols are $1^{-2}2^{12}$ and $1 \cdot 2^{38}$.

C_{18}. The involutions in C_{17} commute with elements of order 3 in C_6 to form products of order 6 in C_{18} whose squares are in C_6. When we solve the congruences $\chi_{18}^i \equiv \chi_{17}^i \pmod 3$ and $\chi_{18}^i \equiv \chi_6{}^i \pmod 2$, taking least residues (mod 6), we obtain the square sum ${}^\circ N_{18}^H = 36$, as expected. Scalar products check out, and $\zeta_{18} = 0$. Type symbols are $1^{-2}2^3 6^3$ and $1 \cdot 2^2 6^{12}$.

C_{19}. For class C_{19} we assign the elements of order 12 whose squares are in C_7. Since $\zeta_7 = 2$, we get ${}^\circ N_{19}^H = 12$. Since just twelve $\chi_7{}^i$ are odd, the corresponding χ_{19}^i are ± 1, and the rest are 0. Since $22_h = 0$, we have $77_h = -1$. Most of the signs of the odd values χ_{19}^i are determined by the relations (6.5), and the rest by orthogonality. Type symbols are $3^{-2}4 \cdot 6^2 12$ and $1^{-1}2 \cdot 3^2 4 \cdot 6(12)^5$.

C_{20}, C_{21}, C_{22}, and C_{23}. Cubes of the elements of C_{19} are elements of order 4 in C_{20}, whose squares are in C_8. As shown above, ${}^\circ H/3840$ elements in non-M-classes have their squares in C_8. At least one class of these must have a centralizer order divisible by 5, so there must be elements of order 20 in non-M-classes. Their squares of order 10 account for another class. There must be elements of order 5 not yet listed that lie in the centralizer of a Sylow 5-subgroup and for which 5^3 divides the centralizer order. Also the complex characters 770_i and 770_j must differ on a pair of inverse classes having elements of the same order. These must be two classes of elements of order 20, which we call C_{21} and C_{22}. Class C_{23} contains their squares, which are of order 10, and whose 5th powers lie in C_8. Since $\chi_{23}^i \equiv \chi_8{}^i \pmod 5$, we take the least residues (mod 5) of the values χ_{23}^i as a first guess for characters of C_{23}. The sum of squares is 20. This is the smallest possible value for $N_k{}^H$ for $k = 21, 22, 23$. Since $4 \cdot 5^3$ divides ${}^\circ N_{24}^H$, its smallest possible value is 500. For these values the sum $\sum 1/{}^\circ N_k{}^H$ checks out to 1. Hence these ${}^\circ N_k{}^H$ cannot be larger. The characters of C_{23} are determined by congruence (mod 5) with C_8. Now the characters in classes C_{21} and C_{22} differ only for 770_i and 770_j, where they are complex numbers θ and $\bar{\theta}$. Since $2\theta\bar{\theta} = 20/2$, we have $\theta\bar{\theta} = 5$. Since θ is a complex sum of 770 twentieth roots of unity, $\theta = \pm 5^{1/2} i$. The congruence $\chi_{23}^i \equiv \chi_{21}^i \pmod 2$ shows that $\chi_{21}^i = \pm 1$, except for the values θ and $\bar{\theta}$ just described. An orthogonality check shows that all non-zero values of 3200_h and 1408_h have been found, so these characters vanish in the last six classes. Then the modular theory shows that for the last four classes, all 5-singular, the characters 175_h, 825_h, 1925_h and 1925_k in the defect 1 block with 3200_h vanish, and also characters 2750_h and 1750_h of defect 0 all vanish. This information, with relations (6.5) and the fact that $\zeta_{21} = 0$, determines the characters for C_{21} and C_{22}. We next determine the characters for C_{20} (mod 30) by observing $\chi_{20}^i \equiv \chi_{19}^i$ (mod 3), $\chi_{20}^i \equiv \chi_8{}^i$ (mod 2) and $\chi_{20}^i \equiv \chi_{21}^i$ (mod 5). Since 2^8 divides ${}^\circ N_{20}^H = 3840$, and $2^9 \mid {}^\circ H$, it follows that 2^{m-1} divides χ_{20}^i if 2^m divides the degree $\chi_1{}^i$. Hence the characters in C_{20} for 3200_h, 1408_h, 896_i and 896_j are divisible by 2^6 and must vanish, while 16 divides 1056_h and 4 divides 2520_h in C_{20}. These are uniquely determined. Relations (6.5) can be used to resolve the few uncertainties among the remaining characters. Check that $\zeta_{20} = 0$. Type symbols for C_{21} and C_{22} are $1^{-1}2^4 \cdot 5^{-1}10 \cdot 20$ and $1 2^{-1}4 \cdot 5 \cdot 10 \cdot 20^3$. Classes C_{20}, C_{22}, C_{23}, and C_{24} all contain powers of elements from C_{21}.

C_{24}. This final class of H contains the elements of the center of a Sylow 5-subgroup of H, which are squares of elements in C_{23}. The congruences $\chi_{24}^i \equiv \chi_{23}^i$ (mod 4) (except in 770_i, 770_j) and $\chi_{24}^i \equiv \chi_1{}^i$ (mod 5) determine the character values for C_{24} (mod 20). If least residues are used, the sum of squares is found to be 460 instead of 500, and $\zeta_{24} = 6$ instead of 26. By changing the value of 1386_h from -9 to 11, the right values are obtained. Formulas (6.5) are verified, and the completed table satisfies the orthogonality conditions. Thus the character table of H was calculated using very little knowledge about the group.

7. Characters of the M'-Classes of H'

The automorphism group H' of the Higman–Sims group H is a group of order $2200 \cdot 8!$ which contains H as a subgroup of index 2, and has a permutation representation with character $100_p = 1_h + 22_h + 77_h$ on the cosets of a subgroup M' which is the automorphism group of $M = M_{22}$. Using only this specific information about H', we compute its characters.

Each of the first eighteen characters d_h of H induces a pair of associated characters d_h and d_h' of H' of the same degree. The alternating character $1_h'$ is negative on the non-H-classes C_h' of H', and we set $d_h' = d_h \times 1_h'$, where the first nonzero value of $d_h - d_h'$ is positive. Thus $d_h \geqslant 0$ on the class of involutions C_1'. To determine whether or not the pairs of characters 154_i and 154_j, 770_i and 770_j, 896_i and 896_j fuse or not in H', and whether the fused representation is of real or symplectic type we examine the differences $\zeta_t' = \zeta_t^{H'} - \zeta_t^H$ between the number of square roots in H' or H of an element of class C_t, and demand that this be nonnegative. Its value is

$$\zeta_t' = \zeta_t^H + (c_1 - 2)(154_i + 154_j)_t + c_2(770_i + 770_j)_t + c_3(896_i + 896_j)_t \tag{7.1}$$

where $c_i = 1$ for real fused characters, $c_i = -1$ for symplectic fused characters, $c_i = 0$ for complex unfused characters, but $c_1 = 2$ if the real characters 154_i and 154_j of H each induce two real irreducible characters of H'. For $t = 14, 18, 19$ the inequalities

$$\zeta_{14}' = 1 + (c_1 - 2)2 + c_2(0) + c_3(2) \geqslant 0, \tag{7.2}$$

$$\zeta_{19}' = 0 + (c_1 - 2)(-2) + c_2(-2) + c_3(0) \geqslant 0 \tag{7.3}$$

$$\zeta_{18}' = 0 + (c_1 - 2)(-2) + c_2(2) + c_3(-4) \geqslant 0 \tag{7.4}$$

imply that $2 - c_1 = c_2 = c_3$. If H' were the direct product of H with an outside involution, no characters fuse and $c_2 = 0$. Otherwise, $c_i = 1$ and the three character pairs fuse into real characters 308_h, 1540_h and 1792_h sof H', respectively, which vanish on classes outside H. Besides these three self-associated characters, H' has 18 pairs of associated characters, all real. Table 3 lists one character from each associated pair.

The three pairs of characters of H which fuse in H' differ in H, respectively, on classes C_{12} and C_{13}, C_{21} and C_{22}, C_2 and C_3, so these pairs of H-classes fuse in H', leaving 21 H-classes in H'. We seek the characters of the 18 classes of H' not in H, denoted $C_1' \cdots C_{18}'$. Of the first ten of these, all but C_8' are represented in M'. Class C_4 of H' splits into two 7-classes C_4 and $C_4{}^*$ of M', and class C_2' of H' splits into two 14-classes C_2' and C_2'' of M'. Each other class of M' is denoted by the symbol C_s' for the class of H' in which it lies.

We next derive the ten centralizer orders ${}^\circ N_s'$ of the non-M-classes C_s' of M' from the centralizer orders ${}^\circ N_t$ of the eight classes C_t of M' in which the squares of C_s' elements are found, and from the number ζ_t' of square roots of an element of C_t that lie in non-M-classes of M'. The latter is found by subtracting from $\zeta_t = \sum_i \chi_t^i$ (since $c_i = 1$) the ratios ${}^\circ N_t/{}^\circ N_s$ for M-classes C_s with squares in C_t. For the eight classes C_t with $\zeta_t' > 0$ we list the values of ζ_t' and ${}^\circ N_t$, the orders ${}^\circ m_s'$ of square roots m_s', and the classes C_s' and centralizer orders ${}^\circ N_s'$ for m_s'.

C_t	C_1	C_4	$C_4{}^*$	C_5	C_6	C_7	C_8	C_9	
ζ_t	1716	1	1	1	6	2	20	4	
$^\circ N_t$	$22 \cdot 8!$	14	14	10	72	24	768	64	(7.5)
$m_s{}'$	2	14	14	10	6	12	4	8	
$C_s{}'$	$C_1{}', C_{10}'$	$C_2{}'$	C_2''	$C_9{}'$	$C_3{}'$	$C_4{}'$	$C_5{}', C_6{}'$	$C_7{}'$	
$N_s{}'$	2688, 640	14	14	10	12	12	96, 64	16	

The last line of the table is $^\circ N_s{}' = {}^\circ N_t/\zeta_t{}'$ in the six cases where this is an integer. In the other two cases, $t = 1$ and 8, $\zeta_t{}' = \sum {}^\circ N_t/{}^\circ N_s{}'$ is a sum of two factors of $^\circ N_t$, to be determined by conditions on the two values of $^\circ N_s{}'$. Only one class of involutions, called $C_1{}'$, contains the 7th powers of 14-elements in $C_2{}'$ or C_2'', and no involutions commute with 11-elements. Only one class of involutions, called C_{10}', contains the 5th powers of 10-elements in $C_9{}'$. Hence $\zeta_1{}' = 1716$ is the sum of a multiple $^\circ N_1/{}^\circ N_1{}'$ of 55 and a multiple $^\circ N_1/{}^\circ N_{10}'$ of 77, and these can only be 330 and 1386. Thus $^\circ N_1{}' = 2688$ and $^\circ N_{10}' = 640$. Similarly only one of the classes of 4-elements, say $C_5{}'$, contains cubes of 12-elements in $C_4{}'$. Hence 3 divides $^\circ N_5{}'$ and $^\circ N_8/{}^\circ N_6{}'$ but not $^\circ N_6{}'$, so $\zeta_8{}' = 20 = 8 + 12$, and $^\circ N_5{}' = 96$, $^\circ N_6{}' = 64$.

Congruence relations (mod p) between the characters of an element and of its p-th power furnish much information about the irreducible characters in the non-M-classes of M'. First, a character is even or odd according as the known character of the square is even or odd. Second, the small size of $^\circ N_s{}'$ in classes $C_2{}'$, C_2'', $C_3{}'$, $C_4{}'$, $C_7{}'$ and $C_9{}'$ (namely 14, 14, 12, 12, 16, 10) forces all the even-valued characters to vanish. Thirdly, since $C_1{}'$ contains the 7th powers of elements of $C_2{}'$, and the cubes of elements in $C_3{}'$, and has its squares in C_1, the character values in $C_3{}'$ must be odd multiples of 7 for 21_m, 231_m and 385_m, multiples of 14 for 154_m and 210_m, and are not divisible by 7 for 45_m, 45_n, 55_m, 99_m. They must be multiples of 3 for degrees 45, 99, 385, and 560, but not for the rest. For 55_m and 99_m they are congruent to ± 1 (mod 14) and for 45_m and 45_n they are congruent to ± 3 (mod 14). The least possible values of the ten characters d_m with positive values on $C_1{}'$ are as follows:

Character:	1_m	21_m	45_m	45_n	55_m	99_m	154_m	210_m	231_m	385_m	(7.6)
Value on $C_1{}'$:	1,	7,	3,	3,	13,	15,	14,	14,	7,	21.	

Since the sum of squares is $1344 = {}^\circ N_1{}'/2$, these are the exact values. Their negatives are the values of the associated characters $d_m{}'$ which are not tabulated. The self associated character 560_m which vanishes is also omitted. The characters 45_m and 45_n have irrational values c and $\bar{c}$ on $C_2{}'$ and C_2'', where $c = (-1 + i\sqrt{7})/2$ is the sum of three seventh roots of unity. The remaining rational values on C_2' are obtained by congruence (mod 7) with C_1' and (mod 2) with C_4. (We omit class C_2'' in listing characters of M' below, since it is inverse to C_2' and fuses with C_2' in H'). Likewise the character values of C_3' are obtained by congruence (mod 3) with C_3' and (mod 2) with C_6.

$$\begin{array}{llllllllllll} C_2': & 1 & 0 & c & \bar{c} & -1 & 1 & 0 & 0 & 0 & 0 \\ C_3': & 1 & 1 & 0 & 0 & 1 & 0 & -1 & -1 & 1 & 0. \end{array} \tag{7.7}$$

In both cases we double the sum of squared absolute values to obtain the centralizer orders ${}^cN_s' = 14$ or 12. Since C_2' is 7-singular and C_3' is 3-singular, we can now determine the $7d1$- and $3d1$-blocks of M' from those of M. Thus we have

$$1_m - 55_m + 99_m - 45_m = 0 \quad \text{in 7-regular classes,} \tag{7.8}$$

$$21_m' - 231_m + 210_m = 0 \quad \text{in 3-regular classes,} \tag{7.9}$$

$$21_m' = -231_m + 210_m \quad \text{in 3-regular classes.}$$

Note that $21_m' = -21_m$ outside M. The product of each of these p-blocks by the alternating character gives an associated p-block of M'.

For the class C_5' of 4-elements that are cubes of 12-elements in C_4', the values must be ± 3 on the character 45_m, 45_n, 99_m, and 385_m of odd degree which vanish on C_4'. The squares of the other six nonzero d_m-characters sum to $96/2 - 4(3^2) = 12$, so the values are ± 2 for even degrees 154 and 210, and ± 1 for the rest. Characters of involutions in C_{10}' are congruent (mod 2) to their degrees and (mod 5) to the characters of 10-elements in C_9'. Hence they are ± 5 for degrees 45, 55, 385, they are ± 4 or ± 6 for 154_m, and ± 1, ± 9 or ± 11 for degrees 21, 99, 231. The values $+9$ or ± 11 for 21_m would imply the values $(9^2 - 21)/2 = 20$ or 50 for $210_m = 21_m^{[1^2]}$, which are too big. Hence 21_m is ± 1 and 210_m is $(1^2 - 21)/2 = -10$ on C_{10}'. The sum of squares in C_{10}' for 99_m, 154_m, and 231_m is $640/2 - 1^2 - 1^2 - 4(5^2) - 10^2 = 118$, so the three remaining squares are $1^2, 6^2, 9^2$. From the $3d1$-block relation (7.9) we find that $21_m = -1$ and $231_m = -9$ on C_{10}', so $99_m{}^2 = 1$. From the $7d1$-block relation (7.8) we get $99_m = -1$ on C_{10}'. The scalar product of C_1' and C_{10}' now forces 154_m to be 6, and determines the signs of the values, ± 5. The character values in C_9' are the least residues (mod 5) of those in C_{10}'. Thus for C_9' and C_{10}' we have

Character:	1_m	21_m	45_m	45_m	55_m	99_m	154_m	210_m	231_m	395_m	
Value on C_9':	1	-1	0	0	0	-1	1	0	1	0,	(7.10)
Value on C_{10}':	1	-1	-5	-5	5	-1	6	-10	-9	5.	

The $5d1$-block relation is now seen to be

$$1_m - 99_m + 231_m - 154_m' + 21_m' = 0 \text{ on 5-regular classes.} \tag{7.11}$$

The character 21_m can now be evaluated on all non-M-classes of M'. It is 7 on C_1', 0 on C_2' and C_2'', and ± 1 on all other classes except for the 4-elements of C_6', where it is an odd number x. Since the sum of $(21_m{}^2 - 1_m{}^2)_s/{}^0N_s'$ vanishes over the C_s', we have

$$(7^2 - 1)/2688 + 2(0 - 1)/14 + (x^2 - 1)/64 = 0. \tag{7.12}$$

Thus $x = \pm 3$. Omitting class C_2'', and inserting an asterisk for class C_8' which is not in M', we have

$$21_m = (7, 0, 1, \pm 1, \pm 1, \pm 3, \pm 1, *, -1, -1).$$

The scalar product with 1_m vanishes only if signs are such that

$$21_m = (7, 0, 1, -1, -1, 3, 1, *, -1, -1). \tag{7.13}$$

The alternating Kronecker square of 21_m is

$$210_m = (14, 0, -1, 1, -2, 2, 0, *, 0, -10). \tag{7.14}$$

From the $3d1$-relation (7.9) we obtain

$$231_m = (7, 0, 1, -1, -1, -1, -1, *, 1, -9). \tag{7.15}$$

Since $21_m{}^2 = 1_m + 21_m + 55_m + 154_m + 210_m$, we have

$$55_m + 154_m = (27, -1, 0, 0, 3, 3, -1, *, 1, 11). \tag{7.16}$$

Previous results show that both 3's split as $1 + 2$. Congruences (mod 3) then split both 0's as $1 - 1$. Thus

$$\begin{aligned} 55_m &= (13, -1, 1, 1, 1, 1, -1, *, 0, 5), \\ 154_m &= (14, 0, -1, -1, 2, 2, 0, *, 1, 6). \end{aligned} \tag{7.17}$$

Now the $5d1$-block relation (7.11) determines

$$99_m = (15, 1, 0, 0, 3, -1, -1, *, -1, -1) \tag{7.18}$$

and the $7d1$-block relation (7.8) determines

$$45_m = (3, (c, \bar{c}), 0, 0, 3, -1, 1, *, 0, -5). \tag{7.19}$$

Finally, since $21_m \times 45_m = 385_m + 560_m$ and $560_m = 0$, we have

$$385_m = (21, 0, 0, 0, -3, -3, 1, *, 0, 5). \tag{7.20}$$

This completes the character table of M' in Table 1.

TABLE I

Characters of the Automorphism Group M' of M_{22}

Class	Element orders												N_k^M
C_1	1	1_m	21_m	45_m	45_n	55_m	99_m	154_m	210_m	231_m	385_m	560_m	$22 \cdot 8!$
C_2	11	1	-1	1	1	0	0	0	1	0	0	-1	11
C_4	7	1	0	c	$\bar{c}$	-1	1	0	0	0	0	0	14
C_4^*	7	1	0	$\bar{c}$	c	-1	1	0	0	0	0	0	14
C_5	5	1	1	0	0	0	-1	1	0	1	0	0	10
C_6	3	1	3	0	0	1	0	1	3	-3	-2	2	72
C_7	6	1	-1	0	0	1	0	1	-1	1	-2	2	24
C_8	2	1	5	-3	-3	7	3	10	2	7	1	-16	768
C_9	4	1	1	1	1	3	3	2	-2	1	1	0	64
C_{10}	4	1	1	1	1	-1	-1	2	-2	-1	1	0	32
C_{11}	8	1	-1	-1	-1	1	1	0	0	-1	1	0	16
C_1'	2	1	7	3	3	13	15	14	14	7	21	·	2688
C_2'	14	1	0	c	$\bar{c}$	-1	1	0	0	0	0	·	14
C_2''	14	1	0	$\bar{c}$	c	-1	1	0	0	0	0	·	14
C_3'	6	1	1	0	0	1	0	-1	-1	1	0	·	12
C_4'	12	1	-1	0	0	1	0	-1	1	-1	0	·	12
C_5'	4	1	-1	3	3	1	3	2	-2	1	-3	·	96
C_6'	4	1	3	1	-1	1	-1	2	2	-1	-3	·	64
C_7'	8	1	1	1	1	-1	1	0	0	-1	1	·	16
C_9'	10	1	-1	0	0	0	-1	1	0	1	0	·	10
C_{10}'	2	1	1	-5	-5	5	-1	6	-10	-9	5	·	640

The characters of H' can now be evaluated on the nine classes of H' represented in M' but not in H. From (7.13) and (7.17) we find the value on these classes of $3(1_m) + 2(21_m) + 55_m$, which is the restriction to M' of the permutation character $100_p = 1_h + 22_h + 77_h$. We then multiply by the centralizer orders ${}^\circ N_s'$ in M' found in (7.5) to obtain the centralizer orders ${}^\circ N_s^{H'}$ for H', except that half the product is used for the fused class C_2'. We describe each class by its cycle symbol for the character 22_h.

H' Class:	C_1'	C_2'	C_3'	C_4'	C_5'	C_6'	C_7'	C_9'	C_{10}',
Class type:	$1^8 2^7$	$1 \cdot 7 \cdot 14$	$1^2 2 3^2 6^2$	$4 \cdot 6 \cdot 12$	$2^3 4^4$	$1^4 2 4^4$	$1^2 4 8^2$	$2 \cdot 10^2$	2^{11},
100_p:	30	2	6	2	2	10	4	1	6,
${}^\circ N_S^{N'}$:	80640	14	72	24	192	640	64	10	3840.

(7.21)

The induce-restrict table (Table 2) from M to H can be used as an induce-restrict table from M' to H', if it is modified to show which of the two associated characters χ_h or χ_h' in $d_m^{H'}$ corresponds to χ_h in $d_m{}^H$. If χ_h occurs once, we indicate a multiplicity 1; whereas if χ_h' occurs instead, we indicate 1′. If χ_h appears twice we show 2, whereas if both χ_h and χ_h' occur we show 2′, and if χ_h' occurs twice we show 2″. To determine which is present, we examine the signs in the involution class C_1'.

We define 22_h to be $1_m + 21_m$ in the M'-classes of H', and readily verify that all the irreducible components of the restriction to M' of $22_h{}^2$ have positive values on C_1'. Thus in the M'-classes of H' we compute

$$77_h = 1_m + 21_m + 55_m\,, \qquad 175_h = 21_m + 154_m\,, \qquad 231_h = 21_m + 210_m\,. \tag{7.22}$$

The composite characters $825_h + 770_h$ and $1925_h + 770_h$ defined by (3.16) and (3.17) have values 139 and 161, respectively, on class C_1' of H', so in formulas (3.15) the characters all must have positive values on C_1', except that the pair $280_m + 280_n$ is fused into 560_m which vanishes on C_1'. Hence 825_h, 770_h, and 1925_h are defined by (3.15) in all M'-classes of H'. However, the character 1925_k in (3.23) has the value $9 \cdot 8 \cdot 7/3 - 8 - 139 = 21$ in C_1', and is a sum of 5-indecomposables when restricted to M'. Hence by (7.11), 154_m goes with $231_m'$, and the decomposition of 1925_k in M'-classes must be

$$1925_k = 154_m + 210_m + 231_m' + 560_m + 385_m + 385_m'. \tag{7.23}$$

The character 1056_h in H was found in (4.2) to belong in a $3d1$-block with 231_h and 825_h, whose values on C_1' are 21 and 69. Since ${}^\circ C_1' = 1100$, and 32 divides 1056, the character on C_1' is a multiple of 8, and cannot be $21 + 69 = 90$. Hence $1056_h = 48$ on C_1' and the $3d1$-chain (4.2) links $231_h'$ with 1056_h in a 3-indecomposable character. This determines 1056_h on the M'-classes. Its decomposition in M' is

$$1056_h \overset{M'}{=} 55_m + 154_m + 231_m + 231_m' + 385_m\,. \tag{7.24}$$

Since the reducible character $693_h + 154_h$ of (4.4) has the value 91 on C_1', all the components of its restriction to M' are positive on C_1', and formulas (4.5) and (4.6) give 693_h and 154_h without modification. Their values 63 and 28 on C_1' are checked by (4.9).

Since 3200 and 1408 are both divisible by 2^7, their character values on the class C_1' of order 1100 are divisible by 2^5. The values of the characters 175_h, 825_h, 1925_h, and 1925_k on C_1' are 21, 69, 91, 21, respectively. The only combinations of signs which give a positive sum divisible by 32 are

TABLE II

The Frobenius Induce-Restrict Table for the Automorphism Groups M' of M_{22} and H' of the Higman–Sims Simple Group H

M' \ H'	1_h	22_h	77_h	175_h	231_h	1056_h	825_h	770_h	1925_h	1925_k	3200_h	1408_h	2750_h	1750_h	693_h	154_h	1386_h	2520_h	308_h	1540_h	1792_h
1_m	1	1	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
21_m	.	1	1	1	1	.	1	1	.	.	.	.	.	.	.	.	.	.	.	.	.
45_m	.	.	.	.	.	.	.	.	.	.	.	.	1	1	.	.	.	.	.	.	.
45_n	.	.	.	.	.	.	.	.	.	.	.	.	1	1	.	.	.	.	.	.	.
55_m	.	.	1	.	.	1	1	1	1	.	.	.	.	.	1	1	.	.	.	.	.
99_m	.	.	.	.	.	.	.	1	1	.	1	1	.	1	1	1	.	.	.	.	.
154_m	.	.	.	1	.	1	1	1	.	1	1	1	.	.	1	.	.	2′	1*	.	.
210_m	.	.	.	.	1	.	1	1	1	1	1	1	1	.	.	.	1	2′	.	1*	.
231_m	.	.	.	.	.	2′	.	1	1	1′	2′	.	.	1	.	.	1	2′	.	.	1*
385_m	.	.	.	.	.	1	1	.	2	2′	3′	1	2′	2	1	.	1′	2′	.	.	1*
560_m	.	.	.	.	.	.	.	.	1*	1*	2*	1*	3*	1*	.	.	1*	1*	.	2	1

Note: Only one character from each associated pair in M' and H' is listed in the table, usually the one with positive values on class C_1'. The table entries 1′, 2′, 3′ mean that one of the components is the associated character, and 0, 1, or 2 components are the one listed. The entries 1*, 2*, 3* mean that both the indicated character and its associate appear in the self-associated character with the indicated multiplicity.

$21 - 69 + 91 + 21 = 64$ or $\pm 21 + 69 + 91 \mp 21 = 160$. The latter gives too large a value to the sum of squares for class C_1' and must be rejected. Hence $3200_h = 64$ on C_1' and the chain (4.11) is valid in all classes of H'. However, the decomposition (4.12) in M must be modified for M'. Since 3200_h is a sum of 5-indecomposables in M', we see by (7.11) that 99_m goes with 231_m but 154_m goes with $231_m'$, so

$$3200_h = 99_m + 154_m + 210_m + 231_m + 231_m' + 2(560_m + 385_m) + 385_m'. \tag{7.25}$$

Since the value of 1408_h given by (4.21) must be a multiple of 32 on class C_1' where 99_m, 154_m, 210_m and 385_m have values 15, 14, 14, 21, the only possible combination of signs is $15 + 14 + 14 + 21 = 64$. Thus the value of 1408_h on C_1' is 64 and (4.21) gives 1408_h for M' classes of H'. Then (4.13) makes $3158_r = 134$, (4.14) makes $1750_h = 70$, and (4.20) makes $2750_h = 20$ on C_1'. Formula (4.21) holds for the decomposition of 1750_h in M', but the formula for 2750_h must be modified by changing one of the 385_m to $385_m'$.

The sum of the squares of the values on H' of those 16 computed characters is 8!; so the characters 1386_h and 2520_h must vanish on C_1'. Hence in (4.24) the pairs of characters of degree 154, 210, 231, and 385 must be replaced by pairs of associated characters whose sum vanishes on all M'-classes of H'. Taking the sum of squares on the other eight M'-classes of H', we find the magnitudes of the character values of 1386_h, and then determine the signs by orthogonality. This completes the characters for the M'-classes of H'.

8. Characters of Non-M'-Classes of H'

We next calculate the orders of centralizers of non-M'-classes of H' by evaluating the number of square roots $\zeta_t = \sum_i \chi_t^i$ of an element in class C_t of H', subtracting the numbers in H-classes, and computing the numbers ζ_t' and ζ_2'' that lie respectively in M'-classes and non-M'-classes of H' not represented in H. The nine non-M'-classes C_s'' have squares in seven classes C_t:

C_t	C_6	C_9	C_{14}	C_{15}	C_{16}	C_{17}	C_{23}	
ζ_t	52	32	2	26	2	72	6	
ζ_t'	10	8	0	0	0	0	0	
ζ_t''	16	8	1	10	2	72	4	
${}^cN_t$	720	512	30	600	40	5760	40	
${}^cx_s$	6	8	30	10	20	4	20	
C_s''	C_{11}', C_{12}'	C_8'	C_{13}'	C_{14}'	C_{15}'	C_{16}'	C_{17}'	C_{18}'
${}^cN_s$	48, 720	64	30	60	20	80	20	20

(8.1)

In each case here the ratio ${}^{\circ}N_t/\zeta_t''$ is an integer, but for $t = 6$ and 23 this integer cannot be ${}^{\circ}N_s$, since it is not a multiple of the order ${}^{\circ}x_s$ of an element in the class. A splitting must occur for these two cases, but not for the other five cases. Just one of the two classes of 6-elements with squares in C_6 contains 5th powers of elements of order 30; so ζ_6'' splits as $15 + 1$ or $10 + 6$. Since the cubes of elements in the other class C_{11}' lie in C_{10}', the integer ${}^{\circ}N_{11}'$ divides 3840 and cannot be 72. It is 48. Since 20 divides both ${}^{\circ}N_{17}'$ and ${}^{\circ}N_{18}'$, and since the sum of their reciprocals is $\zeta_{23}''/{}^{\circ}N_{23} = 1/10$, it follows that both are 20. Thus the remaining centralizer orders for H' are given in the last line of (8.1).

The class C_{11}' was assigned to 6-elements whose cubes are involutions in C_{10}'. These must exist since ${}^{\circ}N_{10}' = 3840$ is divisible by 3. The characters in C_{11}' are congruent (mod 3) to the characters of their cubes in C_{10}' and (mod 2) to the characters of their squares in C_6. Values between -2 and $+2$ which satisfy these conditions (mod 6) have square sum 48 equal to ${}^{\circ}N_{11}'$; so these values are exact.

Next we calculate the character 22_h in all remaining classes. We observe from (3.18), (3.19), and (3.25) that

$$22_h(231_h - 175_h) = 1540_h - 308_h$$
$$= 0 \text{ for non-}H\text{-classes of } H'. \tag{8.2}$$

Since the permutation character 100_p vanishes on non-M'-classes of H', we have

$$77_h = -1_h - 22_h \text{ for non-}M'\text{-classes,} \tag{8.3}$$

$$175_h + 231_h = 22_h^2 - 1_h - 77_h$$
$$= 22_h(22_h + 1_h) \text{ for non-}M'\text{-classes.} \tag{8.4}$$

If x denotes an element of some non-M'-class of H', then

$$22_h(x^2) = 22_h^{[2]}(x) - 22_h^{[1^2]}(x)$$
$$= 1 + 77_h(x) + 175_h(x) - 231_h(x). \tag{8.5}$$

Hence by (8.3) we have

$$22_h(x^2) + 22_h(x) = 175_h(x) - 231_h(x). \tag{8.6}$$

THEOREM 8.1. *The character value $22_h(x)$ for x in a class of H' not represented in H or M' is either equal to 0 or to minus the character $22_h(x^2)$ of the square of the element*:

$$22_h(x) = 0 \text{ or } -22_h(x^2) \text{ for non-}H\text{, non-}M'\text{-classes.} \tag{8.7}$$

The proof follows from (8.2) and (8.6).

In classes C'_{13}, C'_{17}, and C'_{18} the character 22_h is odd; so it cannot be 0. Hence Theorem 8.1 applies and their values of 22_h are $1, -1, -1$, respectively. Since the 30-elements of C'_{13} have their 5th powers in C'_{12} and their cubes in C'_{14}, character 22_h cannot vanish on these classes and must have values -4 and -2 by Theorem 8.1. In the class C'_{15} of 20-elements, where ${}^{c}N'_{15} = 20$, the character values have the same parity as in the class C_{16} that contains the squares. Hence the odd values in C'_{15} are 1 or -1 and the even values all vanish. Thus 22_h is 0 in C'_{15} and also in the class C'_{16} of 4-elements which are 5th powers of elements in C'_{15}. The one remaining value of 22_h for the 8-elements in class C_8' is found to be -2 by forming the scalar product of 22_h with 1_h. The character values of 22_h, the orders of elements, and the class types for the nine H' classes not represented in H or M' are the following:

(8.8)

Class	C_8'	C'_{11}	C'_{12}	C'_{13}	C'_{14}	C'_{15}	C'_{16}	C'_{17}	C'_{18}
Element order	8	6	6	30	10	20	4	20	20
22_h	-2	0	-4	1	-2	0	0	-1	-1
Class Type	$\bar{1}^2 4 8^2$	$2^2 6^3$	$\bar{1}^4 3^4 6$	$\overline{35} \cdot 15/\bar{1}$	$\bar{1}^2 5^2 10$	$\bar{2} \cdot 20$	$\bar{2} \cdot 4^5$	$\bar{1}4^{-1} \cdot 5 \cdot 20$	

Knowing the character 22_h, we then compute 77_h from (8.3), 231_h from $22_h^{[1^2]}$, and 175_h from (8.6), in all classes of (8.8). Then in 3-singular classes we compute 825_h and 1056_h from 231_h by the $3d1$-block relation (7.9), and in 5-singular classes we compute 825_h, 1925_h, 1925_k, and 3200_h from 175_h by the $5d1$-block relation (7.11). Since only classes C_8' and C'_{16} are neither 3-singular nor 5-singular, we easily find the value of 825_h on the classes of (8.8)

$$825_h = (1, 2, -6, -1, -1, 1, 1, 0, 0).$$

Then 770_h, 1925_h, and 1925_k are found for all classes from (6.5), 1056_h is determined in 3-regular classes by (7.9) and 3200_h is found in 5-regular classes from (7.11).

In classes C'_{15} and C'_{16} the only remaining nonzero values are in characters of degree 1408 and 693. These are found by scalar products with C_1' characters to be -1, 4 for 1408_h and 1, 1 for 693_h. The scalar product of 3200_h and 1408_h involves $-1/20 - 4^2/80 = -1/4$ on classes C'_{15} and C'_{16}; so 3200_h and 1408_h must be equal on all other non-H-classes of H'. The character 693_h is now determined by (4.9) and is completely checked by scalar products with class C'_{15}.

Now scalar products with class C_2' determine the values of 2750_h. The vanishing of $45_m{}^H$ on non-M'-classes shows that $1750_h = -2750_h$ on these classes. Scalar products with classes C_3' and C_4' determine 154_h and check the calculations, and scalar products with C_9' determine 1386_h.

TABLE IIIa

Characters of the Automorphism Group H' of the Higman–Sims Simple Group H for Classes Represented in the Subgroup H

Class	Class type	1_h	22_h	77_h	175_h	231_h	1056_h	825_h	770_h	1925_h	$1925_{\bar{h}}$	3200_h	1408_h	2750_h	1750_h	693_h	154_h	1386_h	2520_h	308_h	1540_h	1792_h	$^\circ N_k^{H'}$
C_1	1^{22}																						$2200 \cdot 8!$
$C_{2,3}$	11^2	1	0	0	−1	0	0	0	0	0	0	−1	0	0	1	0	0	0	1	0	0	−1*	11
C_4	$1 \cdot 7^3$	1	1	0	0	0	−1	−1	0	0	0	1	1	−1	0	0	0	0	0	0	0	0	14
C_5	$1^2 5^4$	1	2	2	0	1	1	0	0	0	0	0	−2	0	0	−2	−1	1	0	−2	0	2	50
C_6	$1^4 3^6$	1	4	5	4	6	−6	6	5	−1	−1	−4	4	5	−5	0	1	0	0	2	10	−8	720
C_7	$2^2 3^2 6^2$	1	0	1	0	−2	2	−2	1	−1	−1	0	0	1	−1	0	1	0	0	2	2	0	48
C_8	$1^6 2^8$	1	6	13	15	7	32	25	34	5	5	0	0	−50	−10	21	10	−6	24	20	−28	0	15360
C_9	$1^2 2^2 4^4$	1	2	5	−1	−1	0	1	2	5	−3	0	0	2	6	5	6	−2	−8	−4	−4	0	512
C_{10}	$1^2 2^2 4^4$	1	2	1	3	−1	0	1	−2	−3	1	0	0	2	2	1	−2	−2	0	4	4	0	128
C_{11}	$2 \cdot 4 \cdot 8^2$	1	0	1	−1	−1	0	1	−2	1	1	0	0	0	−2	1	0	0	0	0	0	0	32
$C_{12,13}$	$2 \cdot 4 \cdot 8^2$	1	0	−1	1	−1	0	1	0	1	−1	0	0	0	0	−1	0	0	0	0*	0	0	16
C_{14}	$1^{-1} 3 \cdot 5 \cdot 15$	1	−1	0	−1	1	−1	1	0	−1	−1	1	−1	0	0	0	1	0	0	2	0	2	30
C_{15}	$1^2 5^4$	1	2	−3	5	1	−4	−5	0	5	5	−5	−7	0	0	3	4	6	0	8	0	2	600
C_{16}	$\bar{1}^2 10^2$	1	−2	1	1	1	0	−1	0	1	1	−1	1	0	0	−1	0	−2	0	0	0	2	40
C_{17}	$\bar{1}^2 2^{10}$	1	−2	1	11	−9	0	9	−10	−19	1	−16	16	−10	10	9	10	18	0	−20	20	32	5760
C_{18}	$\bar{1}^2 2 \cdot 6^3$	1	−2	1	2	0	0	0	−1	−1	1	2	−2	−1	1	0	1	0	0	−2	2	−4	72
C_{19}	$\bar{3}^2 4 \cdot 12$	1	0	−1	0	0	0	0	1	−1	1	0	0	1	−1	0	1	0	0	−2	−2	0	24
C_{20}	$\bar{1}^6 4^4$	1	−6	5	15	15	0	−15	−14	5	−35	0	0	10	−10	21	−2	6	24	−20	−20	0	7680
$C_{21,22}$	$\bar{1} \cdot 4^{-1} \bar{5} \cdot 20$	1	−1	0	0	0	0	0	1	0	0	0	0	0	0	1	−2	1	−1	0	0*	0	20
C_{23}	$1 \cdot 2^{-2} 5 \cdot 10^2$	1	1	−2	0	2	2	0	−1	0	0	0	0	0	0	1	0	−1	−1	0	2	0	40
C_{24}	$1^{-3} 5^5$	1	−3	2	0	6	6	0	−5	0	0	0	8	0	0	−7	4	11	−5	8	−10	−8	1000

Note: The entries indicated by an asterisk in each of the self-associated characters 308_h, 1540_h, and 1792_h split in H as follows: $0 = 2 - 2$ for 308_h, $0 = 5^{1/2}i - 5^{1/2}i$ in 1540_h, and $-1 = \frac{1}{2}(-1 + 11^{1/2}i) + \frac{1}{2}(-1 - 11^{1/2}i)$ for 1792_h.

TABLE IIIb

Characters of the Automorphism Group H' of the Higman–Sims Group H for classes not Represented in the Subgroup H

Class	Class type	1_h	22_h	77_h	175_h	231_h	1056_h	825_h	770_h	1925_h	1925_k	3200_h	1408_h	2750_h	1750_h	693_h	154_h	1386_h	2520_h	$^{\circ}N_k^{H'}$
C_1'	$1^8 2^7$	1	8	21	21	21	48	69	70	91	21	64	64	20	70	63	28	0	0	80640
C_2'	$1 \cdot 7 \cdot 14$	1	1	0	0	0	−1	−1	0	0	0	1	1	−1	0	0	0	0	0	14
C_3'	$1^2 2 \cdot 3^2 6^2$	1	2	3	0	0	0	0	1	1	−3	−2	−2	−1	1	0	1	0	0	72
C_4'	$4 \cdot 6 \cdot 12$	1	0	1	−2	0	0	0	−1	1	1	0	0	1	−1	0	1	0	0	24
C_5'	$2^3 4^4$	1	0	1	1	−3	0	−3	2	−5	1	0	0	4	2	3	4	0	0	192
C_6'	$1^4 2 \cdot 4^4$	1	4	5	5	5	0	5	6	−5	5	0	0	0	−10	−1	0	4	0	640
C_7'	$1^2 4 \cdot 8^2$	1	2	1	1	1	0	1	−2	−1	1	0	0	2	2	−1	−2	−2	0	64
C_8'	$\bar{1}^2 4 \cdot 8^2$	1	−2	1	1	1	0	1	−2	−1	1	0	0	−2	2	−1	2	2	0	64
C_9'	$2 \cdot 10^2$	1	0	0	0	−1	1	0	0	0	0	0	0	0	0	0	−1	1	0	10
C_{10}'	2^{11}	1	0	5	5	−11	16	5	−10	−5	5	0	0	−20	−10	15	4	−24	0	3840
C_{11}'	$2^2 6^3$	1	0	−1	2	−2	−2	2	−1	1	−1	0	0	1	−1	0	1	0	0	48
C_{12}'	$\bar{1}^4 3^4 6$	1	−4	3	6	6	6	−6	−5	1	−9	4	4	5	−5	0	1	0	0	720
C_{13}'	$\bar{1}^{-1}\bar{3} \cdot \bar{5} \cdot 15$	1	1	−2	1	1	1	−1	0	1	1	−1	−1	0	0	0	1	0	0	30
C_{14}'	$\bar{1}^2 5^2 10$	1	−2	1	1	1	−2	−1	0	1	1	−1	−1	0	0	3	−2	0	0	60
C_{15}'	$\bar{2} \cdot 20$	1	0	−1	−1	1	0	1	0	−1	−1	1	−1	0	0	1	0	0	0	20
C_{16}'	$\bar{2} \cdot 4^5$	1	0	−1	−1	1	0	1	0	−1	−1	−4	4	0	0	1	0	0	0	80
C_{17}'	$\bar{1} \cdot 4^{-1} 5 \cdot 20$	1	−1	0	0	0	0	0	1	0	0	0	0	0	0	−1	0	−1	$5^{1/2}$	20
C_{18}'	$\bar{1} \cdot 4^{-1} 5 \cdot 20$	1	−1	0	0	0	0	0	1	0	0	0	0	0	0	−1	0	−1	$-5^{1/2}$	20

Note: The associates of these characters are not listed. Their values are the negatives of the values in this table. The self-associated characters 308_h, 1540_h and 1792_h are 0 in these classes.

Orthogonality checks by classes show that 2520_h must vanish on all non-H-classes of H' except the pair C'_{17} and C'_{18}. Here 2520_h assumes the irrational real values $\pm\sqrt{5}$.

This completes the character table of the automorphism group H' of the Higman Sims simple group.

9. Permutation Characters and Subgroups

The character $176_p = 1_h + 175_h$ is found to be a non-negative integral character on H satisfying all the obvious requirements for a permutation character of H. However, it is negative on class C_4' of 12-elements of H'. This suggests the existence of a subgroup G of index 176 in H, which would have index 352 in H' with permutation character $1_h + 1_h' + 175_h + 175_h'$ in H', but no subgroup of index 176 in H'. As a further check, the nonzero values of 176_p on classes C_k divide the corresponding centralizer orders ${}^\circ N_k{}^H$ to produce values of ${}^\circ N_k{}^G$ for a subgroup G of index 176 and order 252 000. We restrict the characters of H and split them to find the table of irreducible characters of G. These resemble the characters of the simple group $U = PSU_3(5)$ of order 126,000 [2], and we look for a permutation character of H of degree 352. We find two of these:

$$1_h + 175_h + 22_h + 154_i\,,$$
$$1_h + 175_h + 22_h + 154_j\,. \qquad (9.1)$$

These differ only in the classes C_{12} and C_{13} of elements of order 8; so it appears that H contains two nonconjugate subgroups each isomorphic with $PSU_3(5)$. The induce-restrict table is easily found, since the characters of U are known and those of H can be split in U. The supposed subgroup G of H contains the alternating group A_7 with index 100. Graham Higman [6] used this subgroup G, rather than M_{22}, in discovering independently a simple group later shown to be isomorphic to the simple group of D. G. Higman and C. C. Sims.

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