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n-covariation, generalized Dirichlet processes and calculus with respect to finite cubic variation processes

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Abstract

In this paper, we introduce first a natural generalization of the concept of Dirichlet process, providing significant examples. The second important tool concept is the n-covariation and the related n-variation. The n-variation of a continuous process and the n-covariation of a vector of continuous processes, are defined through a regularization procedure. We calculate explicitly the n-variation process, when it exists, of a martingale convolution. For processes having finite cubic variation, a basic stochastic calculus is developed. We prove an Itô formula and we study existence and uniqueness of the solution of a stochastic differential equation, in a symmetric-Stratonovich sense, with respect to those processes.

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1. Introduction

In the last 20 years, many authors have tried to develop a stochastic calculus beyond semimartingales. The strategy of this paper consists in 'mimicking' a "pathwise theory" for the same purpose. Pathwise type integrals are defined very often using discretization, as limit of Riemann sums: an interesting survey on the subject is a book of Dudley and Norvaisa (1999). They emphasize a big historical literature in the deterministic case.

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The first contribution in the stochastic framework has been provided by Föllmer (1981); through this significant and simply written contribution, the author wished to discuss integration with respect to a Dirichlet process X, that is to say a local martingale plus a zero quadratic variation (or sometimes zero energy) process.

In the sequel this approach has been continued by Bertoin (1986); an important break through has been realized by Lyons (1998). This author is able to integrate pathwise when the integrator process has finite p-variation paths, for $p > 1$, provided those paths fulfill a suitable condition on Lévy stochastic area. Recently, Bass et al. (2002) implement this theory for constructing Wong–Zakai approximations for solving SDEs directed by reversible Markov processes.

Since 1991, Russo and Vallois have developed a regularization procedure, whose philosophy is similar to the discretization. Their techniques are similar to a pathwise approach but are not truly pathwise. They make large use of uniform convergence in probability (*ucp*) related topology. More recently, several papers have followed those techniques, see for instance Russo and Vallois (1993, 1995, 1996), Wolf (1997) and Errami and Russo (1998).

All those articles aimed at formulating an efficient set of calculus rules allowing to relate classical and non-classical models in stochastic analysis. In particular they partially covered the following fields.

- the anticipating calculus of Skorohod type,
- the enlargement of filtrations tools,
- the case of Dirichlet processes,
- the case of Gaussian process.

Until now the framework was restricted to the case when the integrator is a finite quadratic variation process and the techniques are particularly suitable, we believe also for teaching purposes, when the integrator is continuous. This paper constitutes a first attempt to implement their tools, and a true stochastic calculus, for the case when the integrator has a finite n-variation process, where n is an integer greater than 2. Our study is motivated by many examples coming from the literature, among those one can refer to the following examples.

- The case when the integrator has the following convolution form $X_t = \int_0^t G(t,s)\,\mathrm{d}M_s$, M being a local martingale, where the kernel G is a random field.
- Typical examples having a finite n-variation (even in a strong sense) as the fractional Brownian motion B^H with $H \geqslant \frac{1}{n}$. Other examples can be constructed mixing fractional Brownian motions.
- The iterated Brownian motion (a double-sided Brownian motion indexed by an independent Brownian motion).

This paper approach does not follow the line of Lyons and coauthors. First of all, since our approach is only a "fac-simile" of a pathwise approach, processes having a finite n-(strong) variation in our case, they have finite p-variation for p strictly bigger than n. For instance, by Bass et al. (2002), the typical Markov process as diffusion process is Dirichlet, therefore it is a finite quadratic variation process. In the Lyons approach, it has a finite p-variation for $p>2$. Moreover, we aim at developing a calculus which does not only operate with the n-variation but also with the n-covariation of a vector of processes without any further assumption on the nature of the integrator process.

This paper is followed by a companion, by Gradinaru et al. (2001), in which one examines calculus for a fractional Brownian motion with Hurst index $H \geqslant \frac{1}{4}$, which is a typical (Gaussian) process having a finite 4-variation and in the "pathwise sense" a finite p-variation with p greater than 4.

About fractional Brownian motion we have to acknowledge a large amount of work. This process is, for instance, a cubic finite variation process if the Hurst index $H \geqslant \frac{1}{3}$ and it has a fourth finite variation for $H \geqslant \frac{1}{4}$. It is a semimartingale if and only if $H=\frac{1}{2}$, i.e. when it is a classical Brownian motion. Integration with respect to general Gaussian processes has been attacked using Malliavin calculus techniques (Skorohod integrals), see for instance Decreusefond and Ustunel (1998), Carmona and Coutin (1998), and Alos et al. (1999, 2001). Those techniques are quite powerful and they allow to treat integration with respect to processes, whose variation is larger than 2. However, they cannot be easily related to Riemann sums limits and for the moment, they cannot reach the barrier $H=\frac{1}{4}$. The regularization or discretization technique for those processes has been recently performed by Errami and Russo (1998), Feyel and De La Pradelle (1999), Klingenhöfer and Zähle (1999), Russo and Vallois (2000), and Zähle (1998, 2001) in the case of finite quadratic variation ($H \geqslant \frac{1}{2}$). The rough path approach of Lyons has been adapted to the fractional Brownian motion case when $H>\frac{1}{4}$ by Coutin and Qian (2002).

We briefly discuss now the content of the paper. In Section 2 we define the concept of n-variation of a continuous process and n-covariation of n continuous processes, for $n \geqslant 2$. We also introduce the notion of *weak Dirichlet process* which is essentially the sum of a continuous local $(\mathscr{F}_t)$-martingale plus a process which is "orthogonal" to those martingales. Examples of such processes are precisely the convolution of martingales as indicated before, but also $C^{0,1}$-functions of semimartingales. Both provide examples of non-finite quadratic variation processes.

A large part of the paper focuses on the convolution of a local martingale. The n-variation of that and related processes is explicitly given. If the martingale is a Brownian motion, our Proposition 2.17 constitutes a generalization of Hu–Meyer formula which appears, for instance, in Ben Arous (1989), Hu and Meyer (1988), Bardina and Jolis (2000), and Solé and Utzet (1990). In the case $n=2$, we discuss finite quadratic variation processes which are not Dirichlet.

The second part of the paper is constituted by Sections 3 and 4 and concerns, in fact, finite strong cubic variation processes. A process $X=\{X_t,\ t\in[0,1]\}$ will be said to have a finite (strong) cubic variation (or 3-variation), denoted by $[X;3]$, equals to Y if

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_0^{\cdot}(X_{s+\varepsilon}-X_s)^3\,\mathrm{d}s = Y \quad ucp,$$

$$\sup_{0<\varepsilon\leqslant 1}\frac{1}{\varepsilon}\int_0^1|X_{s+\varepsilon}-X_s|^3\,\mathrm{d}s<+\infty \quad \text{a.s.} \tag{1.1}$$

In Section 3, we prove an Itô formula with related calculus. If a process X is of finite strong cubic variation, we will prove that for every $f \in C^3$,

$$f(X_t) = f(X_0) + \int_0^t f'(X_s)\mathrm{d}^\circ X_s - \frac{1}{12}\int_0^t f^{(3)}(X_s)\,\mathrm{d}[X,X,X]_s, \tag{1.2}$$

where, $\int_0^t f'(X_s)\mathrm{d}^\circ X$ is the symmetric integral (an extension of Stratonovich integral) defined as the *ucp* limit of

$$\frac{1}{2\varepsilon}\int_0^{\cdot} f'(X_s)(X_{s+\varepsilon} - X_{s-\varepsilon})\,\mathrm{d}s.$$

In Section 4, we study an SDE of symmetric type driven by a bounded variation process and a finite cubic variation process performing Doss–Sussmann method.

Even if the results of sections 3 and 4 do not cover the whole existing literature, they are significant because they do not make other assumptions as those on the variation. In the companion paper by Gradinaru et al. (2001), the Itô formula in the spirit (1.2) type extends to the case of 4-variation but the Gaussian assumption on the process is fundamental. In that paper, a significant application of the notion of n-covariation is given showing that the 4-covariation $[g(B), B, B, B]$ exists when g is a continuous function and B is a fractional Brownian motion with Hurst index $H \geqslant \frac{1}{4}$ and it is related to the local time of B.

2. n-covariation and n-variation processes

Throughout this paper all the processes, are assumed to be continuous, indexed by the *time* variable t in [0, 1] and defined on the same complete probability space $(\Omega, \mathscr{F}, P)$ equipped with a filtration $\mathscr{F} = \{\mathscr{F}_t,\ t \in [0,1]\}$ satisfying the usual assumptions. Two continuous processes X and Y which are indistinguishable will be considered equal. Clearly, if $X(t) = Y(t)$ a.s. for all $t \in [0, 1]$, X and Y are then indistinguishable. We recall, also, that a sequence of continuous processes $\{H_n(t),\ t \in [0,1]\}$ converges in the sense of the uniform convergence in probability (*ucp*) if there exists a process H such that the sequence of random variable $\sup_{0 \leqslant t \leqslant 1} |H_n(t) - H(t)|$ converges to 0 in probability. Martingales will stand for local continuous martingales. For convenience the processes may be extended to the real line by continuity.

We denote by BV the space of continuous functions which have bounded variation on [0, 1]. We equip BV with the metrizable topology that is associated with the following convergence. A sequence (v_n) in BV converges to a function v if and only if $v_n(0) \to v(0)$ and $\mathrm{d}v_n \to \mathrm{d}v$ holds with respect to the weak $*$-topology.

Along the paper, for every process X, we will freely interchange $X(t)$ and X_t.

Remark 2.1. Let (v_n) be a sequence in BV such that

$$\sup_n \int_0^1 \mathrm{d}|v_n| < \infty. \tag{2.1}$$

Let v_n^+, v_n^- be the increasing functions such that

$$v_n = v_n^+ - v_n^- \quad \text{and} \quad |v_n| = v_n^+ + v_n^-.$$

Then there is a subsequence (n_k) such that $(v^+_{n_k})$ and $(v^-_{n_k})$ converge in BV. In fact, (2.1) implies

$$\sup_n \int_0^1 \mathrm{d}v_n^{\pm} < \infty.$$

By the Helly extraction argument, there is a subsequence (n_k) such that $(v^+_{n_k})$ and $(v^-_{n_k})$ converge, respectively, to some v^1 and v^2. In particular, the subsequence $(|v_{n_k}|)$ of the total variations converges in BV to $v^1 + v^2$.

2.1. Definitions, notations and basic calculus

Let $n \geqslant 2$, and $(X^1, X^2, \dots, X^n)$ be a vector of continuous processes. For any $\varepsilon > 0$ and $t \in [0,1]$, we set

$$[X^1, X^2, \dots, X^n]_\varepsilon(t) = \frac{1}{\varepsilon}\int_0^t \prod_{k=1}^n (X^k_{s+\varepsilon} - X^k_s)\,\mathrm{d}s,$$

and

$$\|[X^1, X^2, \dots, X^n]_\varepsilon\| = \frac{1}{\varepsilon}\int_0^1 \prod_{k=1}^n |X^k_{s+\varepsilon} - X^k_s|\,\mathrm{d}s.$$

If $[X^1, X^2, \dots, X^n]_\varepsilon$ converges *ucp*, when $\varepsilon \to 0$, then the limiting process is called the n-covariation (process) of the vector $(X^1, X^2, \dots, X^n)$, and denoted $[X^1, X^2, \dots, X^n]$. If furthermore

$$\sup_{0<\varepsilon\leqslant 1} \|[X^1, X^2, \dots, X^n]_\varepsilon\| := \|[X^1, X^2, \dots, X^n]\| < +\infty \tag{2.2}$$

we will say that it exists in the *strong sense*. Otherwise we will only say that it exists.

If the processes $\{X^k,\ k = 1, 2, \dots, n\}$ are all equal to a real valued process X, then we will simple denote $[X; n]$ the n-covariation process of the considered vector. This will be called the n-variation (process) of X. If $n = 2$ it is the quadratic variation and denoted simply by $[X]$ or $[X, X]$, see for instance Russo and Vallois (1995, 2000). Cubic variation will often indicate 3-variation. If X has a quadratic (resp. strong cubic) variation, such process will stand for *finite quadratic* (resp. *strong cubic*) *variation* process.

Remark 2.2. (1) By definition, the n-covariation is a continuous process.

(2) The map $(X^1, X^2, \dots, X^n) \to [X^1, X^2, \dots, X^n]$, when it is well defined, is a multi-linear symmetric application with values in the space of real valued continuous processes.

(3) If n is even then the existence in the strong sense of the n-variation is equivalent to the existence.

Definition 2.3. A vector $(X^1, X^2, \dots, X^m)$ of continuous processes is said to have all its *mutual* (resp. *strong*) *n-covariations* if $[X^{i_1}, X^{i_2}, \dots, X^{i_n}]$ exists (resp. exists in the strong sense) for any choice (even with repetition) of indices $i_1, i_2, \dots, i_n \in \{1, 2, \dots, m\}$.

Remark 2.4. If $n=2$ and $(X^1,\dots,X^m)$ has all its mutual brackets (or 2-covariations) then, using Remark 2.2(3) and polarization, $[X^i,X^j]$, $i,j=1,2,\dots,m$, exist in the strong sense. In particular, this happens when $X^1,X^2,\dots,X^m$ are $\mathscr{F}$-semimartingales.

Proposition 2.5. *If (2.2) holds then $[X^1,X^2,\dots,X^n]$ has bounded variation whenever it exists.*

Proof. According to Assumption (2.2), ω a.s., the total variations of the measures $[X^1,X^2,\dots,X^n]_\varepsilon$ are bounded. Then since $[X^1,X^2,\dots,X^n]$ exists, using Remark 2.1, it must be, ω a.s., of bounded variation. $\square$

Remark 2.6. (1) If n is even then $[X;n]$ is an increasing continuous process.

(2) If for every $k=1,2,\dots,n$, $\|[X^k;n]\|$ is finite then $[X^1,X^2,\dots,X^n]$ exists in the strong sense whenever it exists. Moreover by Hölder inequality, we have

$$|[X^1,X^2,\dots,X^n]|^n \leqslant \|[X^1;n]\|\ \|[X^2;n]\|\cdots\|[X^n;n]\|.$$

(3) If the n-variation $[X;n]$ exists in the strong sense for some n, then $[X;m]=0$ for all $m>n$. In particular, for any semimartingale S, $[S;n]=0$ for all $n\geqslant 3$.

(4) Suppose that $[X;n]$ exists in the strong sense, then for every continuous process Y and every $m>n$ such that $[Y;m]$ exists in the strong sense, we have

$$[X,\underbrace{Y,Y,\dots,Y}_{(m-1)\text{times}}]=0.$$

In fact,

$$\begin{aligned}
&|[X,\underbrace{Y,Y,\dots,Y}_{(m-1)\text{times}}]_\varepsilon(t)|\\
&\leqslant \|[X,Y,\dots,Y]_\varepsilon\|\\
&=\frac{1}{\varepsilon}\int_0^1 |X_{s+\varepsilon}-X_s|\,|Y_{s+\varepsilon}-Y_s|^{m-1}\,\mathrm{d}s\\
&\leqslant \left(\frac{1}{\varepsilon}\int_0^1 |X_{s+\varepsilon}-X_s|^n\,\mathrm{d}s\right)^{1/n}\left(\frac{1}{\varepsilon}\int_0^1 |Y_{s+\varepsilon}-Y_s|^{n(m-1)/(n-1)}\,\mathrm{d}s\right)^{(n-1)/n}\\
&\leqslant \left(\sup_{s\in[0,1]}|Y_{s+\varepsilon}-Y_s|^{(m-n)/(n-1)}\right)^{(n-1)/n}\|[X;n]_\varepsilon\|^{1/n}\|[Y;m]_\varepsilon\|^{(n-1)/n},
\end{aligned}$$

whose limit, using the uniform continuity of the process Y in [0, 1], is equal to zero.

(5) If $(X^1,\dots,X^n)$ has a strong n-covariation, then for every vector $(Y^1,Y^2,\dots,Y^m)$ of continuous processes, $(X^1,\dots,X^n,Y^1,Y^2,\dots,Y^m)$ has its strong $(n+m)$-covariation equal to zero.

(6) Let $(X^1,\dots,X^n)$ be a vector having a strong n-covariation and Y a continuous process. Then

$$\frac{1}{\varepsilon}\int_0^\cdot Y_s \prod_{k=1}^n (X^k_{s+\varepsilon}-X^k_s)\,\mathrm{d}s = \int_0^\cdot Y_s\,\mathrm{d}[X^1,X^2,\dots,X^n]_\varepsilon(s),$$

converges *ucp* to

$$\int_0^\cdot Y\,\mathrm{d}[X^1,X^2,\dots,X^n],$$

because, ω *a.s.*, $([X^1,X^2,\dots,X^n]_\varepsilon)$ converges in BV, up to a subsequence, to $[X^1,\dots,X^n]$.

We have a stability result through C^1 transformation.

Proposition 2.7. *Let $F^1,F^2,\dots,F^n$ be n functions in $C^1(\mathbb{R}^n)$. Let $X=(X^1,X^2,\dots,X^n)$ be a vector of continuous processes having all its strong mutual n-covariations. Then the vector $(F^1(X),F^2(X),\dots,F^n(X))$ have the same property and*

$$[F^1(X),\dots,F^n(X)](t) = \sum_{1\leqslant i_1,\dots,i_n\leqslant n}\int_0^t \partial_{i_1}F^1(X)\cdots\partial_{i_n}F^n(X)\,\mathrm{d}[X^{i_1},\dots,X^{i_n}].$$

Proof. Let $\varepsilon>0$. For every $t\in[0,1]$ we express

$$[F^1(X),\dots,F^n(X)]_\varepsilon(t) = \sum_{1\leqslant i_1\cdots,i_n\leqslant n}\int_0^t \partial_{i_1}F^1(X_s)\cdots\partial_{i_n}F^n(X_s)\,\mathrm{d}[X^{i_1},\dots,X^{i_n}]_\varepsilon(s)+R_\varepsilon(t),$$

where R_ε is a rest which converges *ucp* to zero because of the uniform continuity of the derivatives of $F^1,\dots,F^m$ on compacts. On the other hand, Remark 2.6(6) shows that

$$\int_0^\cdot \partial_{i_1}F^1(X_s)\cdots\partial_{i_n}F^n(X_s)\,\mathrm{d}[X^{i_1},\dots,X^{i_n}]_\varepsilon(s), \tag{2.3}$$

converges *ucp*, for every $1\leqslant i_1,\dots,i_n\leqslant n$, to

$$\int_0^\cdot \partial_{i_1}F^1(X)\cdots\partial_{i_n}F^n(X)\,\mathrm{d}[X^{i_1},\dots,X^{i_n}]. \tag{2.4}$$

On the other hand, a similar argument allows to show that $|F^i(X);n|$ is finite for every $1\leqslant i\leqslant n$.

Remark 2.6(2) implies then the strong existence. □

2.2. Some basic examples

In the literature many examples arise for justifying the introduction of the concept of n-covariation. We discuss some of them.

(a) *The iterated Brownian motion*. Let B^1 be a two-sided classical Brownian motion and B^2 an independent Brownian motion. It is easy to see that $B^1_{B^2}$ has a finite 4-variation equals to t.
(b) *The fractional Brownian motion*. A classical example of Gaussian process is given by the fractional Brownian motion B^H of Hurst index H. We recall that B^H is a mean zero with covariance.

$$\mathrm{Cov}(B^H_u, B^H_v) = \frac{1}{2}(u^{2H} + v^{2H} - |u-v|^{2H}) \quad \text{for } u, v \geqslant 0.$$

Remark 2.8. Let n be a positive integer. Using standard linear regression arguments as in (Russo and Vallois, 2000) one can easily prove the following:

(i) If $H \geqslant \frac{1}{n}$, B^H has a strong n-variation. If $H > \frac{1}{n}$ then the n-variation vanishes.
(ii) If $H = \frac{1}{2n}$, B^H has a $2n$-variation equals to $C_n t$ where C_n is the $2n$-moment of a standard $N(0,1)$ variable.
(iii) If $H = \frac{1}{2n-1}$ then the $2n-1$-variation of B^H is zero.

(c) *The martingale convolution case*. In the following subsections we aim at calculating the n-variation of processes of the type

$$X = \left\{ X(t) = \int_0^t G(t,s)\,\mathrm{d}M(s); \quad t \in [0,1] \right\}, \tag{2.5}$$

where $M = \{M(t),\ t \geqslant 0\}$ is a local $(\mathscr{F}_t)$-continuous martingale and $G : \{0 \leqslant s \leqslant t \leqslant 1\} \to \mathbb{R}$, is a continuous $\mathscr{F}_0$-measurable random field, which we prolongate to $\mathbb{R}^2$ by setting,

$$G(t,s) = G(s,s) \quad \text{if } s \geqslant t.$$

The convolution case, i.e. $X(t) = \int_0^t \mathscr{G}(t-s)\,\mathrm{d}M(s)$, where $\mathscr{G}$ is a $\mathscr{F}_0$-measurable continuous process, will be a particular case setting $G(t,s) = \mathscr{G}(t-s)$. We remark that the process X is not in general a semimartingale unless that $\mathscr{G}$ is enough regular. It is, for instance, the case when $\mathscr{G}$ has paths in $W^{1,2}_{\mathrm{loc}}$. When M is a Brownian motion and $\mathscr{G}$ deterministic, Goldys and Musiela (1998) have shown that this is a necessary and sufficient condition. We remark that Brzeźniak et al. (2001) gives necessary and sufficient conditions on $\mathscr{G}$ so that X is continuous.

Later we will evaluate the n-variation of such a process, but we will first need the concept of *weak Dirichlet process*.

2.3. Weak Dirichlet processes

In our framework a process $(X_t)_{t\geqslant 0}$ will be said $(\mathscr{F}_t)$-*Dirichlet process* if it is the sum of an $(\mathscr{F}_t)$-local martingale M plus a zero quadratic variation $(\mathscr{F}_t)$-adapted process A, see for instance Russo and Vallois (2000).

We will say that X is a *weak* $(\mathscr{F}_t)$-*Dirichlet process* if it is the sum of an $(\mathscr{F}_t)$-local martingale M plus a process A such that

$$[A,N] \equiv 0 \quad \text{for every local } (\mathscr{F}_t)\text{-martingale } N, \tag{2.6}$$

A will also be said to be a *weak zero energy* process.

Clearly, an $(\mathscr{F}_t)$-Dirichlet process is a weak $(\mathscr{F}_t)$-Dirichlet process.

Remark 2.9. The decomposition of an $(\mathscr{F}_t)$-weak Dirichlet process is unique if we require for instance $M_0 = 0$.

Examples of such processes arise in several situations; for instance, if X is an $(\mathscr{F}_t)$-semimartingale, $f \in C^1(\mathbb{R})$, $f(X)$ is an $(\mathscr{F}_t)$-Dirichlet process; if $u:\mathbb{R}_+\times\mathbb{R}\to\mathbb{R}$ is continuous such that $\partial u/\partial x$ exists and is continuous, then $(u(t,X_t))_{t\geqslant 0}$ is a weak Dirichlet process. This property has been proved by Gozzi and Russo (2002).

Another example is furnished by the following result.

Proposition 2.10. *Let M be a local $(\mathscr{F}_t)$-continuous martingale, $G:\{0\leqslant s\leqslant t\leqslant 1\}\to\mathbb{R}$ be a continuous $(\mathscr{F}_s)$-measurable random field. We set*

$$X_t = \int_0^t G(t,s)\,\mathrm{d}M_s. \tag{2.7}$$

Then X is a weak $(\mathscr{F}_t)$-Dirichlet process with decomposition $N+Z$ where $N_t = \int_0^t G(s,s)\,\mathrm{d}M_s$ is the martingale part and the remainder Z is a weak zero energy process.

Proof. Setting $Z_t = \int_0^t H(t,s)\,\mathrm{d}M_s$, with $H(t,s) = G(t,s) - G(s,s)$, we have to show that $[Z,Y] = 0$ for every local $(\mathscr{F}_t)$-continuous martingale Y.

Let so Y be a local $(\mathscr{F}_t)$-continuous martingale, $\varepsilon > 0$ and $t \in [0,1]$. We have

$$\begin{aligned}
[Z,Y]_\varepsilon(t) &= \frac{1}{\varepsilon}\int_0^t (Y_{s+\varepsilon} - Y_s)(Z_{s+\varepsilon} - Z_s)\,\mathrm{d}s \\
&= \frac{1}{\varepsilon}\int_0^t (Y_{s+\varepsilon} - Y_s)\left(\int_0^s (H(s+\varepsilon,u) - H(s,u))\,\mathrm{d}M_u\right.
\end{aligned}$$

$$\left.+\int_s^{s+\varepsilon} H(s+\varepsilon,u)\,\mathrm{d}M_u\right)\mathrm{d}s$$

$$:= I_1(\varepsilon,t)+I_2(\varepsilon,t),$$

where

$$\begin{aligned}
I_1(\varepsilon,t) &= \frac{1}{\varepsilon}\int_0^t (Y_{s+\varepsilon}-Y_s)\int_0^s (H(s+\varepsilon,u)-H(s,u))\,\mathrm{d}M_u\,\mathrm{d}s \\
&= \frac{1}{\varepsilon}\int_0^t \int_s^{s+\varepsilon}\int_0^s (H(s+\varepsilon,u)-H(s,u))\,\mathrm{d}M_u\,\mathrm{d}Y_v\,\mathrm{d}s \\
&= \frac{1}{\varepsilon}\int_0^t \int_0^{v\wedge t}\int_{0\vee u\vee v-\varepsilon}^{v\wedge t} 1_{\{v\leqslant t+\varepsilon\}}(H(s+\varepsilon,u)-H(s,u))\,\mathrm{d}s\,\mathrm{d}M_u\,\mathrm{d}Y_v,
\end{aligned}$$

using stochastic Fubini's theorem in the last equation. Using localization arguments, we will reduce to the case where Y and M are square integrable martingales. Then Doob's inequality, continuity of H and the fact that $H(s,s)=0$, for every $s\in[0,1]$, show that $I_1(\varepsilon,t)$ converges *ucp* to zero. It remains to calculate the limit of $I_2(\varepsilon,\cdot)$. Recall that

$$I_2(\varepsilon,t)=\frac{1}{\varepsilon}\int_0^t (Y_{s+\varepsilon}-Y_s)\int_s^{s+\varepsilon} H(s+\varepsilon,u)\,\mathrm{d}M_u\,\mathrm{d}s.$$

Using Cauchy–Schwarz inequality, we have

$$\begin{aligned}
&E\left(\sup_{t\in[0,1]}|I_2(\varepsilon,t)|\right)\\
&\leqslant E\left(\frac{1}{\varepsilon}\int_0^1 |Y_{s+\varepsilon}-Y_s|\left|\int_s^{s+\varepsilon} H(s+\varepsilon,u)\,\mathrm{d}M_u\right|\mathrm{d}s\right)\\
&\leqslant E\left(\frac{1}{\varepsilon}\left[\int_0^1 (Y_{s+\varepsilon}-Y_s)^2\,\mathrm{d}s\right]^{1/2}\left[\int_0^1\left(\int_s^{s+\varepsilon} H(s+\varepsilon,u)\,\mathrm{d}M_u\right)^2\mathrm{d}s\right]^{1/2}\right)\\
&\leqslant \left[E\frac{1}{\varepsilon}\int_0^1 (Y_{s+\varepsilon}-Y_s)^2\,\mathrm{d}s\right]^{1/2}\left[E\int_0^1\frac{1}{\varepsilon}\left(\int_s^{s+\varepsilon} H(s+\varepsilon,u)\,\mathrm{d}M_u\right)^2\mathrm{d}s\right]^{1/2}\\
&= \left[E\left(\frac{1}{\varepsilon}\int_0^1 (Y_{s+\varepsilon}-Y_s)^2\,\mathrm{d}s\right)\right]^{1/2}\left[E\int_0^1\frac{1}{\varepsilon}\int_u^{u+\varepsilon} H(s,u)^2\,\mathrm{d}s\,\mathrm{d}[M]_u\right]^{1/2},
\end{aligned}$$

where in the last equality we use again stochastic Fubini's theorem. It is clear that $E\frac{1}{\varepsilon}\int_0^1 (Y_{s+\varepsilon}-Y_s)^2\,\mathrm{d}s$ converges to $E([Y]_1)$. The continuity of H and the fact that $H(s,s)=0$, for every $s\in[0,1]$, together with dominated convergence theorem show that $I_2(\varepsilon,\cdot)$ converges *ucp* to zero. □

Corollary 2.11. *Let* $(G(\cdot,s))$ *be a continuous* $(\mathscr{F}_s)$*-adapted random field. Let* $Z_t = \int_0^t G(t,s)\,\mathrm{d}M_s$*. Then, for every local* $(\mathscr{F}_t)$*-martingale* Y *we have*

$$[Z,Y]_t = \int_0^t G(s,s)\,\mathrm{d}[M,Y]_s.$$

We are now interested in evaluating 2-covariations of martingale convolution processes of the same nature as X introduced in (2.5).

From now on, for the sequel of this Section 2, we decompose the process X as

$$X = N + Z \tag{2.8}$$

with N will stand for the local $(\mathscr{F}_t)$-continuous martingale given by

$$N_t = \int_0^t G(s,s)\,\mathrm{d}M_s, \quad t \in [0,1]$$

and Z for the $(\mathscr{F}_t)$-adapted process,

$$Z_t = \int_0^t H(t,s)\,\mathrm{d}M_s = \int_0^1 H(t,s)\,\mathrm{d}M_s$$

with $H(t,s) = G(t,s) - G(s,s)$, $(t,s) \in [0,1]^2$. For $(t,s) \notin [0,1]^2$, H will be extended putting zero.

We formulate the following assumption:

(H2) $[G(\cdot,u), G(\cdot,v)]_\varepsilon(t)$,

converges *ucp* for (u,v,t) belonging to $[0,1]^3$. In particular, $[G(\cdot,u), G(\cdot,v)]$ exists (in the strong sense) for all $(u,v) \in [0,1]^2$.

Proposition 2.12. *Under assumption* (H2) *we set*

$$A(t) = A_1(t) + A_2(t), \quad t \in [0,1],$$

where

$$A_1(t) = \int_0^t [G(\cdot,s); G(\cdot,s)]_1\,\mathrm{d}[M]_s,$$

$$A_2(t) = 2\int_0^t \int_0^{s_2} [G(\cdot,s_1); G(\cdot,s_2)]_1\,\mathrm{d}M_{s_1}\,\mathrm{d}M_{s_2}.$$

Then

$$[Z,Z] = A \quad \text{and} \quad [Z,N] = 0. \tag{2.9}$$

In particular $[X,X] = [N] + A$.

Remark 2.13. Since $[Z,Z]$ and A_1 are increasing processes, A_2 is forced to be of bounded variation.

Corollary 2.14. *The process* X *in Proposition* 2.10 *is a finite quadratic variation process which is not a Dirichlet process unless* $[Z] = 0$.

Remark 2.15. *By Proposition* 2.10, *X is a weak $(\mathscr{F}_t)$-Dirichlet process.*

Proof of the Corollary 2.14. Let again consider the decomposition $N+Z$ of Proposition 2.10. If X were Dirichlet by the uniqueness of weak Dirichlet decomposition, one would get $[Z]=0$. □

We remark that under (H2), $[Z,Z]$ can be written as

$$[Z,Z](t)=2\int_0^t\int_0^{s_2}[G(\cdot,s_1);G(\cdot,s_2)]_1\,\mathrm{d}^\circ M_{s_1}\,\mathrm{d}^\circ M_{s_2}, \tag{2.10}$$

where d° means that the integral is in Stratonovich sense.

In general let $f(t,s)$; $(t,s)\in[0,1]^2$, be an $\mathscr{F}_0$-measurable continuous random field. We set

$$I_2^\circ(f)(t)=\int_0^t\int_0^{s_2}f(s_1,s_2)\,\mathrm{d}^\circ M_{s_1}\,\mathrm{d}^\circ M_{s_2}.$$

Classical Itô–Stratonovich calculus (Remark 3.2(2)) implies that,

$$I_2^\circ(f)(t)=\int_0^t\int_0^{s_2}f(s_1,s_2)\,\mathrm{d}M_{s_1}\,\mathrm{d}M_{s_2}+\frac{1}{2}\int_0^t f(s,s)\,\mathrm{d}[M]_s. \tag{2.11}$$

2.4. Gaussian case

We suppose that the martingale M is a Brownian motion $W=\{W_t,\ t\in[0,1]\}$, so that

$$X_t=\int_0^t G(t,s)\,\mathrm{d}W_s. \tag{2.12}$$

Under assumption (H2), Proposition 2.12 gives the following expression for the 2-variation (quadratic variation):

$$[X]_t=\int_0^t G(s,s)^2\,\mathrm{d}s+A_t,$$

where $A=A_1+A_2$ with

$$A_1(t)=\int_0^t[G(\cdot,s);G(\cdot,s)]_1\,\mathrm{d}s$$

$$A_2(t)=2\int_0^t\int_0^{s_2}[G(\cdot,s_1);G(\cdot,s_2)]_1\,\mathrm{d}W_{s_1}\,\mathrm{d}W_{s_2},\qquad t\in[0,1].$$

We will now make the link with the study of the quadratic variation of a Gaussian process given by Russo and Vallois (2000).

Russo and Vallois (2000) considered a (mean zero) Gaussian process with covariance function

$$K(u,v)=E(X_u X_v),\qquad u,v\in[0,1].$$

If X is of the form (2.12) then obviously,

$$K(u,v)=\int_0^{u\wedge v} G(u,s)G(v,s)\,\mathrm{d}s.$$

Russo and Vallois (2000) defined the concept of 2-planar variation for K which was given by

$$\lim_{\varepsilon,\delta\to 0}\frac{1}{\varepsilon\delta}\int_{[0,1]^2}(\triangle_{\varepsilon,\delta}K(u,v))^2\,\mathrm{d}u\,\mathrm{d}v, \tag{2.13}$$

with

$$\triangle_{\varepsilon,\delta}K(u,v)=K(u+\varepsilon,v+\delta)+K(u,v)-K(u,v+\delta)-K(u+\varepsilon,v),$$

provided that the limit in (2.13) exists for any $t\in[0,1]$. By Russo and Vallois (2000) the concept of energy process $\mathrm{En}(X)$ was defined as,

$$\mathrm{En}(X)(t)=\lim_{\varepsilon\to 0}E\left(\frac{1}{\varepsilon}\int_0^t (X_{s+\varepsilon}-X_s)^2\,\mathrm{d}s\right).$$

It was easily shown that

$$\mathrm{En}(X)(t)=\lim_{\varepsilon\to 0}\int_0^t \triangle_{\varepsilon,\varepsilon}K(s,s)\,\mathrm{d}s. \tag{2.14}$$

Remark 2.16. (1) A careful analysis on (2.13) and (2.14) shows the following properties.

(a) The 2-planar variation of K equals

$$4E\int_0^t\int_0^{s_2}[G(\cdot,s_1);G(\cdot,s_2)]_1^2\,\mathrm{d}s_1\,\mathrm{d}s_2=E(A_2^2(t)).$$

(b)

$$\begin{aligned}\mathrm{En}(X)(t)&=E\int_0^t G(s,s)^2\,\mathrm{d}s+\int_0^t E[G(\cdot,s);G(\cdot,s)]_1\,\mathrm{d}s\\&=E\left[\int_0^{\cdot} G(s,s)\,\mathrm{d}W_s\right](t)+E(A_1(t)).\end{aligned}$$

(2) For illustration consider the case $G(t,s)=B_{t-s}$, where B is a Brownian motion independent of W, which can be considered $\mathscr{F}_0$-measurable. Then the process $\{X_t=\int_0^t B_{t-s}\,\mathrm{d}W_s,\ t\in[0,1]\}$ is well defined and Proposition 2.12 gives that,

$$[X,X](t)=\frac{t^2}{2}.$$

In fact in this case $A_2(t)=0$ since $[B,B_{\cdot-s}]=0$ for all $s\in[0,1]$.

2.5. n-variation of martingale convolutions

To extend this calculus in order to evaluate the n-variation process of X, we will need an explicit expression of $(N)^n$ for any $n \geqslant 3$ and N continuous $(\mathscr{F}_t)$-martingale. As we will see, this expression will somehow generalize (2.11).

Notation. Let $n \geqslant 3$, $k \in \{1, 2, \ldots, [n/2]\}$. We denote by $\sigma = (\sigma^1, \ldots, \sigma^{n-k})$ a permutation of $\{1, 2, \ldots, n-k\}$ such that the first k elements of σ^{-1} are chosen arbitrarily among $\{1, 2, \ldots, n-k\}$ and the $n-2k$ remaining are taken at the sequel. We denote by Σ_k^n that family of permutations σ. We remark that its cardinal is given by $C_{n-k}^k = (n-k)!/k!(n-2k)!$.

Let Y be a finite quadratic variation process, and $k \in \{1, 2, \ldots, [n/2]\}$. With $\sigma \in \Sigma_k^n$ we associate

$$\sigma_Y = \sigma(\underbrace{[Y], \ldots, [Y]}_{k\text{times}}, \underbrace{Y, \ldots, Y}_{n-2k\text{times}}) = (\sigma_Y^1, \ldots, \sigma_Y^{n-k}),$$

we remark that, for all $l \in \{1, 2, \ldots, n-k\}$,

$$\sigma_Y^l = \begin{cases} [Y] & \text{if } \sigma(l) \in \{1, 2, \ldots, k\}, \\ Y & \text{if } \sigma(l) \in \{k+1, k+2, \ldots, n-k\}. \end{cases} \tag{2.15}$$

We denote by $P_k^n(Y)$ the set of σ_Y where $\sigma \in \Sigma_k^n$.

Now we give a generalization of Hu–Meyer formula. For the proof see the Appendix A.

Proposition 2.17. *Let $n \geqslant 3$ and $\{f(s_1, \ldots, s_n);\ (s_1, \ldots, s_n) \in [0,1]^n\}$ be a continuous, symmetric and $\mathscr{F}_0$-measurable random field. We set,*

$$I_n^\circ(f)(t) := \int_0^t \int_0^{s_n} \cdots \int_0^{s_2} f(s_1, \ldots, s_n)\, \mathrm{d}^\circ M_{s_1} \cdots \mathrm{d}^\circ M_{s_n}.$$

Then

$$I_n^\circ(f)(t) = \sum_{k=0}^{[n/2]} \frac{1}{2^k} \sum_{\sigma \in \Sigma_k^n} \int_0^t \int_0^{s_{n-k}} \cdots \int_0^{s_2} f(s_{\sigma^{-1}(1)}, s_{\sigma^{-1}(1)}, \ldots, s_{\sigma^{-1}(k)}, s_{\sigma^{-1}(k)},$$

$$s_{\sigma^{-1}(k+1)}, s_{\sigma^{-1}(k+2)}, \ldots, s_{\sigma^{-1}(n-k)})\, \mathrm{d}\sigma_M^1(s_1) \cdots \mathrm{d}\sigma_M^{n-k}(s_{n-k}). \tag{2.16}$$

Corollary 2.18. *Let $M = \{M_t, t \in [0,1]\}$ be any $(\mathscr{F}_t)$-continuous martingale, then for all $n \geqslant 3$,*

$$(M_t)^n = \sum_{k=0}^{[n/2]} \frac{n!}{2^k} \sum_{\sigma \in \Sigma_k^n} \int_0^t \int_0^{s_{n-k}} \cdots \int_0^{s_2} \mathrm{d}\sigma_M^1(s_1) \cdots \mathrm{d}\sigma_M^{n-k}(s_{n-k}). \tag{2.17}$$

Proof. Itô–Stratonovich formula shows that

$$(M_t)^n = n! I_n^\circ(1)(t),$$

so that we can apply Proposition 2.17. □

As a consequence of Proposition 2.17 we also obtain the following.

Proposition 2.19. *If the martingale M is a Brownian motion W then* (2.16) *and* (2.17) *become, respectively,*

$$I_n^\circ(f)(t) =$$

$$\sum_{k=0}^{[n/2]} \frac{1}{2^k} \int_0^t \int_0^{s_{n-k}} \cdots \int_0^{s_{k+2}} \int_0^{\mathbf{t}} \int_0^{s_k} \cdots \int_0^{s_2} f(s_1, s_1, \ldots, s_k, s_k, s_{k+1}, s_{k+2}, \ldots, s_{n-k})$$

$$\times \mathrm{d}s_1 \cdots \mathrm{d}s_k \, \mathrm{d}W_{s_{k+1}} \cdots \mathrm{d}W_{s_{n-k}}.$$

and

$$(W_t)^n = \sum_{k=0}^{[n/2]} \frac{n! t^k}{2^k k!} \int_0^t \int_0^{s_{n-2k}} \cdots \int_0^{s_2} \mathrm{d}W_{s_1} \cdots \mathrm{d}W_{s_{n-2k}}, \quad t \in [0,1].$$

Proof. $(W_t)^n$ follows immediately from the first expression. The evaluation of $I_n^\circ(f)$ is given in Appendix A. □

Using Proposition 2.17 and classical convergence properties of Itô integrals we get the following.

Corollary 2.20. *Let $(F_\varepsilon(s_1, \ldots, s_n); (s_1, \ldots, s_n) \in [0,1]^n)_{\varepsilon > 0}$ be a sequence of $\mathscr{F}_0$ -measurable continuous and symmetric random fields which converges ucp to a continuous random field F when ε goes to* 0. *Then*

$$\int_0^t \int_0^{s_n} \cdots \int_0^{s_2} F_\varepsilon(s_1, \ldots, s_n) \, \mathrm{d}^\circ M_{s_1} \cdots \mathrm{d}^\circ M_{s_n}$$

$$\rightarrow_{\varepsilon \to 0} \int_0^t \int_0^{s_n} \cdots \int_0^{s_2} F(s_1, \ldots, s_n) \, \mathrm{d}^\circ M_{s_1} \cdots \mathrm{d}^\circ M_{s_n}$$

where the convergence holds ucp.

Lemma 2.21. *Let $1 \leqslant p \leqslant n$. The covariation $[\underbrace{N, \ldots N}_{p \text{ times}}, \underbrace{Z, \ldots Z}_{(n-p) \text{ times}}]$ is identically zero.*

Proof. (i) Suppose first $p \geqslant 2$. The existence of $[N; N]$ in the strong sense and Remark 2.6(5) imply

$$[\underbrace{N, \ldots N}_{p \text{ times}}, \underbrace{Z, \ldots Z}_{(n-p) \text{ times}}] = 0 \quad \text{for all } p \geqslant 2$$

(ii) It remains to discuss the case $p=1$. This will follow by an adaptation of the proof of Proposition 2.10. □

We are now ready to calculate the n-variation of process X, under the following assumption:

$$(\mathrm{H}n) \quad [G(\cdot,s_1),\dots,G(\cdot,s_n)]_\varepsilon(t)$$

converges *ucp* for $(s_1,s_2,\dots,s_n,t)$ belonging to $[0,1]^{n+1}$. In particular, $[G(\cdot,s_1),\dots,G(\cdot,s_n)]$ exists for all $(s_1,\dots,s_n)\in[0,1]^n$, which constitutes the natural generalization of (H2).

Theorem 2.22. *Under assumption (Hn) we set*

$$A(t)=n!\int_0^t\int_0^{s_n}\cdots\int_0^{s_2}[G(\cdot,s_1),\dots,G(\cdot,s_n)](1)\,\mathrm{d}^\circ M_{s_1}\cdots\mathrm{d}^\circ M_{s_n},$$

$t\in[0,1]$. *Then*

$$[X;n]=[Z;n]=A,$$

where, we recall that $Z_t=\int_0^t(G(t,s)-G(s,s))\,\mathrm{d}M_s$.

Proof. We recall the decomposition of the process X,

$$X_t=N_t+Z_t,\quad t\in[0,1],$$

where $Z=\{\int_0^1 H(t,s)\,\mathrm{d}M_s,\ t\in[0,1]\}$, with $H(t,s)=G(t,s)-G(s,s)$, is a continuous $(\mathscr{F}_t)$-adapted process. We recall that

$$H(t,s)=0\quad\text{for all } 0\leqslant t\leqslant s\leqslant 1.$$

(i) We first prove that $[Z;n]=A$. Now we take $\varepsilon>0$ and $t\in[0,1]$ and

$$\begin{aligned}[Z;n]_\varepsilon(t)&=\frac{1}{\varepsilon}\int_0^t(Z_{s+\varepsilon}-Z_s)^n\,\mathrm{d}s\\&=\frac{1}{\varepsilon}\int_0^t\left(\int_0^1(H(s+\varepsilon,u)-H(s,u))\,\mathrm{d}M_u\right)^n\mathrm{d}s.\end{aligned}$$

We set

$$F_{s,\varepsilon}(u):=H(s+\varepsilon,u)-H(s,u);\ \text{ and } N_{s,\varepsilon}(v):=\int_0^v F_{s,\varepsilon}(u)\,\mathrm{d}M_u$$

for all $s<t$, and $u,v\in[0,1]$. This gives

$$[Z;n]_\varepsilon(t)=\frac{1}{\varepsilon}\int_0^t(N_{s,\varepsilon}(1))^n\,\mathrm{d}s.$$

Let $v\in[0,1]$. Itô–Stratonovich formula gives

$$(N_{s,\varepsilon}(v))^n=n!\int_0^v\int_0^{s_n}\cdots\int_0^{s_2}\prod_{k=1}^n(H(s+\varepsilon,s_k)-H(s,s_k))^2\,\mathrm{d}^\circ M_{s_1}\cdots\mathrm{d}^\circ M_{s_n}.$$

Using the stochastic Fubini's theorem we get

$$[Z;n]_\varepsilon(t) = n! \int_0^1 \int_0^{s_n} \cdots \int_0^{s_2} [H(\cdot,s_1),\cdots,H(\cdot,s_n)]_\varepsilon(t)\mathrm{d}^\circ M_{s_1}\cdots \mathrm{d}^\circ M_{s_n}.$$

Since $[G(\cdot,s_1),\ldots,G(\cdot,s_n)]$ exists and it is equal to $[H(\cdot,s_1),\ldots,H(\cdot,s_n)]$, Corollary 2.20 and the fact that,

$$[H(\cdot,s_1),H(\cdot,s_2),\ldots,H(\cdot,s_n)](t) = 0, \qquad s_n > t,$$

allow to show that $[Z;n] = A$.

(ii) It remains to show that $[X;n]=[Z;n]$. Using the multi-linearity of the n-covariation map we have that

$$[X;n] = \sum_{p=1}^{n} C_n^p [\underbrace{N,\ldots,N}_{p \text{ times}}, \underbrace{Z,\ldots,Z}_{(n-p) \text{ times}}] + [Z;n]$$

The result follows by (i) in this proof and Lemma 2.21. □

Under suitable assumptions, applying Theorem 2.22 and Proposition 2.17 we obtain the following illustration.

Example 2.23. (1) The cubic variation of the process X is given by

$$\begin{aligned}
[X;3](t) &= 3! \int_0^t \int_0^{s_3} \int_0^{s_2} [G(.,s_1);G(.,s_2);G(.,s_3)](t)\,\mathrm{d}^\circ M_{s_1}\,\mathrm{d}^\circ M_{s_2}\,\mathrm{d}^\circ M_{s_3}\\
&= 3! \int_0^t \int_0^{s_3} \int_0^{s_2} [G(.,s_1);G(.,s_2);G(.,s_3)](t)\,\mathrm{d}M_{s_1}\,\mathrm{d}M_{s_2}\,\mathrm{d}M_{s_3}\\
&\quad + \frac{3!}{2} \int_0^t \int_0^{s_2} [G(.,s_1);G(.,s_1);G(.,s_2)](t)\,\mathrm{d}[M]_{s_1}\,\mathrm{d}M_{s_2}\\
&\quad + \frac{3!}{2} \int_0^t \int_0^{s_2} [G(.,s_1);G(.,s_2);G(.,s_2)](t)\,\mathrm{d}M_{s_1}\,\mathrm{d}[M]_{s_2}.
\end{aligned}$$

(2) Suppose that the martingale M is a Brownian Motion W. Then the 4-variation of the process X are given by

$$\begin{aligned}
[X;4](t) &= 4! \int_0^t \int_0^{s_4} \int_0^{s_3} \int_0^{s_2} [G(.,s_1);G(.,s_2);G(.,s_3);G(.,s_4)](t)\\
&\quad \times \mathrm{d}^\circ W_{s_1}\,\mathrm{d}^\circ W_{s_2}\,\mathrm{d}^\circ W_{s_3}\,\mathrm{d}^\circ W_{s_4}\\
&= 4! \int_0^t \int_0^{s_4} \int_0^{s_3} \int_0^{s_2} [G(.,s_1);G(.,s_2);G(.,s_3);G(.,s_4)](t)
\end{aligned}$$

$$\times \mathrm{d}W_{s_1}\,\mathrm{d}W_{s_2}\,\mathrm{d}W_{s_3}\,\mathrm{d}W_{s_4}$$

$$+\frac{4!}{2}\int_0^t\int_0^{s_3}\int_0^t [G(.,s_1);G(.,s_1);G(.,s_2);G(.,s_3)](t)\,\mathrm{d}s_1\,\mathrm{d}W_{s_2}\,\mathrm{d}W_{s_3}$$

$$+\frac{4!}{2^2}\int_0^t\int_0^{s_2} [G(.,s_1);G(.,s_1);G(.,s_2);G(.,s_2)](t)\,\mathrm{d}s_1\,\mathrm{d}s_2.$$

We observe that those variations can also be obtained using Hu–Meyer formula.

2.6. True convolution case

Suppose that $G(t,s)=\mathscr{G}(t-s)$ for all $\{0\leqslant s,t\leqslant 1\}$. The process X is then given by

$$X(t)=\int_0^t \mathscr{G}(t-s)\,\mathrm{d}M_s.$$

Assumption (Hn) becomes here:

$$(\mathrm{H}'n)\quad [\theta_{s_1}\mathscr{G},\theta_{s_2}\mathscr{G},\ldots,\theta_{s_n}\mathscr{G}]_\varepsilon(t),$$

where $\theta_s\mathscr{G}(t)=\mathscr{G}(t-s)$, converges uniformly for $(s_1,s_2,\ldots,s_n,t)$ belonging to $[0,1]^{n+1}$. In particular, $[\theta_{s_1}\mathscr{G},\ldots,\theta_{s_n}\mathscr{G}]$ exists for all $(s_1,\ldots,s_n)\in[0,1]^n$.

Corollary 2.24. *Under assumption* (Hn′) *if the following process*,

$$A(t)=\int_0^t\int_0^{s_n}\cdots\int_0^{s_2}[\theta_{s_1}\mathscr{G},\ldots,\theta_{s_n}\mathscr{G}](t)\,\mathrm{d}^\circ M_{s_1}\cdots\mathrm{d}^\circ M_{s_n},$$

is continuous, then the process X has the following decomposition,

$$X=\mathscr{G}(0)M+Z,$$

where $Z(t)=\int_0^t(\mathscr{G}(t-s)-\mathscr{G}(0))\,\mathrm{d}M(s)$ and $[Z;n]=A$.

3. Stochastic calculus with respect to finite cubic variation continuous processes

We are interested in a stochastic calculus with respect to finite strong cubic variation (or 3-variation) continuous processes.

In this stochastic calculus, the symmetric integral will play a similar role of the forward integral in the case of the stochastic calculus with respect to finite quadratic variation continuous processes, by Russo and Vallois (2000). We start by recalling, from Russo and Vallois (1995), the definition and some properties.

3.1. Symmetric integral

Definition 3.1. Let X, Y be two continuous processes. For any $\varepsilon > 0$ and $t \in [0,1]$ we set,

$$I_\varepsilon^\circ(Y, \mathrm{d}X)(t) = \frac{1}{2\varepsilon} \int_0^t Y_s(X_{s+\varepsilon} - X_{s-\varepsilon})\,\mathrm{d}s.$$

If the process $I_\varepsilon^\circ(Y, \mathrm{d}X)$ converge *ucp*, when ε goes to zero, then the limit will be denoted by $\int_0^t Y \mathrm{d}^\circ X$, and called the *symmetric* integral.

Remark 3.2. (1) It is easy to prove that the symmetric integral $\int_0^t Y \mathrm{d}^\circ X$, if it exists, is the *ucp* limit of

$$J_\varepsilon^\circ(Y, \mathrm{d}X)(t) = \frac{1}{2\varepsilon} \int_0^t (Y_{s+\varepsilon} + Y_s)(X_{s+\varepsilon} - X_s)\,\mathrm{d}s.$$

(2) The symmetric integral $\int_0^t Y \mathrm{d}^\circ X$ coincides with the Stratonovich one when X and Y are two semimartingales. More precisely,

$$\int_0^t Y\,\mathrm{d}^\circ X = \int_0^t Y\,\mathrm{d}X + \frac{1}{2}[Y, X]. \tag{3.1}$$

(3) If the process X is of bounded variation then $\int_0^t Y\,\mathrm{d}^\circ X$ is well defined, it is equal to the integral $\int_0^t Y\,\mathrm{d}X$ in Stieltjes sense, and has bounded variation.

(4) By definition, the symmetric integral is a continuous process. If both processes X and Y are $\{\mathscr{F}_t;\ t \in [0,1]\}$-adapted then, since the filtration satisfies the usual assumptions, the integral process $\int_0^\cdot Y\,\mathrm{d}^\circ X$, if it exists, is an adapted process.

(5) We have an integration by parts formula,

$$\int_0^t Y\,\mathrm{d}^\circ X = YX(t) - YX(0) - \int_0^t X\,\mathrm{d}^\circ Y,$$

provided that one of the two integrals exists.

3.2. Itô formulae

We recall that, e.g. Russo and Vallois (2000), in the case where X is a continuous process with finite quadratic variation and $f \in C^2$, we have

$$f(X_t) = f(X_0) + \int_0^t f'(X_s)\,\mathrm{d}^\circ X.$$

In the case of finite strong cubic variation continuous processes we have the following.

Proposition 3.3. *Let X be a real valued process with finite strong cubic variation, and $f \in C^3$. Then*

$$f(X_t) = f(X_0) + \int_0^t f'(X_s)\,\mathrm{d}^\circ X_s - \frac{1}{12} \int_0^t f^{(3)}(X_s)\,\mathrm{d}[X, X, X]_s. \tag{3.2}$$

Remark 3.4. (1) *In particular the symmetric integral above exists.*
(2) *Using Proposition* 2.7, *Eq.* (3.2) *is equivalent to*

$$f(X_t) = f(X_0) + \int_0^t f'(X_s)\,\mathrm{d}^\circ X_s - \frac{1}{12}[f''(X),X,X](t).$$

Proof. Recall a Taylor-type formula,

$$f(b) = f(a) + f'(a)(b-a) + \frac{1}{2}f''(a)(b-a)^2 + \frac{1}{6}f^{(3)}(a)(b-a)^3 + R(a,b)(b-a)^3, \tag{3.3}$$

for every $a,b \in \mathbb{R}$, where,

$$R(a,b) = \int_0^1 \frac{\alpha^2}{2}(f^{(3)}(\alpha a + (1-\alpha)b) - f^{(3)}(a))\,\mathrm{d}\alpha.$$

Let $\varepsilon > 0$ and $s \in [0,1]$. Applying (3.3) we get that

$$f(X_{s+\varepsilon}) = f(X_s) + f'(X_s)(X_{s+\varepsilon} - X_s) + \frac{1}{2}f''(X_s)(X_{s+\varepsilon} - X_s)^2 + \frac{1}{6}f^{(3)}(X_s)(X_{s+\varepsilon} - X_s)^3 + R(X_s, X_{s+\varepsilon})(X_{s+\varepsilon} - X_s)^3 \tag{3.4}$$

and

$$f(X_s) = f(X_{s+\varepsilon}) - f'(X_{s+\varepsilon})(X_{s+\varepsilon} - X_s) + \frac{1}{2}f''(X_{s+\varepsilon})(X_{s+\varepsilon} - X_s)^2 - \frac{1}{6}f^{(3)}(X_{s+\varepsilon})(X_{s+\varepsilon} - X_s)^3 - R(X_{s+\varepsilon}, X_s)(X_{s+\varepsilon} - X_s)^3. \tag{3.5}$$

Calculating the difference between (3.4) and (3.5), dividing by 2ε and integrating over $[0,t]$ we get,

$$\begin{aligned}\frac{1}{\varepsilon}\int_0^t (f(X_{s+\varepsilon}) - f(X_s))\,\mathrm{d}s &= \frac{1}{2\varepsilon}\int_0^t (f'(X_{s+\varepsilon}) + f'(X_s))(X_{s+\varepsilon} - X_s)\,\mathrm{d}s \\ &\quad - \frac{1}{4\varepsilon}\int_0^t (f''(X_{s+\varepsilon}) - f''(X_s))(X_{s+\varepsilon} - X_s)^2\,\mathrm{d}s \\ &\quad + \frac{1}{12\varepsilon}\int_0^t (f^{(3)}(X_{s+\varepsilon}) + f^{(3)}(X_s))(X_{s+\varepsilon} - X_s)^3\,\mathrm{d}s \\ &\quad + \frac{1}{2\varepsilon}\int_0^t (R(X_s, X_{s+\varepsilon}) + R(X_{s+\varepsilon}, X_s))(X_{s+\varepsilon} - X_s)^3\mathrm{d}s.\end{aligned}$$

First, since $f(X)$ is a continuous process, we observe that

$$\frac{1}{\varepsilon}\int_0^t (f(X_{s+\varepsilon}) - f(X_s))\,\mathrm{d}s$$

converges *ucp* to $f(X_t) - f(X_0)$. By definition of cubic variation,

$$\frac{1}{4\varepsilon}\int_0^t (f''(X_{s+\varepsilon}) - f''(X_s))(X_{s+\varepsilon} - X_s)^2\,\mathrm{d}s,$$

converges *ucp* to $\frac{1}{4}[f''(X),X,X](t)$. We have,

$$\begin{aligned}&\frac{1}{12\varepsilon}\int_0^t (f^{(3)}(X_{s+\varepsilon}) + f^{(3)}(X_s))(X_{s+\varepsilon} - X_s)^3\,\mathrm{d}s\\ &\quad= \frac{1}{6\varepsilon}\int_0^t f^{(3)}(X_s)(X_{s+\varepsilon} - X_s)^3\,\mathrm{d}s + \frac{1}{12}[f^{(3)}(X),X,X,X]_\varepsilon(t).\end{aligned}$$

The first term converges *ucp* to $\frac{1}{6}\int_0^t f^{(3)}(X)\,\mathrm{d}[X,X,X]$ by Remark 2.6(6). The second one converges to the bracket $[f^{(3)}(X),X,X,X]$ that vanishes following Remark 2.6(5). Moreover, using the uniform continuity of both f'' and X on compacts, and the fact that $[X;3]$ exists strongly, we have that

$$\frac{1}{\varepsilon}\int_0^t (R(X_s,X_{s+\varepsilon}) + R(X_{s+\varepsilon},X_s))(X_{s+\varepsilon} - X_s)^3\,\mathrm{d}s$$

converges *ucp* to 0. Finally, using Remark 3.2(1)

$$\frac{1}{2\varepsilon}\int_0^t (f'(X_{s+\varepsilon}) + f'(X_s))(X_{s+\varepsilon} - X_s)\,\mathrm{d}s$$

is forced to converge *ucp* to $\int_0^t f'(X)\,\mathrm{d}^\circ X$. So the result follows. □

We give now multi-dimensional extension of Proposition 3.3. For this aim we introduce some notations.

Notations and Definitions. Let $X = (X^1, X^2, \ldots, X^n)$, and $Y = (Y^1, Y^2, \ldots, Y^n)$ be two vectors of continuous processes. We set

$$I_\varepsilon^\circ(Y,\cdot\mathrm{d}X)(t) = \frac{1}{2\varepsilon}\sum_{k=1}^n \int_0^t Y_s^k (X_{s+\varepsilon}^k - X_{s-\varepsilon}^k)\,\mathrm{d}s.$$

If the process $I_\varepsilon^\circ(Y,\cdot\mathrm{d}X)$ converges *ucp*, when ε goes to zero, then the limiting process will be denoted by $\int_0^\cdot Y\cdot\mathrm{d}^\circ X$. This integral is in the spirit of Chatelain and Stricker (1995) for semimartingales.

If for every $k = 1, 2, \ldots, n$, $\int_0^\cdot Y^k\,\mathrm{d}^\circ X^k$ is well defined, then

$$\int_0^\cdot Y\cdot\mathrm{d}^\circ X = \sum_{k=1}^n \int_0^\cdot Y^k\,\mathrm{d}^\circ X^k.$$

Let $Z=(Z^{i,j})_{1\leqslant i,j\leqslant n}$ be a $\mathbb{R}^{n\times n}$ matrix of continuous processes. We set,

$$[X,Z,Y^{\mathrm{T}}]_\varepsilon(t)=\frac{1}{\varepsilon}\sum_{i,j=1}^{n}\int_0^t (X^i_{s+\varepsilon}-X^i_s)(Z^{i,j}_{s+\varepsilon}-Z^{i,j}_s)(Y^j_{s+\varepsilon}-X^j_s)\,\mathrm{d}s,$$

where Y^{T} is the transposition of vector Y. If the process $[X,Z,Y^{\mathrm{T}}]_\varepsilon$ converges *ucp*, when ε goes to zero, then the limiting process, denoted by $[X,Z,Y^{\mathrm{T}}]$, will define the 3-*covariation* of (X,Z,Y). If, furthermore,

$$\|[X,Z,Y^{\mathrm{T}}]\|:=\sup_{0<\varepsilon\leqslant 1}\frac{1}{\varepsilon}\sum_{i,j=1}^{n}\int_0^1 |X^i_{s+\varepsilon}-X^i_s\|Z^{i,j}_{s+\varepsilon}-Z^{i,j}_s\|Y^j_{s+\varepsilon}-X^j_s|\,\mathrm{d}s<\infty,$$

then we will say that $[X,Z,Y^{\mathrm{T}}]$ exists in the strong sense.

If, for every $i,j=1,2,\ldots,n$, the 3-covariation process (or strong 3-covariation) $[X^i,Z^{i,j},Y^j]$ exists, then $[X,Z,Y^{\mathrm{T}}]$ strongly exists and it is equal to $\sum_{i,j}[X^i,Z^{i,j},Y^j]$.

If F is a function of class C^1, we set $\nabla F(X)=(\partial_1F(X),\ldots,\partial_nF(X))$, and Hess $F(X)=(\partial_{i,j}F(X))_{\substack{1\leqslant i\leqslant n\\ 1\leqslant j\leqslant n}}$ in the case when F is of class C^2.

Proposition 3.5. *Let $F\in C^3(\mathbb{R}^n)$ and $X=(X^1,X^2,\ldots,X^n)$ be a vector of continuous processes having all its mutual strong 3-covariations. Then*

$$F(X_t)=F(X_0)+\int_0^t \nabla F(X)\cdot\mathrm{d}^\circ X-\frac{1}{12}[X,\mathrm{Hess}\,F(X),X^{\mathrm{T}}](t).$$

In particular the symmetric integral above exists.

Remark 3.6. If $\int_0^\cdot \partial_kF(X)\,\mathrm{d}^\circ X^k$ exists for all $k=1,2,\ldots,n$, then $F(X)$ is given explicitly by,

$$F(X_t)=F(X_0)+\sum_{i=1}^{n}\int_0^t \partial_iF(X)\,\mathrm{d}^\circ X^i-\frac{1}{12}\sum_{1\leqslant i,j,k\leqslant n}\int_0^t \partial_{ijk}F(X)\,\mathrm{d}[X^i,X^j,X^k].$$

Proposition 3.7. *Let $X=(X^1,X^2,\ldots,X^m)$ be a vector of continuous processes having its mutual strong 3-covariations and $S=(S^1,\ldots,S^p)$ a vector of continuous processes having its mutual 2-covariations. We set $Y=(X,S)$. Let $F\in C^{3,2}(\mathbb{R}^m\times\mathbb{R}^p)$. Then*

$$F(Y_t)=F(Y_0)+\int_0^t \nabla F(Y)\cdot\mathrm{d}^\circ Y-\frac{1}{12}\sum_{1\leqslant i,j,k\leqslant m}\int_0^t \partial_{ijk}F(Y)\,\mathrm{d}[X^i,X^j,X^k]. \tag{3.6}$$

Remark 3.8. If we replace S with V, a vector of bounded variation processes then (3.6) holds even when F belongs to $C^{3,1}(\mathbb{R}^m\times\mathbb{R}^p)$ only. Moreover, if $\int_0^t \partial_iF(Y)\,\mathrm{d}^\circ X^i$ exists for all $i=1,\ldots,m$, then

$$F(Y_t)=F(Y_0)+\sum_{i=1}^{m+p}\int_0^t \partial_iF(Y)\,\mathrm{d}^\circ Y^i-\frac{1}{12}\sum_{1\leqslant i,j,k\leqslant m}\int_0^t \partial_{ijk}F(Y)\,\mathrm{d}[X^i,X^j,X^k].$$

3.3. On chain-rule formulae

Let $X = (X^1, X^2, \ldots, X^n)$ be a vector of continuous processes, and Z a real process. Suppose that (Z, X) has all its mutual strong 3-covariations. The aim here, is to evaluate integrals of the type $\int_0^t Z \, d^\circ \varphi(X)$, where $\varphi \in C^3(\mathbb{R}^n)$.

Proposition 3.9. *If $\int_0^\cdot Z \nabla \varphi(X) \cdot d^\circ X$ exists then $\int_0^\cdot Z \, d^\circ \varphi(X)$ is well defined and it is given by*

$$\int_0^t Z \, d^\circ \varphi(X) = \int_0^t Z \nabla \varphi(X) \cdot d^\circ X - \frac{1}{4} [X, Z \nabla \varphi(X), X^{\mathrm{T}}](t)$$
$$+ \frac{1}{6} \int_0^t Z \, d[X, \operatorname{Hess} \varphi(X), X^{\mathrm{T}}], \quad t \in [0,1]. \tag{3.7}$$

Remark 3.10. (1) Using Proposition 2.7, (3.7) is explicitly given by

$$\int_0^t Z \, d^\circ \varphi(X) = \int_0^t Z \nabla \varphi(X) \cdot d^\circ X - \frac{1}{4} \sum_{1 \leqslant i,j \leqslant n} \int_0^t \partial_{ij} \varphi(X) \, d[Z, X^i, X^j]$$
$$- \frac{1}{12} \sum_{1 \leqslant i,j,k \leqslant n} \int_0^t Z \partial_{ijk} \varphi(X) \, d[X^i, X^j, X^k]. \tag{3.8}$$

Recall that, when all the integrals $\int_0^t Z \partial_i \varphi(X) \, d^\circ X^i$, $i = 1, \ldots, n$, are well defined then, $\int_0^t Z \nabla \varphi(X) \cdot d^\circ X = \sum_{i=1}^n \int_0^t Z \partial_i \varphi(X) \, d^\circ X^i$.

(2) If X is a real valued process with finite strong cubic variation, and φ a C^3 function such that $\int_0^t Z \varphi'(X) \, d^\circ X$ exists, then (3.8) becomes

$$\int_0^t Z \, d^\circ \varphi(X) = \int_0^t Z \varphi'(X) \, d^\circ X - \frac{1}{4} \int_0^t \varphi''(X) \, d[Z, X, X]$$
$$- \frac{1}{12} \int_0^t Z \varphi^{(3)}(X) \, d[X, X, X]. \tag{3.9}$$

(3) As an application, we get an integration by parts formula. Let (X, Y, Z) be a vector of continuous processes having all its mutual strong 3-covariations. Suppose that, $\int_0^t ZX \, d^\circ Y$ and $\int_0^t ZY \, d^\circ X$ exist. Setting $\varphi(x, y) = xy$, Proposition 3.9 gives

$$[X, Y, Z](t) = 2 \left(\int_0^t ZX \, d^\circ Y + \int_0^t ZY \, d^\circ X - \int_0^t Z \, d^\circ X \, Y \right).$$

On the other hand, it is not difficult to show, directly, that previous formula holds when two of previous integrals exist.

Proof. We suppose that $n = 1$. The proof of the general case is similar. Let $\varepsilon > 0$, and $s \in [0,1]$. We multiply, respectively, Eqs. (3.4) and (3.5) (identifying f with φ) by Z_s

and $Z_{s+\varepsilon}$. Then calculating the difference, dividing by 2ε and integrating over $[0,t]$ we get

$$\frac{1}{2\varepsilon}\int_0^t (Z_{s+\varepsilon}+Z_s)(\varphi(X_{s+\varepsilon})-\varphi(X_s))\,\mathrm{d}s$$

$$=\frac{1}{2\varepsilon}\int_0^t (Z_{s+\varepsilon}\varphi'(X_{s+\varepsilon})+Z_s\varphi'(X_s))(X_{s+\varepsilon}-X_s)\,\mathrm{d}s$$

$$-\frac{1}{4\varepsilon}\int_0^t (Z_{s+\varepsilon}\varphi''(X_{s+\varepsilon})-Z_s\varphi''(X_s))(X_{s+\varepsilon}-X_s)^2\,\mathrm{d}s$$

$$+\frac{1}{12\varepsilon}\int_0^t (Z_{s+\varepsilon}\varphi^{(3)}(X_{s+\varepsilon})+Z_s\varphi^{(3)}(X_s))(X_{s+\varepsilon}-X_s)^3\,\mathrm{d}s$$

$$+\frac{1}{2\varepsilon}\int_0^t (Z_sR(X_s,X_{s+\varepsilon})+Z_{s+\varepsilon}R(X_{s+\varepsilon},X_s))(X_{s+\varepsilon}-X_s)^3\,\mathrm{d}s.$$

The term on the left-hand side of the equality converges *ucp* to $\int_0^t Z\,\mathrm{d}^\circ\varphi(X)$. Using similar arguments as those in the proof of the Proposition 3.3 we see that, the first term on the right-hand side converges to $\int_0^t Z\,\mathrm{d}^\circ\varphi'(X)$; the second one converges to $-\frac{1}{4}[Z\varphi''(X),X,X]$ and Proposition 2.7 tells that

$$[Z\varphi''(X),X,X](t)=\int_0^t \varphi''(X)\,\mathrm{d}[Z,X,X]+\int_0^t Z\varphi^{(3)}(X)\,\mathrm{d}[X,X,X].$$

Using the fact that the third term converges to,

$$\frac{1}{6}\int_0^t Z\varphi^{(3)}(X)\,\mathrm{d}[X,X,X]$$

and the last term to zero we get (3.9). □

We give a small generalization of Proposition 3.9.

Remark 3.11. Let $X=(X^1,X^2,\ldots,X^n)$ be a vector of continuous processes, and Z a continuous process such that (Z,X) has all its mutual strong 3-covariations. Let $S=(S^1,\ldots,S^m)$ be a vector of continuous processes with finite mutual 2-covariations (resp. with bounded variation). We set $Y=(X,S)$. If for every function $\varphi\in C^{3,2}(\mathbb{R}^{n+m})$ (resp. $\in C^{3,1}(\mathbb{R}^{n+m})$)

$$\int_0^t Z\partial_i\varphi(Y)\,\mathrm{d}^\circ Y^i \text{ exist for all } i=1,\ldots,n+m;$$

(resp. $\int_0^t Z\partial_i\varphi(Y)\,\mathrm{d}^\circ X^i$ exist for all $i=1,\ldots,n$) then $\int_0^\cdot Z\,\mathrm{d}^\circ\varphi(Y)$ exists and is given by

$$\int_0^t Z\,\mathrm{d}^\circ\varphi(Y)=\sum_{i=1}^{n+m}\int_0^t Z\partial_i\varphi(Y)\,\mathrm{d}^\circ Y^i-\frac{1}{4}\sum_{1\leqslant i,j\leqslant n}\int_0^t \partial_{ij}\varphi(Y)\,\mathrm{d}[Z,X^i,X^j]$$

$$-\frac{1}{12}\sum_{1\leqslant i,j,k\leqslant n}\int_0^t Z\partial_{ijk}\varphi(Y)\,\mathrm{d}[X^i,X^j,X^k].$$

3.4. Generalized symmetric vector cubic variation process

Let X be a real valued continuous process with finite strong cubic variation. Using Proposition 3.3, we easily see that the integral $\int_0^{\cdot} f(X)\,\mathrm{d}^{\circ}X$ is well defined for every $f \in C^2$. However, if $X = (X^1, X^2, \dots, X^n)$ is a vector of continuous processes, the existence of its mutual strong 3-covariations, is not a sufficient condition for guaranteeing the existence of $\int_0^t f(X)\,\mathrm{d}^{\circ}X^k$; $k = 1, 2, \dots, n$. For this reason, we need the concept of symmetric vector cubic variation (SVCV) process.

Definition 3.12. A vector of continuous processes $X = (X^1, X^2, \dots, X^n)$ is a SVCV process if the following assumptions are fulfilled.

(i) $\|[X^i, \dots, X^i]\| < \infty, \quad i = 1, 2, \dots, n,$
(ii) $\int_0^{\cdot} f(X)\,\mathrm{d}^{\circ}X^i$ exists for every $f \in C^2(\mathbb{R}^n)$, $i = 1, 2, \dots, n$,
(iii)
$$\left[\int_0^{\cdot} f_1(X)\,\mathrm{d}^{\circ}X^i, \int_0^{\cdot} f_2(X)\,\mathrm{d}^{\circ}X^j, \int_0^{\cdot} f_3(X)\,\mathrm{d}^{\circ}X^k\right]$$
$$= \int_0^{\cdot} f_1(X) f_2(X) f_3(X)\,\mathrm{d}[X^i, X^j, X^k],$$
for every $f_1, f_2, f_3 \in C^2(\mathbb{R}^n)$ and $1 \leqslant i, j, k \leqslant n$.

Remark 3.13. If $(X^1, \dots, X^n)$ is an SVCV process, then, in particular, it has all its strong 3-mutual covariations. This is a consequence of Remark 3.10(3) and Proposition 3.9.

Now we state some results that we will need in the next section.

Lemma 3.14. *Let $X = (X^1, X^2, \dots, X^m)$ be an SVCV process, and $\varphi : \mathbb{R}^m \to \mathbb{R}^n$ a function of class C^3. We set*
$$Y = (\varphi_1(X), \varphi_2(X), \dots, \varphi_n(X)) = \varphi(X).$$
Then Y is again an SVCV process.

Proof. We will prove (i)–(iii) of Definition 3.12.

(i) It follows by similar arguments to the proof of Proposition 2.7.
(ii) For every $f \in C^2(\mathbb{R}^n)$ and $i = 1, 2, \dots, n$, $\int_0^{\cdot} f(Y)\,\mathrm{d}^{\circ}Y^i$ exists using the fact that X is an SVCV process and Remark 3.10(1).
(iii) Let $f_k \in C^2(\mathbb{R}^n)$, $k = 1, 2, 3$. We apply successively Proposition 3.9 with $Z = f_1(\varphi(X))$, $f_2(\varphi(X))$ and $f_3(\varphi(X))$ and we obtain
$$\int_0^{\cdot} f_k(Y)\,\mathrm{d}^{\circ}Y^i = \sum_{j=1}^{m} \int_0^{\cdot} f_k(\varphi(X))\partial_j\varphi_i(X)\,\mathrm{d}^{\circ}X^j + V_i(\cdot)$$
for every $i = 1, \dots, n$ and $k = 1, 2, 3$, where V_i is a bounded variation continuous process. Consequently, (iii) follows from the fact that X is an SVCV process, Remark 2.6(3) and (4). $\square$

Proposition 3.15. *Let X be a real valued continuous process with finite strong cubic variation and $V=(V^1,V^2,\dots,V^n)$ a vector of bounded variation processes. Then (X,V) is an SVCV process.*

Proof. We can verify points (i)–(iii) of the definition of SVCV process.

Point (i) is obvious. Concerning (ii), let $f\in C^2(\mathbb{R}^{1+n})$. Remark 3.2(3) says that $\int_0^\cdot f(X,V)\,\mathrm{d}^\circ V^i$, $i=1,\dots,n$ coincide with the classical Stieltjes integrals. On the other hand, we set $F(x,v)=\int_0^x f(y,v)\,\mathrm{d}y$, where $v=(v_1,\dots,v_n)$; F belongs to $C^{3,2}(\mathbb{R}^{1+n})$. Remark 3.8 (Itô formula) tells that $\int_0^t f(X,V)\,\mathrm{d}^\circ X$ is well defined by

$$\int_0^t f(X,V)\,\mathrm{d}^\circ X = F(X_t,V_t)+A(t),$$

where

$$A(t) = -F(X_0,V_0) - \sum_{i=2}^{n+1}\int_0^t \partial_i F(X_s,V_s)\,\mathrm{d}^\circ V_s^i - \frac{1}{12}\int_0^t \frac{\partial^2 f}{\partial x^2}(X_s,V_s)\mathrm{d}[X,X,X], \quad t\in[0,1].$$

Now we prove (iii). Let $g,h\in C^2(\mathbb{R}^{1+n})$. Since

$$\int_0^\cdot f(X_s,V_s)\,\mathrm{d}^\circ V^i, \int_0^\cdot g(X_s,V_s)\,\mathrm{d}^\circ V^i \quad \text{and} \quad \int_0^\cdot h(X_s,V_s)\,\mathrm{d}^\circ V^i,$$

$i=1,\dots,n$, are bounded variation processes, Remark 2.6(4) tells that it remains to prove

$$\left[\int_0^\cdot f(X,V)\,\mathrm{d}^\circ X, \int_0^\cdot g(X,V)\,\mathrm{d}^\circ X, \int_0^\cdot h(X,V)\,\mathrm{d}^\circ X\right] = \int_0^\cdot f(X,V)g(X,V)h(X,V)\,\mathrm{d}[X,X,X]. \tag{3.10}$$

As at the beginning of the proof, we can write

$$\int_0^t g(X,V)\,\mathrm{d}^\circ X = G(X_t,V_t)+B(t),$$

$$\int_0^t h(X,V)\,\mathrm{d}^\circ X = H(X_t,V_t)+C(t), \quad t\in[0,1],$$

where $G(x,v)=\int_0^x g(y,v)\,\mathrm{d}y$ and $H(x,v)=\int_0^x h(y,v)\,\mathrm{d}y$. Processes B and C have bounded variation. Using Remark 2.6(4) and Proposition 2.7 we obtain (3.10). □

Corollary 3.16. *A finite strong cubic variation process is an SVCV process.*

As a consequence of Itô formula (Remark 3.8), we have the following result.

Remark 3.17. Let ξ be a finite strong cubic variation process. We denote by $\mathscr{V}(\xi)$ the class of processes

$$X_t = X_0 + \int_0^t \varphi(\xi, V^1, \dots, V^n)\, \mathrm{d}^\circ \xi + V_t^0,$$

where $n \in \mathbb{N}^*$, $V^1, \dots, V^n, V^0$, are bounded variation processes and $\varphi \in C^{2,1}(\mathbb{R}^{1+n})$. $\mathscr{V}(\xi)$ coincides with the set of processes $\{\psi(\xi_t, V_t^1, \dots, V_t^m),\ t \in [0,1]\}$, where $m \in \mathbb{N}^*$, $V^1, \dots, V^m$ are bounded variation processes and $\psi \in C^{3,1}(\mathbb{R}^{1+m})$.

We conclude this section with a useful lemma which provides a chain-rule formula for differentiating integral processes.

Lemma 3.18. *Let ξ, Z be two continuous processes, $V = (V^1, \dots, V^n)$ a vector of bounded variation processes. We suppose that (ξ, Z, V) is an SVCV process. Let $\varphi = (\varphi(r, v)) \in C^{2,1}(\mathbb{R}^{1+n})$, where $v = (v_1, \dots, v_n)$. We set*

$$X_t = \int_0^t \varphi(\xi, V)\, \mathrm{d}^\circ \xi, \quad t \in [0,1].$$

Then the integral process $\int_0^\cdot Z\, \mathrm{d}^\circ X$ exists and it is given by

$$\int_0^\cdot Z\, \mathrm{d}^\circ X = \int_0^\cdot Z\varphi(\xi, V)\, \mathrm{d}^\circ \xi - \frac{1}{4} \int_0^\cdot \frac{\partial \varphi}{\partial r}(\xi, V)\, \mathrm{d}[Z, \xi, \xi]. \tag{3.11}$$

Proof. We set $\phi(r, v) = \int_0^r \varphi(u, v)\, \mathrm{d}u$. ϕ is obviously of class $C^{3,1}$. Since (ξ, V) has all 3-strong covariations, applying Proposition 3.5 we get,

$$\phi(\xi_t, V_t) = \phi(\xi_0, V_0) + \int_0^t \varphi(\xi, V)\, \mathrm{d}^\circ \xi + \sum_{i=1}^n \int_0^t \frac{\partial \phi}{\partial v_i}(\xi, V)\, \mathrm{d}V^i$$
$$- \frac{1}{12} \int_0^t \frac{\partial^2 \varphi}{\partial r^2}(\xi, V)\, \mathrm{d}[\xi, \xi, \xi].$$

Since (ξ, Z, V) is an SVCV process and by mean of Lemma 3.14,

$$\int_0^t Z\, \mathrm{d}^\circ X = \int_0^t Z\, \mathrm{d}^\circ \phi(\xi, V) - \sum_{i=1}^n \int_0^t Z \frac{\partial \phi}{\partial v_i}(\xi, V)\, \mathrm{d}V^i$$
$$+ \frac{1}{12} \int_0^t \frac{\partial^2 \varphi}{\partial r^2}(\xi, V)\, \mathrm{d}[\xi, \xi, \xi]. \tag{3.12}$$

On the other hand, Remark 3.11 tells that,

$$\int_0^t Z\, \mathrm{d}^\circ \phi(\xi, V) = \int_0^t Z\, \varphi(\xi, V)\, \mathrm{d}^\circ \xi + \sum_{i=1}^n \int_0^t Z \frac{\partial \phi}{\partial v_i}(\xi, V)\, \mathrm{d}V^i$$
$$- \frac{1}{4} \int_0^t \frac{\partial \varphi}{\partial r}(\xi, V)\, \mathrm{d}[Z, \xi, \xi] - \frac{1}{12} \int_0^t Z_s \frac{\partial^2 \varphi}{\partial r^2}(\xi, V)\, \mathrm{d}[\xi, \xi, \xi]. \tag{3.13}$$

(3.12) and (3.13) show (3.11). □

Remark 3.19. Let $(X^2,\dots,X^n)$ be a vector of bounded variation processes and X^1 a finite strong cubic variation process. Then the conclusion of Lemma 3.14 holds even when $\varphi \in C^{3,1}(\mathbb{R}\times\mathbb{R}^{n-1})$.

4. On an SDE which is driven by finite cubic variation continuous processes

We aim here to study stochastic differential equations driven by finite strong cubic variation continuous process. We will operate with Doss–Sussmann (Doss, 1977; Sussmann, 1977) transformation.

Let $\xi=\{\xi_t,\ t\in[0,1]\}$ (resp. $V=\{V_t,\ t\in[0,1]\}$) be a real process with finite strong cubic variation (resp. bounded variation).

We are interested in one equation of the type:

$$\begin{cases} \mathrm{d}X_t=\sigma(X_t)\circ\mathrm{d}\xi_t+b(t,X_t)\,\mathrm{d}V_t, \\ X_0=\alpha, \end{cases} \tag{4.1}$$

where $\sigma:\mathbb{R}\to\mathbb{R}$ (resp. $b:[0,1]\times\mathbb{R}\to\mathbb{R}$) is of class $C^3(\mathbb{R})$ (resp. continuous), such that, σ' σ'' are bounded. We suppose moreover that b is locally Lipschitz with linear growth (uniformly in t, with respect to the second variable x), α any random variable $\mathscr{F}_1$-measurable.

Definition 4.1. A real process X will be called *solution of* Eq. (4.1) if the following assumptions are fulfilled:

(i) $X_0=\alpha$,
(i) (X,ξ) is an SVCV process,
(iii) For every $Z=\psi(X,\xi)$, where $\psi\in C^\infty(\mathbb{R}^2)$, we have,

$$\int_0^t Z\,\mathrm{d}^\circ X=\int_0^t Z\sigma(X)\,\mathrm{d}^\circ\xi-\frac{1}{4}\int_0^t\sigma\sigma'(X)\,\mathrm{d}[Z,\xi,\xi] + \int_0^t Z_s b(s,X_s)\,\mathrm{d}V_s. \tag{4.2}$$

Remark 4.2. (1) If X is a solution of (4.1), taking $Z=1$, we observe in particular that X solves the integral equation

$$X_t=\alpha+\int_0^t\sigma(X)\,\mathrm{d}^\circ\xi+\int_0^t b(s,X_s)\,\mathrm{d}V_s.$$

(2) If X is a solution of (4.1) then (4.2) remains true for every $Z=\psi(X,\xi)$, with $\psi\in C^2(\mathbb{R}^2)$.

In fact, using Banach Steinhaus theorem (Dunford and Schwarz, 1967, Chapter II.1), and Proposition 2.7 we can prove that

$$\psi \mapsto \int_0^t \psi(X,\xi)\,\mathrm{d}^\circ X = \int_0^t \psi(X,\xi)\,\sigma(X)\,\mathrm{d}^\circ\xi - \frac{1}{4}\int_0^t \sigma\sigma'(X)\,\mathrm{d}[\psi(X,\xi),\xi,\xi]$$
$$+ \int_0^t \psi(X_s,\xi_s)b(s,X_s)\,\mathrm{d}V_s,$$

is linear and continuous operator with values in the space of continuous processes, equipped with the uniform convergence in probability. So by regularizing and passing to limit we have (iii) for every $\psi \in C^2$.

Let $F:[0,1]\times\mathbb{R}\to\mathbb{R}$ be the flow generated by σ, defined as the solution of the following equation:

$$\begin{cases} \dfrac{\partial F}{\partial r}(r,x) = \sigma(F(r,x)), \\ F(0,x) = x. \end{cases} \tag{4.3}$$

Since σ is of class C^3, for any $r\in\mathbb{R}$, $F(r,.)$ is a C^3-diffeomorphism on $\mathbb{R}$. We set

$$H(r,x) = F^{-1}(r,x), \tag{4.4}$$

where the inverse is taken with respect to the variable x. H is again of class C^3.

We state, first, the uniqueness result.

Proposition 4.3. *There is at most one solution to* (4.1). *Moreover if X is a solution of* (4.1), *then it is equal to $F(\xi,Y)$ where Y is the unique solution to the following equation*:

$$Y_t = H(\xi_0,\alpha) + \int_0^t \frac{\partial H}{\partial x}(\xi_s,F(\xi_s,Y_s))b(s,F(\xi_s,Y_s))\,\mathrm{d}V_s$$
$$+\frac{1}{12}\int_0^t(\sigma\sigma'^2(F(\xi_s,Y_s))+\sigma^2\sigma''(F(\xi_s,Y_s)))\,\frac{\partial H}{\partial x}(\xi_s,F(\xi_s,Y_s))\,\mathrm{d}[\xi,\xi,\xi]_s. \tag{4.5}$$

Proof. We recall some relations involving F and H established by Russo and Vallois (2000).

$$\frac{\partial H}{\partial r}(r,x) = -\sigma(x)\frac{\partial H}{\partial x}(r,x). \tag{4.6}$$

$$\frac{\partial^2 H}{\partial r\partial x}(r,x) = -\sigma'(x)\,\frac{\partial H}{\partial x}(r,x) - \sigma(x)\,\frac{\partial^2 H}{\partial x^2}(r,x). \tag{4.7}$$

Deriving those relations, we can prove the following:

$$\frac{\partial^2 H}{\partial r^2}(r,x) = \sigma\sigma'(x)\frac{\partial H}{\partial x}(r,x) + \sigma^2(x)\frac{\partial^2 H}{\partial x^2}(r,x), \tag{4.8}$$

$$\frac{\partial^3 H}{\partial r \partial x^2}(r,x) = -\sigma''(x)\frac{\partial H}{\partial x}(r,x) - 2\sigma'(x)\frac{\partial^2 H}{\partial x^2}(r,x) - \sigma(x)\frac{\partial^3 H}{\partial x^3}(r,x), \tag{4.9}$$

$$\begin{aligned}\frac{\partial^3 H}{\partial r^2 \partial x}(r,x) &= (\sigma'^2(x) + \sigma\sigma''(x))\frac{\partial H}{\partial x}(r,x)\\ &\quad + 3\sigma\sigma'(x)\frac{\partial^2 H}{\partial x^2}(r,x) + \sigma^2(x)\frac{\partial^3 H}{\partial x^3}(r,x).\end{aligned} \tag{4.10}$$

$$\begin{aligned}\frac{\partial^3 H}{\partial r^3}(r,x) &= -(\sigma\sigma'^2(x) + \sigma^2\sigma''(x))\frac{\partial H}{\partial x}(r,x)\\ &\quad - 3\sigma^2\sigma'(x)\frac{\partial H^2}{\partial x^2}(r,x) - \sigma^3(x)\frac{\partial^3 H}{\partial x^3}(r,x).\end{aligned} \tag{4.11}$$

Now, let X be a solution of (4.1) and set $Y = H(\xi, X)$. Obviously $X = F(\xi, Y)$. Since (ξ, X) is an SVCV process, in particular all its mutual strong 3-covariations exist; H is of class C^3 so Proposition 3.5 tells then that

$$\begin{aligned}Y_t &= Y_0 + \int_0^t \frac{\partial H}{\partial r}(\xi, X)\,\mathrm{d}^\circ\xi + \int_0^t \frac{\partial H}{\partial x}(\xi, X)\,\mathrm{d}^\circ X\\ &\quad - \frac{1}{12}[(\xi, X), \mathrm{Hess}\,H(\xi, X), (\xi, X)^{\mathrm{T}}](t),\end{aligned}$$

where $Y_0 = H(\xi_0, \alpha)$ and

$$\begin{aligned}&[(\xi, X), \mathrm{Hess}\,H(\xi, X), (\xi, X)^{\mathrm{T}}](t)\\ &\quad = \int_0^t \frac{\partial^3 H}{\partial r^3}(\xi_s, X_s)\,\mathrm{d}[\xi,\xi,\xi]_s + 3\int_0^t \frac{\partial^3 H}{\partial r^2 \partial x}(\xi_s, X_s)\,\mathrm{d}[\xi,\xi,X]_s\\ &\qquad + 3\int_0^t \frac{\partial^3 H}{\partial r \partial x^2}(\xi_s, X_s)\,\mathrm{d}[\xi,X,X]_s + \int_0^t \frac{\partial^3 H}{\partial x^3}(\xi_s, X_s)\,\mathrm{d}[X,X,X]_s.\end{aligned}$$

X being a solution of (4.1), we choose $Z = \partial H/\partial x$ in (4.2); using Remark 4.2(2) we obtain

$$\begin{aligned}&\int_0^t \frac{\partial H}{\partial x}(\xi, X)\,\mathrm{d}^\circ X\\ &\quad = \int_0^t \frac{\partial H}{\partial x}(\xi, X)\sigma(X)\,\mathrm{d}^\circ\xi\end{aligned}$$

$$-\frac{1}{4}\int_0^t \sigma\sigma'(X)\,\mathrm{d}\left[\frac{\partial H}{\partial x}(\xi,X),\xi,\xi\right] + \int_0^t \frac{\partial H}{\partial x}(\xi_s,X_s)b(s,X_s)\,\mathrm{d}V_s$$

$$= \int_0^t \frac{\partial H}{\partial x}(\xi,X)\sigma(X)\,\mathrm{d}^\circ\xi - \frac{1}{4}\int_0^t \sigma\sigma'(X)\frac{\partial^2 H}{\partial x^2}(\xi,X)\,\mathrm{d}[X,\xi,\xi]$$

$$-\frac{1}{4}\int_0^t \sigma\sigma'(X)\frac{\partial^2 H}{\partial r\partial x}(\xi,X)\,\mathrm{d}[\xi,\xi,\xi] + \int_0^t \frac{\partial H}{\partial x}(\xi_s,X_s)b(s,X_s)\,\mathrm{d}V_s,$$

where in the second equality we use Proposition 2.7.

Using Remark 4.2(1) and the fact that (ξ,X) an SVCV process we get

$$\mathrm{d}[\xi,\xi,X]_s = \sigma(X_s)\,\mathrm{d}[\xi,\xi,\xi]_s,$$

$$\mathrm{d}[\xi,X,X]_s = \sigma^2(X_s)\,\mathrm{d}[\xi,\xi,\xi]_s,$$

$$\mathrm{d}[X,X,X]_s = \sigma^3(X_s)\,\mathrm{d}[\xi,\xi,\xi]_s.$$

So, identities (4.6), (4.9), (4.10) and (4.11) show that

$$[(\xi,X),\mathrm{Hess}\,H(\xi,X),(\xi,X)^{\mathrm{T}}](t)$$
$$= \int_0^t (2\sigma\sigma'^2(X_s) - \sigma^2\sigma''(X_s))\frac{\partial H}{\partial x}(\xi_s,X_s)\,\mathrm{d}[\xi,\xi,\xi]_s.$$

and

$$\int_0^t \frac{\partial H}{\partial x}(\xi,X)\,\mathrm{d}^\circ X = \int_0^t \frac{\partial H}{\partial x}(\xi_s,X_s)b(s,X_s)\,\mathrm{d}V_s$$
$$- \int_0^t \frac{\partial H}{\partial r}(\xi,X)\,\mathrm{d}^\circ\xi + \frac{1}{4}\int_0^t \sigma\sigma'^2(X)\frac{\partial H}{\partial x}(\xi,X)\,\mathrm{d}[\xi,\xi,\xi].$$

This gives that

$$Y_t = Y_0 + \int_0^t \frac{\partial H}{\partial x}(\xi_s,X_s)\,b(s,X_s)\,\mathrm{d}V_s$$
$$+ \frac{1}{12}\int_0^t (\sigma\sigma'^2(X_s) + \sigma^2\sigma''(X_s))\frac{\partial H}{\partial x}(\xi_s,X_s)\,\mathrm{d}[\xi,\xi,\xi]_s.$$

X being equal to $F(\xi,Y)$, Y is then a solution of (4.5).

The proof will be concluded by the following remark. □

Remark 4.4. Eq. (4.5) is in fact a random differential equation which is driven by bounded variation processes and it has a unique solution.

In fact, in the proof of Proposition 5.3 of Russo and Vallois (2000) there are elements to prove that,

$$(t,x) \mapsto (\sigma\sigma'^2(F(\xi_t,x)) + \sigma^2\sigma''(F(\xi_t,x)))\,\frac{\partial H}{\partial x}\,(\xi_t,F(\xi_t,x))$$

$$(t,x) \mapsto b(t,F(\xi_t,x))\,\frac{\partial H}{\partial x}\,(\xi_t,F(\xi_t,x))$$

belong, ω a.s., to the LL class constituted by locally Lipschitz and of linear growth functions. Therefore, Eq. (4.5) has exactly one solution because of classical propositions of Protter (1990).

We can now state the most important result of this section.

Theorem 4.5. *Let ξ (resp. V) be a finite strong cubic (resp. bounded) variation real process. Let $\sigma:\mathbb{R}\to\mathbb{R}$ (resp. $b:[0,1]\times\mathbb{R}\to\mathbb{R}$) be of class $C^3(\mathbb{R})$ (resp. continuous), such that σ', σ'' are bounded. We suppose moreover that b is locally Lipschitz with linear growth (uniformly in t, with respect to the second variable x), α any random variable. Let Y be the unique solution to (4.5) and F the flow generated by σ. Then $X=F(\xi,Y)$ is the unique solution of (4.1).*

Proof. Recall that Y is a bounded variation continuous process. We have to prove that $X=F(\xi,Y)$ solves (4.1), i.e. (i)–(iii) of Definition 4.1.

(i) $X_0=F(\xi_0,H(\xi_0,\alpha))=\alpha$ because of (4.4).
(ii) Using Proposition 3.15 and Lemma 3.14, we get that $F(\xi,Y)$ is an SVCV process.
(iii) Let $\psi\in C^\infty(\mathbb{R}^2)$ and $Z=\psi(X,\xi)$. We apply Proposition 3.9 and Remark 3.10(1) to get

$$\int_0^t Z\,\mathrm{d}^\circ X = \int_0^t Z\,\frac{\partial F}{\partial r}\,(\xi,Y)\,\mathrm{d}^\circ\xi + \int_0^t Z\,\frac{\partial F}{\partial x}\,(\xi,Y)\,\mathrm{d}Y$$
$$-\frac{1}{4}\int_0^t \frac{\partial^2 F}{\partial r^2}\,(\xi,Y)\,\mathrm{d}[Z,\xi,\xi] - \frac{1}{12}\int_0^t Z\,\frac{\partial^3 F}{\partial r^3}\,(\xi,Y)\,\mathrm{d}[\xi,\xi,\xi].$$

We remark that $\int_0^t Z\,\frac{\partial F}{\partial r}(\xi,Y)\,\mathrm{d}^\circ\xi$ exists since (Z,ξ,Y) is an SVCV process by Lemma 3.14.

We need a few relations involving F and H. Since $F(r,H(r,x))=x$, taking the derivative with respect to x we obtain

$$\frac{\partial F}{\partial x}\,(r,H(r,x))\,\frac{\partial H}{\partial x}\,(r,x)=1. \tag{4.12}$$

We apply the operator $\partial/\partial r$ to the first identity of (4.3):

$$\frac{\partial^2 F}{\partial r^2}=\sigma\sigma'(F(r,x)). \tag{4.13}$$

$$\frac{\partial^3 F}{\partial r^3}=\sigma\sigma'^2(F(r,x))+\sigma^2\sigma''(F(r,x)). \tag{4.14}$$

Using (4.3), (4.12)–(4.14), and the fact that Y is the solution of (4.5), we have

$$\int_0^t Z_s \,\mathrm{d}^\circ X_s = \int_0^t Z_s\,\sigma(X_s)\,\mathrm{d}^\circ \xi_s + \int_0^t Z_s b(s,X_s)\,\mathrm{d}V_s$$
$$-\frac{1}{4}\int_0^t \sigma\sigma'(X)\,\mathrm{d}[Z,\xi,\xi].$$

This implies that X solves (4.1). □

4.1. On the integral equation

The definition we gave, of a solution to the differential problem (4.1), may appear unusual. One may ask if the following integral problem is well stated:

$$X(t) = \alpha + \int_0^t \sigma(X)\,\mathrm{d}^\circ \xi + \int_0^t b(s,X_s)\,\mathrm{d}V_s. \tag{4.15}$$

For this integral equation, it is hard to imagine that uniqueness will hold in the class of all continuous processes, even adapted if α is $\mathscr{F}_0$-measurable. However uniqueness (and existence) will be shown in the class $\mathscr{V}(\xi)$, defined in Remark 3.17, of processes

$$X_t = X_0 + \int_0^t \varphi(\xi_s, V_s^1,\ldots,V_s^n)\,\mathrm{d}^\circ \xi_s + V_t^0,$$

where $\varphi = \varphi(r,v) \in C^{2,1}(\mathbb{R}^{1+n})$ for some positive integer n and bounded variation processes $V^1,\ldots,V^n$, V^0.

Proposition 4.6. *Integral Eq.* (4.15) *has a unique solution in the class* $\mathscr{V}(\xi)$; *this one coincides with the solution of differential problem* (4.1).

Proof. Existence is provided by Theorem 4.5 setting $Z=1$. In order to prove uniqueness we will show that a solution to (4.15) in $\mathscr{V}(\xi)$ will solve problem (4.1).

Let X be such a solution. Clearly we have $X_0 = \alpha$. Using Proposition 3.15, Remark 3.19, Lemma 3.14 and Remark 3.17, it follows that (X,ξ) is an SVCV process. It remains to prove (iii) of Definition 4.1.

Let $Z = \psi(X,\xi)$, where ψ a C^∞ function.

$$\int_0^t Z\,\mathrm{d}^\circ X = \int_0^t Z_s\,\mathrm{d}^\circ \left(\alpha + \int_0^s \sigma(X)\,\mathrm{d}^\circ \xi + \int_0^s b(u,X_u)\,\mathrm{d}V_u\right)$$
$$= \int_0^t Z_s\,\mathrm{d}^\circ \left(\int_0^s \sigma(X)\,\mathrm{d}^\circ \xi\right) + \int_0^t Z_s\, b(s,X_s)\,\mathrm{d}V_s.$$

We observe in fact that (ξ,Z,X) is an SVCV process; Remark 3.17 says that X is of the form $\psi(\xi,V^1,\ldots,V^m)$; therefore, Lemma 3.18 tells that

$$\int_0^t Z_s\,\mathrm{d}^\circ \left(\int_0^s \sigma(X)\,\mathrm{d}^\circ \xi\right)$$

$$= \int_0^t Z\sigma(X)\,\mathrm{d}^\circ\xi - \frac{1}{4}\int_0^t \frac{\partial\psi}{\partial r}(\xi, V^1, \ldots, V^m)\sigma'(X)\,\mathrm{d}[Z,\xi,\xi]$$

$$= \int_0^t Z\sigma(X)\,\mathrm{d}^\circ\xi - \frac{1}{4}\int_0^t \sigma'(X)\,\mathrm{d}[Z,X,\xi],$$

where in the second equality we use Proposition 1. Using the SVCV properties and (4.15), we get that

$$\mathrm{d}[Z,X,\xi] = \sigma(X)\,\mathrm{d}[Z,\xi,\xi].$$

So,

$$\int_0^t Z\,\mathrm{d}^\circ X = \int_0^t Z\sigma(X)\,\mathrm{d}^\circ\xi - \frac{1}{4}\int_0^t \sigma\sigma'(X)\,\mathrm{d}[Z,\xi,\xi] + \int_0^t Z_s b(s,X_s)\,\mathrm{d}V_s.$$

(iii) is then established. □

4.2. Example: linear SDE

Let ξ be a process with finite strong cubic variation, and α any random variable. We consider the following equation:

$$\mathrm{d}X_t = X_t \circ \mathrm{d}\xi_t,$$

$$X_0 = \alpha. \tag{4.16}$$

In this case, the flow is given by $F(r,x) = x\exp(r)$, and $H(r,x) = x\exp(-r)$. The ordinary differential equation (4.5) becomes,

$$Y_t = \alpha\exp(-\xi_0) + \frac{1}{12}\int_0^t Y_s\,\mathrm{d}[\xi,\xi,\xi]_s,$$

whose unique solution is the following process:

$$Y_t = \alpha\exp(-\xi_0)\exp\left(\frac{1}{12}[\xi,\xi,\xi]_t\right), \quad t \in [0,1].$$

This gives, using Theorem 4.5, that the unique solution of (4.16) is given by

$$X_t = \alpha\exp(-\xi_0)\exp\left(\frac{1}{12}[\xi,\xi,\xi](t) + \xi(t)\right), \quad t \in [0,1].$$

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Appendix A.

Proof of Proposition 2.17. We will proceed by recurrence. We will start with $n=3$, however, the case $n=2$ may be considered done by (2.11). Using Itô–Stratonovich calculus we get

$$\begin{aligned} I_3^\circ(f)(t) &= \int_0^t \int_0^{s_3} \int_0^{s_2} f(s_1,s_2,s_3)\,\mathrm{d}^\circ M_{s_1}\,\mathrm{d}^\circ M_{s_2}\,\mathrm{d}^\circ M_{s_3} \\ &= \int_0^t \int_0^{s_3} \int_0^{s_2} f(s_1,s_2,s_3)\,\mathrm{d}M_{s_1}\,\mathrm{d}^\circ M_{s_2}\,\mathrm{d}^\circ M_{s_3} \\ &:= I_1 + I_2, \end{aligned}$$

where

$$I_1 = \int_0^t \left(\int_0^{s_3} \int_0^{s_2} f(s_1,s_2,s_3)\,\mathrm{d}M_{s_1}\,\mathrm{d}M_{s_2} \right) \mathrm{d}^\circ M_{s_3}$$

$$I_2 = \frac{1}{2} \int_0^t \left(\left[\int_0^{\cdot} f(s_1,\cdot,s_3)\,\mathrm{d}M_{s_1}, M \right]_{s_3} \right) \mathrm{d}^\circ M_{s_3}.$$

Now

$$\begin{aligned} I_1 &= \int_0^t \int_0^{s_3} \int_0^{s_2} f(s_1,s_2,s_3)\,\mathrm{d}M_{s_1}\,\mathrm{d}M_{s_2}\,\mathrm{d}M_{s_3} \\ &\quad + \frac{1}{2} \left[\int_0^{\cdot} \int_0^{s_2} f(s_1,s_2,\cdot)\,\mathrm{d}M_{s_1}\,\mathrm{d}M_{s_2}, M \right](t). \end{aligned}$$

Using Corollary 2.11, with $G(t,s_2) = \int_0^{s_2} f(s_1,s_2,t)\,\mathrm{d}M_{s_1}$, we get

$$\begin{aligned} I_1 &= \int_0^t \int_0^{s_3} \int_0^{s_2} f(s_1,s_2,s_3)\,\mathrm{d}M_{s_1}\,\mathrm{d}M_{s_2}\,\mathrm{d}M_{s_3} \\ &\quad + \frac{1}{2} \int_0^t \int_0^{s_2} f(s_1,s_2,s_2)\,\mathrm{d}M_{s_1}\,\mathrm{d}[M]_{s_2}; \end{aligned}$$

moreover, using again the same Corollary with $G(t,s_1) = f(s_1,t,s_3)$, we get

$$\left[\int_0^{\cdot} f(s_1,\cdot,s_3)\,\mathrm{d}M_{s_1}, M \right]_t = \int_0^t f(s_1,s_1,s_3)\,\mathrm{d}[M]_{s_1}.$$

This implies that

$$\begin{aligned} I_2 &= \frac{1}{2} \int_0^t \int_0^{s_3} f(s_1,s_1,s_3)\,\mathrm{d}[M]_{s_1}\,\mathrm{d}^\circ M_{s_3} \\ &= \frac{1}{2} \int_0^t \int_0^{s_2} f(s_1,s_1,s_2)\,\mathrm{d}[M]_{s_1}\,\mathrm{d}M_{s_2}. \end{aligned}$$

This gives

$$
\begin{aligned}
I_3^{\circ}(f)(t) = & \int_0^t \int_0^{s_3} \int_0^{s_2} f(s_1,s_2,s_3)\,\mathrm{d}M_{s_1}\,\mathrm{d}M_{s_2}\,\mathrm{d}M_{s_3} \\
& + \frac{1}{2}\int_0^t \int_0^{s_2} f(s_1,s_2,s_2)\,\mathrm{d}M_{s_1}\,\mathrm{d}[M]_{s_2} \\
& + \frac{1}{2}\int_0^t \int_0^{s_2} f(s_1,s_1,s_2)\,\mathrm{d}[M]_{s_1}\,\mathrm{d}M_{s_2}.
\end{aligned}
$$

Now let $n \geqslant 3$, and suppose that the statement of Proposition 2.17 holds for all $m \leqslant n$. We have

$$
\begin{aligned}
I_{n+1}^{\circ}(f)(t) &= \int_0^t \int_0^{s_{n+1}} \cdots \int_0^{s_2} f(s_1,\ldots,s_{n+1})\,\mathrm{d}^{\circ}M_{s_1}\cdots\mathrm{d}^{\circ}M_{s_{n+1}} \\
&= \int_0^t \left(\int_0^{s_{n+1}} \cdots \int_0^{s_2} f(s_1,\ldots,s_{n+1})\,\mathrm{d}^{\circ}M_{s_1}\cdots\mathrm{d}^{\circ}M_{s_n}\right)\mathrm{d}M_{s_{n+1}} \\
&\quad + \frac{1}{2}\left[\int_0^{\cdot} \int_0^{s_n} \cdots \int_0^{s_2} f(s_1,\ldots,s_n,\cdot)\,\mathrm{d}^{\circ}M_{s_1}\cdots\mathrm{d}^{\circ}M_{s_{n-1}}\,\mathrm{d}M_{s_n}, M\right]_t \\
&:= I_1 + I_2,
\end{aligned}
$$

where s_n plays the role of t.

Using induction hypothesis for $m = n$, and renaming the variable s_{n+1} with s_{n+1-k}, we have

$$
\begin{aligned}
I_1 = \sum_{k=0}^{[n/2]} \frac{1}{2^k} \sum_{\sigma\in\Sigma_k^n} \int_0^t \int_0^{s_{n+1-k}} \int_0^{s_{n-k}} \cdots \int_0^{s_2} & f(s_{\sigma^{-1}(1)}, s_{\sigma^{-1}(1)}, \ldots, s_{\sigma^{-1}(k)}, s_{\sigma^{-1}(k)}, \\
& s_{\sigma^{-1}(k+1)}, s_{\sigma^{-1}(k+2)}, \ldots, s_{\sigma^{-1}(n-k)}, s_{n+1-k})\mathrm{d}\sigma_M^1(s_1)\cdots\mathrm{d}\sigma_M^{n-k}(s_{n-k})\,\mathrm{d}M_{s_{n+1-k}};
\end{aligned}
$$

$\sigma \in \Sigma_k^n$ is extended trivially to $\sigma \in \Sigma_k^{n+1}$ setting $\sigma^{n+1-k} = n+1-k$.

Using Corollary 2.11 with

$$
G(t,s_n) = \int_0^{s_n} \cdots \int_0^{s_2} f(s_1,\ldots,s_n,t)\,\mathrm{d}^{\circ}M_{s_1}\cdots\mathrm{d}^{\circ}M_{s_{n-1}}
$$

we get

$$
I_2 = \frac{1}{2}\int_0^t \left(\int_0^{s_n} \cdots \int_0^{s_2} f(s_1,\ldots,s_n,s_n)\,\mathrm{d}^{\circ}M_{s_1}\cdots\mathrm{d}^{\circ}M_{s_{n-1}}\right)\mathrm{d}[M]_{s_n}.
$$

Using this time the induction hypothesis for $m = n-1$ and renaming the variable s_n with s_{n-l}, we obtain

$$
I_2 = \sum_{l=0}^{[(n-1)/2]} \frac{1}{2^{l+1}} \sum_{\sigma\in\Sigma_l^{n-1}} \int_0^t \int_0^{s_{n-l}} \int_0^{s_{(n-1)-l}} \cdots \int_0^{s_2}
$$

$$\times f(s_{\sigma^{-1}(1)}, s_{\sigma^{-1}(1)}, \ldots, s_{\sigma^{-1}(l)}, s_{\sigma^{-1}(l)},$$

$$s_{\sigma^{-1}(l+1)}, s_{\sigma^{-1}(l+2)}, \ldots, s_{\sigma^{-1}(n-1-l)}, s_{n-l}, s_{n-l})$$

$$\times \mathrm{d}\sigma_M^1(s_1)\cdots \mathrm{d}\sigma_M^{n-1-l}(s_{n-1-l})\,\mathrm{d}[M]_{s_{n-l}}.$$

We change the variable l with $k=l+1$, so $1 \leqslant k \leqslant [(n-1)/2]+1=[(n+1)/2]$ which gives that

$$I_2 = \sum_{k=1}^{[(n+1)/2]} \frac{1}{2^k} \sum_{\sigma\in\Sigma_{k-1}^{n-1}} \int_0^t \int_0^{s_{(n+1)-k}} \cdots \int_0^{s_2}$$

$$\times f(s_{\sigma^{-1}(1)}, s_{\sigma^{-1}(1)}, \ldots, s_{\sigma^{-1}(k-1)}, s_{\sigma^{-1}(k-1)},$$

$$s_{\sigma^{-1}(k)}, s_{\sigma^{-1}(k+1)}, \ldots, s_{\sigma^{-1}(n-k)}, s_{n+1-k}, s_{n+1-k})$$

$$\times \mathrm{d}\sigma_M^1(s_1)\cdots \mathrm{d}\sigma_M^{(n-1)-(k-1)}(s_{n-k})\,\mathrm{d}[M]_{s_{(n+1)-k}}.$$

For given $\sigma\in\Sigma_{k-1}^{n-1}$, we define $\tilde\sigma\in\Sigma_k^{n+1}$ by

$$\tilde\sigma^i = \begin{cases} \sigma^i & \text{for } i=1,2,\ldots n-k,\\ k & \text{for } i=n+1-k. \end{cases}$$

Recall that we have to prove

$$I_1+I_2 = \sum_{k=0}^{[(n+1)/2]} \frac{1}{2^k} \sum_{\sigma\in\Sigma_k^{n+1}} \int_0^t \int_0^{s_{(n+1)-k}} \cdots \int_0^{s_2}$$

$$\times f(s_{\sigma^{-1}(1)}, s_{\sigma^{-1}(1)}, \ldots, s_{\sigma^{-1}(k)}, s_{\sigma^{-1}(k)},$$

$$s_{\sigma^{-1}(k+1)}, s_{\sigma^{-1}(k+2)}, \ldots, s_{\sigma^{-1}((n+1)-k)})$$

$$\times \mathrm{d}\sigma_M^1(s_1)\cdots \mathrm{d}\sigma_M^{n-k}(s_{n-k})\,\mathrm{d}\sigma_M^{(n+1)-k}(s_{n+1-k}).$$

To obtain it we have just to see that

$$P_0^{n+1}(M) = \left\{ \left(\underbrace{M, M, \ldots, M}_{(n+1)\ \text{times}} \right) \right\} = \left\{ \sigma \left(\underbrace{M, M, \ldots, M}_{(n+1)\ \text{times}} \right) \right\},$$

where $\sigma^i = i$, $i=1,\ldots,n+1$. For all $1 \leqslant k \leqslant [(n+1)/2]$, we have

$$P_k^{n+1}(M) = \left\{ \left(\sigma(\underbrace{[M],\ldots,[M]}_{k\ \text{times}}, \underbrace{M,\ldots,M}_{n-2k\ \text{times}}), M \right); \quad \sigma\in\Sigma_k^n \right\}$$

$$\cup\left\{\left(\sigma(\underbrace{[M],\ldots,[M]}_{(k-1)\ \text{times}},\quad\underbrace{M,\ldots,M}_{(n-1)-2(k-1)\ \text{times}}\quad),[M]\right);\ \sigma\in\Sigma_{k-1}^{n-1}\right\}$$

$$=\{(\sigma_M,M);\sigma\in\Sigma_k^n\}\cup\{(\sigma_M,[M]);\ \sigma\in\Sigma_{k-1}^{n-1}\},$$

where previous union is disjoint. □

Remark. We recall that the cardinality of $P_k^n(M)$ (resp. $P_{k-1}^{n-1}(M)$) equals C_{n-k}^k (resp. C_{n-k}^{k-1}). Therefore, the cardinality of $P_k^{n+1}(M)$ is C_{n+1-k}^k as expected by combinatorial calculus.

Proof of Proposition 2.19. It remains to prove the following property: for all $n\geqslant 3$, and every $k=1,\ldots,[n/2]$,

$$\begin{aligned}L_k^n(t) := &\sum_{\sigma\in\Sigma_k^n}\int_0^t\int_0^{s_{n-k}}\cdots\int_0^{s_2}\\ &\times f(s_{\sigma^{-1}(1)},s_{\sigma^{-1}(1)},\ldots,s_{\sigma^{-1}(k)},s_{\sigma^{-1}(k)},s_{\sigma^{-1}(k+1)},\ldots,s_{\sigma^{-1}(n-k)})\\ &\times \mathrm{d}\sigma_W^1(s_1)\cdots\mathrm{d}\sigma_W^{n-k}(s_{n-k})\\ = &\int_0^t\int_0^{s_{n-k}}\cdots\int_0^{s_{k+2}}\int_0^{\mathbf{t}}\int_0^{s_k}\cdots\int_0^{s_2}\\ &\times f(s_1,s_1,\ldots,s_k,s_k,s_{k+1},s_{k+2},\ldots,s_{n-k})\\ &\times \mathrm{d}s_1\cdots\mathrm{d}s_k\,\mathrm{d}W_{s_{k+1}}\cdots\mathrm{d}W_{s_{n-k}}.\end{aligned}$$

We will prove by induction. We start proving the case $n=3$. We have,

$$P_0^3(W)=\{(W,W,W)\}$$

$$P_1^3(W)=\{([W],W),(W,[W])\}.$$

This gives

$$L_0^3(t)=\int_0^t\int_0^{s_3}\int_0^{s_2} f(s_1,s_2,s_3)\,\mathrm{d}W_{s_1}\,\mathrm{d}W_{s_2}\,\mathrm{d}W_{s_3},$$

and

$$\begin{aligned}L_1^3(t)&=\int_0^t\int_0^{s_2} f(s_1,s_1,s_2)\,\mathrm{d}s_1\,\mathrm{d}W_{s_2}+\int_0^t\int_0^{s_2} f(s_1,s_2,s_2)\,\mathrm{d}W_{s_1}\,\mathrm{d}s_2\\ &=\int_0^t\int_0^{s_2} f(s_1,s_1,s_2)\,\mathrm{d}s_1\,\mathrm{d}W_{s_2}+\int_0^t\int_{s_1}^t f(s_1,s_2,s_2)\,\mathrm{d}s_2\,\mathrm{d}W_{s_1}\end{aligned}$$

$$= \int_0^t \int_0^{s_2} f(s_1, s_1, s_2)\, \mathrm{d}s_1\, \mathrm{d}W_{s_2} + \int_0^t \int_{s_2}^t f(s_1, s_1, s_2)\, \mathrm{d}s_1\, \mathrm{d}W_{s_2}$$

$$= \int_0^t \int_0^t f(s_1, s_1, s_2)\, \mathrm{d}s_1\, \mathrm{d}W_{s_2}.$$

This means that the Proposition is true for $n = 3$.

Now let $n \geqslant 3$, and suppose that the statement of the Proposition holds for all $m \leqslant n$. Recall that, for all $1 \leqslant k \leqslant [(n+1)/2]$, we have

$$P_k^{n+1}(W) = \{(\sigma_W, W); \sigma \in \Sigma_k^n\} \cup \{(\sigma_W, [W]); \sigma \in \Sigma_{k-1}^{n-1}\},$$

where the union is disjoint. Therefore

$$L_k^{n+1}(t) = I_1 + I_2,$$

where

$$I_1 = \sum_{\sigma \in \Sigma_k^n} \int_0^t \left(\int_0^s \int_0^{s_{n-k}} \cdots \int_0^{s_2} \right.$$
$$\times f(s_{\sigma^{-1}(1)}, s_{\sigma^{-1}(1)}, \ldots, s_{\sigma^{-1}(k)}, s_{\sigma^{-1}(k)}, s_{\sigma^{-1}(k+1)}, \ldots, s_{\sigma^{-1}(n-k)}, s)$$
$$\left. \times\ \mathrm{d}\sigma_W^1(s_1) \cdots \mathrm{d}\sigma_W^{n-k}(s_{n-k}) \right) \mathrm{d}W_s.$$

and

$$I_2 = \sum_{\sigma \in \Sigma_{k-1}^{n-1}} \int_0^t \left(\int_0^s \int_0^{s_{n-k}} \cdots \int_0^{s_2} \right.$$
$$\times f(s_{\sigma^{-1}(1)}, \ldots, s_{\sigma^{-1}(k-1)}, s_{\sigma^{-1}(k-1)}, s_{\sigma^{-1}(k)}, \ldots, s_{\sigma^{-1}(n-k)}, s, s)$$
$$\left. \times\ \mathrm{d}\sigma_W^1(s_1) \cdots \mathrm{d}\sigma_W^{n-k}(s_{n-k}) \right) \mathrm{d}s.$$

Using the hypothesis for $m = n$, and renaming s with s_{n+1-k}, we have

$$I_1 = \int_0^t \int_0^{s_{n+1-k}} \int_0^{s_{n-k}} \cdots \int_0^{s_{k+2}} \int_0^{\mathbf{s_{n+1-k}}} \int_0^{s_k} \cdots \int_0^{s_2}$$
$$\times f(s_1, s_1, \ldots, s_k, s_k, s_{k+1}, s_{k+2}, \ldots, s_{n+1-k})$$
$$\times \mathrm{d}s_1 \cdots \mathrm{d}s_k \mathrm{d}W_{s_{k+1}} \cdots \mathrm{d}W_{s_{n-k}}\, \mathrm{d}W_{s_{n+1-k}}.$$

Using now the induction hypothesis for $m = n - 1$, we have

$$I_2 = \int_0^t \int_0^s \int_0^{s_{n-k}} \cdots \int_0^{s_{k+1}} \int_0^{\mathbf{s}} \int_0^{s_{k-1}} \cdots \int_0^{s_2}$$
$$\times f(s_1, s_1, \ldots, s_{k-1}, s_{k-1}, s_k, s_{k+1}, \ldots, s_{n-k}, s, s)$$

$$\times \mathrm{d}s_1 \cdots \mathrm{d}s_{k-1}\,\mathrm{d}W_{s_k} \cdots \mathrm{d}W_{s_{n-k}}\,\mathrm{d}s$$

$$= \int_0^t \int_{s_{n-k}}^t \int_0^{s_{n-k}} \cdots \int_0^{s_{k+1}} \int_0^{\mathbf{s}} \int_0^{s_{k-1}} \cdots \int_0^{s_2}$$

$$\times f(s_1, s_1, \ldots, s_{k-1}, s_{k-1}, s_k, s_{k+1}, \ldots, s_{n-k}, s, s)$$

$$\times \mathrm{d}s_1 \cdots \mathrm{d}s_{k-1}\,\mathrm{d}W_{s_k} \cdots \mathrm{d}W_{s_{n-k-1}}\,\mathrm{d}s\,\mathrm{d}W_{s_{n-k}}$$

$$= \int_0^t \int_0^{s_{n-k}} \cdots \int_0^{s_{k+1}} \int_{s_{n-k}}^t \int_0^{\mathbf{s}} \int_0^{s_{k-1}} \cdots \int_0^{s_2}$$

$$\times f(s_1, s_1, \ldots, s_{k-1}, s_{k-1}, s_k, s_{k+1}, \ldots, s_{n-k}, s, s)$$

$$\times \mathrm{d}s_1 \cdots \mathrm{d}s_{k-1}\,\mathrm{d}s\,\mathrm{d}W_{s_k} \cdots \mathrm{d}W_{s_{n-k-1}}\,\mathrm{d}W_{s_{n-k}}.$$

Renaming s with s_k, s_k with $s_{k+1}, \ldots, s_{n-k}$ with s_{n+1-k}, we obtain

$$I_1 = \int_0^t \int_0^{s_{n+1-k}} \cdots \int_0^{s_{k+2}} \int_{\mathbf{s_{n+1-k}}}^t \int_0^{s_k} \int_0^{s_{k-1}} \cdots \int_0^{s_2}$$

$$\times f(s_1, s_1, \ldots, s_k, s_k, s_{k+1}, s_{k+2}, \ldots, s_{n+1-k})$$

$$\times \mathrm{d}s_1 \cdots \,\mathrm{d}s_k\,\mathrm{d}W_{s_{k+1}} \cdots \mathrm{d}W_{s_{n+1-k}}.$$

This gives that

$$I_1 + I_2 = \int_0^t \int_0^{s_{n+1-k}} \cdots \int_0^{s_{k+2}} \int_0^{\mathbf{t}} \int_0^{s_k} \int_0^{s_{k-1}} \cdots \int_0^{s_2}$$

$$\times f(s_1, s_1, \ldots, s_k, s_k, s_{k+1}, s_{k+2}, \ldots, s_{n+1-k})$$

$$\times \mathrm{d}s_1 \cdots \,\mathrm{d}s_k\,\mathrm{d}W_{s_{k+1}} \cdots \mathrm{d}W_{s_{n+1-k}}. \qquad \square$$

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